Selfish Routing
Price of Anarchy and Coordination Mechanisms

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Outline

Introduction

Price of Anarchy

Question

Answer?

Theorem

Construction
Price of Anarchy and Coordination Mechanisms

Global Optimum versus Selfish Behavior

consider a situation with many independent agents, e.g., traffic
Nash equilibrium = each agent optimizes its own fate
Global optimum = a solution of minimum cost
Price of Anarchy = \( \max \frac{\text{Cost of a Nash Equilibrium}}{\text{Cost of Global Optimum}} \)

Koutsoupias/Papadimitriou (99)

Coordination Mechanism = increase of costs that makes selfish agents behave differently
Routing

Basic Notation I

- $G = (V, E)$, a network, $s = \text{source}$, $t = \text{sink}$
- want to send $r$ units of flow from $s$ to $t$
- $f = \text{a flow of rate } r$
- $f_e = \text{flow across edge } e$

The cost of a flow

$$C(f) = \sum_e \text{cost of } e \text{ at flow } f_e \cdot f_e$$

Observe: Cost (latency) of an edge depends on flow across it
Routing

**Basic Notation II**
- $\ell_e(x) =$ latency (cost) of $e$ as a function of flow over $e$
- affine cost functions: $\ell_e(x) = a_e x + b_e$ with $a_e \geq 0$ and $b_e \geq 0$

**The cost of a flow**

$$C(f) = \sum_{e} \ell_e(f_e)f_e$$
Optimal Flow

- Cost of upper link \( x \) = 0 \( x \) + 1, cost of lower link \( x \) = 1 \( x \) + 0
- \( f^* = f^*(r) = \) optimum flow for rate \( r = \) flow of minimum cost
- Here: \( C(f^*) = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} \)
- Opt-flow minimizes \( f_1 \cdot f_1 + 1 \cdot f_2 \) subject to \( r = f_1 + f_2, f_i \geq 0 \)
  - Marginal costs are identical; here \( \frac{d}{dx} x^2 |_{x=1/2} = \frac{d}{dx} x |_{x=1/2} \)
- Selfish agents will deviate from optimum flow
Nash Flow

- Nash flow = no gain by deviating infinitesimally, i.e., all used edges have the same latency
- \( f^N = f^N(r) = \text{Nash flow for rate } r \)
- here: \( C(f^N) = 1 \cdot 0 + 1 \cdot 1 = 1 \)

Price of Anarchy

\[
\text{PoA} = \max_{r > 0} \frac{C(f^N(r))}{C(f^*(r))} \geq \frac{C(f^N(1))}{C(f^*(1))} = \frac{1}{\frac{3}{4}} = \frac{4}{3}
\]
Remarks

- $C(f^N(r))$ and $C(f^*(r))$ are piecewise quadratic functions in $r$
- $PoA$ is quotient of piecewise quadratic functions in $r$
Roughgarden/Tardos (02): aff. costs, \( PoA \leq \frac{4}{3} \)

proof for two links: assume Nash and Opt both use both links

let \( L = \ell_1(f_1^N) = \ell_2(f_2^N) \) and assume \( f_1^* \leq f_1^N \)

\[
C^N - C^* = L(f_1^N + f_2^N) - \ell_1(f_1^*)f_1^* - \ell_2(f_2^*)f_2^*
\]

\[
= L(f_1^* + f_2^*) - \ell_1(f_1^*)f_1^* - \ell_2(f_2^*)f_2^*
\]

\[
= \left( \ell_1(f_1^N) - \ell_1(f_1^*) \right) f_1^* + \left( \ell_2(f_2^N) - \ell_1(f_2^*) \right) f_2^*
\]

\[
\leq \left( \ell_1(f_1^N) - \ell_1(f_1^*) \right) f_1^*
\]

\[
\leq \frac{\ell_1(f_1^N)f_1^N}{4} \leq \frac{C^N}{4}
\]

and hence \( (1 - \frac{1}{4})C^N \leq C^* \). Thus \( C^N \leq \frac{4}{3}C^* \).
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= L(f_1^* + f_2^*) - \ell_1(f_1^*)f_1^* - \ell_2(f_2^*)f_2^*
= \left( \ell_1(f_1^N) - \ell_1(f_1^*) \right) f_1^* + \left( \ell_2(f_2^N) - \ell_1(f_2^*) \right) f_2^*
\leq \left( \ell_1(f_1^N) - \ell_1(f_1^*) \right) f_1^*
\leq \frac{\ell_1(f_1^N) f_1^N}{4} \leq \frac{C^N}{4}
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proof for two links: assume Nash and Opt both use both links

let $L = \ell_1(f_1^N) = \ell_2(f_2^N)$ and assume $f_1^* \leq f_1^N$

\[
C^N - C^* = L(f_1^N + f_2^N) - \ell_1(f_1^*)f_1^* - \ell_2(f_2^*)f_2^*
\]

\[
= L(f_1^* + f_2^*) - \ell_1(f_1^*)f_1^* - \ell_2(f_2^*)f_2^*
\]

\[
= \left(\ell_1(f_1^N) - \ell_1(f_1^*)\right)f_1^* + \left(\ell_2(f_2^N) - \ell_1(f_2^*)\right)f_2^*
\]

\[
\leq \left(\ell_1(f_1^N) - \ell_1(f_1^*)\right)f_1^*
\]

\[
\leq \frac{\ell_1(f_1^N)f_1^N}{4} \leq \frac{C^N}{4}
\]

and hence $(1 - \frac{1}{4})C^N \leq C^*$. Thus $C^N \leq \frac{4}{3}C^*$.
The Key Inequality

\[
\left( \ell_1(f_1^N) - \ell_1(f_1^*) \right) f_1^* \leq \frac{\ell_1(f_1^N) f_1^N}{4}
\]

(Correa/Schulz/Stier-Moses, 08)
The Question

Summary

For affine costs, the price of anarchy can be as large as $4/3$, but is never larger.

Question

Can we reduce the price of anarchy by a coordination mechanism? In particular, by taxes or tolls? In other words

- underlying network is unchanged
- we increase the cost (latency) of some edges.
- this leads to a change of behavior of selfish agents
- such that total cost goes down
- although cost of new Nash flow is computed with respect to increased costs!!!!
Question rephrased

Can making edges more expensive reduce the overall cost by leading to “better” behavior of selfish agents?
Engineered Price of Anarchy (ePoA)

- replace $\ell_e$ by $\hat{\ell}_e$ with $\hat{\ell}_e(x) \geq \ell_e(x)$ for all $x$.
- $\hat{C}^N = \hat{C}^N(r) = \text{cost of Nash flow of rate } r \text{ for } \hat{\ell} \text{ computed with respect to } \hat{\ell}$
- Are there $\hat{\ell}$ such that for all $r$

$$ePoA(r) = \frac{\hat{C}^N(r)}{C^*(r)} < \frac{4}{3}?$$

- Observe: $\hat{C}^N$ is with respect to increased costs, $C^*$ is with respect to original costs. We want a solution that works for all $r$. 
The Answer is clearly NO

Obviously, increasing edge costs can never decrease total cost

A Negative Result

If the $\hat{\ell}$ are continuous, then $\hat{C}^N(r) \geq C^N(r)$ for all $r$
and hence $ePoA(r) \geq PoA(r)$ for all $r$
A Non-Solution: Marginal Cost Pricing

\[ \hat{\ell}(x) = \frac{d}{dx} \ell(x) x = 2ae x + be \]

- Nash flow for marginal cost latencies = optimal flow for original latencies
- but \( \hat{C}^N(1) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1 \) and hence \( ePoA(1) \geq \frac{4}{3} \)
- \( \hat{C}^N(\epsilon) = 2\epsilon^2 = 2C^*(\epsilon) \) and hence \( ePoA(\epsilon) = 2 \).
The Answer might be **Yes**

- Nash flow = Optimal flow for all $r$ and
- $\hat{C}^N = C^*$ for all $r$
- Thus $ePoA = 1$
Braess’ Paradox

At rate $r = 1$,
- Opt routes 1/2 each along upper and lower path: $C^*(1) = 3/2$
- Nash routes 1 along path $x \to 0 \to x$: $C^N(1) = 2$
- deleting the edge of cost zero, i.e., setting its cost to $\infty$, makes the optimum flow a Nash flow, i.e., $\hat{C}^N(1) = 3/2$
- generally, $\hat{\ell}(x) = 0$ for $x \leq 2/3$ and $\infty$ ow

In Stuttgart, after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.
A Theorem

For any network of \( k \) parallel links, there are modified latency functions \( \hat{\ell}_1 \) to \( \hat{\ell}_k \) with \( \hat{\ell}_i \geq \ell_i \) such that

\[
\frac{\hat{C}^N(r)}{C^*(r)} \leq c_k < \frac{4}{3} \quad \text{for all } r.
\]

- \( c_2 \leq 5/4 \) by an easy argument
- \( c_2 \leq 1.192 \) by an involved argument
- \( c_k \to 4/3 \) for \( k \to \infty \)
Open Problems

- improved upper bounds
  - improve upper bound for $c_2$?
  - is there a construction with $c_k \leq c < 4/3$ for all $k$
- lower bounds: we know $c_2 \geq 1.02$.
- general networks instead of parallel links
- more general cost functions, e.g., polynomial cost functions
- atomic flow, i.e., flow consists of units of fixed size instead of infinitesimal units
Two Links, $b_1 < b_2$

- Nash starts to use the second link at $r = r_2^N = \frac{b_2-b_1}{a_1}$
- worst-case PoA is at this rate, flows are:

$$\text{Nash: } (r, 0) \quad \text{Opt: } (f_1^*, f_2^*) = (f_1^*, r - f_1^*)$$

$$\text{PoA} \leq \frac{4 + 4R}{3 + 4R}$$

where $R = a_2/a_1$
The Key Inequality Revised

flows are:  \( (r, 0) \) \hspace{1cm} \text{Opt:} \ (f_1^*, f_2^*) = (f_1^*, r - f_1^*)

\[ \ell_2(x) = a_2 x + b_2 \]

\[ \ell_1(x) = a_1 x + b_1 \]

Opt saves the red area, but pays the blue area. \( \frac{\text{red} - \text{blue}}{\text{cyan}} \leq \ldots \)
Two Links: Engineered Price of Anarchy

\[ PoA \leq \frac{4 + 4R}{3 + 4R} \text{ where } R = \frac{a_2}{a_1} \]

**The benign case:** \( R \geq 1/4 \)

Then \( PoA \leq \frac{5}{4} \)

We do nothing, i.e. \( \hat{\ell}_i = \ell_i \) for all \( i = 1, 2 \).

**The non-benign case:** \( R < 1/4 \)

see next slide
Non-benign Case: \( R = \frac{a_2}{a_1} < \frac{1}{4} \)

- second link is much more efficient than first
- Nash is hurt since it uses second link only at \( r_2^N \).
- we modify \( \ell_1 \) as follows (\( \ell_2 \) stays unchanged)

\[
\hat{\ell}_1(x) = \begin{cases} 
\ell_1(x) & \text{for } x \leq r_2^* \\
\infty & \text{for } x > r_2^*
\end{cases}
\]

- this limits the flow on link 1 to \( r_2^* \).

\[ ePoA \leq 1 + R \leq \frac{5}{4} \]
2 Links: Advanced Solution

- in the non-benign case (with modified threshold)
- we modify $\ell_1$ as follows ($\ell_2$ stays unchanged)

$$
\hat{\ell}_1(x) = \begin{cases} 
\ell_1(x) & \text{for } x \leq x_1 \text{ or } x > x_2 \\
\ell_1(x_2) & \text{for } x_1 < x \leq x_2 
\end{cases}
$$

- this forces Nash to use second link early, but also allows Nash to use both links at high rates
$k$ Links

- highest link is unchanged
- consider any link which is not the highest:
- if there is no higher link that is much more efficient, we leave it unchanged
- if there is a higher link that is much more efficient, we modify the cost function such that the higher link is used earlier.
Conclusion

- first study of coordination mechanisms for routing games
- we show that coordination mechanisms improve price of anarchy for networks of parallel links.
- many open problems
  - improved upper bounds
    - what is $c_2$?
    - is there a construction with $c_k < 4/3 - \epsilon$ for all $k$
  - lower bounds: is $ePoA > 1$ for the case of two links?
  - general networks instead of parallel links
  - more general cost functions, e.g., polynomial cost functions
  - atomic flow, i.e., flow consists of units of fixed size instead of infinitesimal units