Reachability Substitutes for Planar Digraphs

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Joint work with
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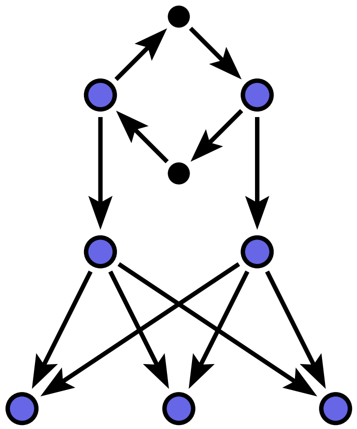
Martin Skutella (Universität Dortmund)

Reachability Substitutes

Given a digraph $G = (V, E)$ with a set of vertices U marked "interesting".



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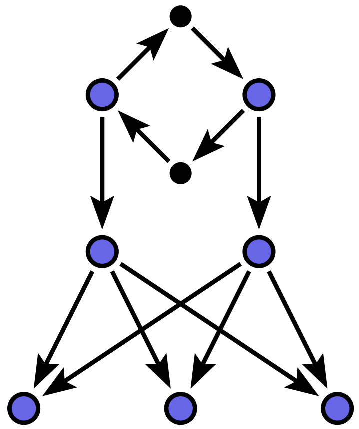

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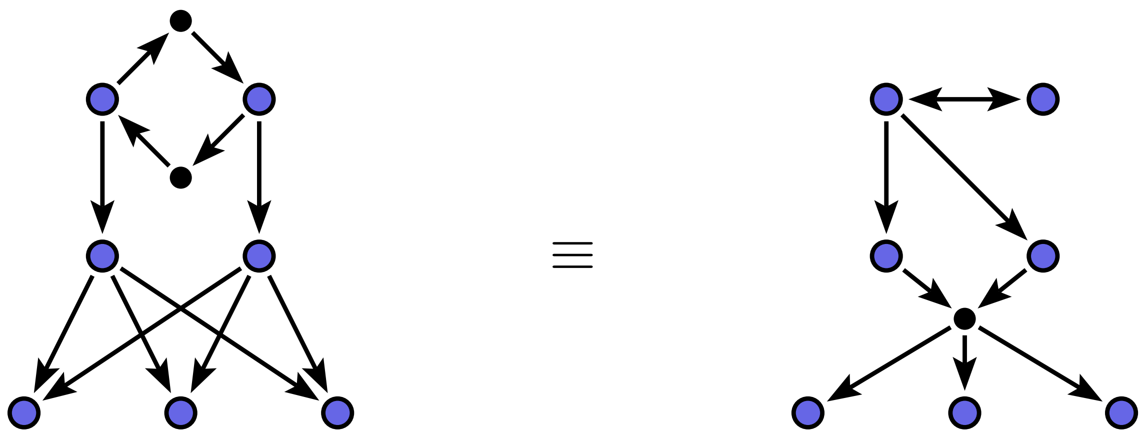
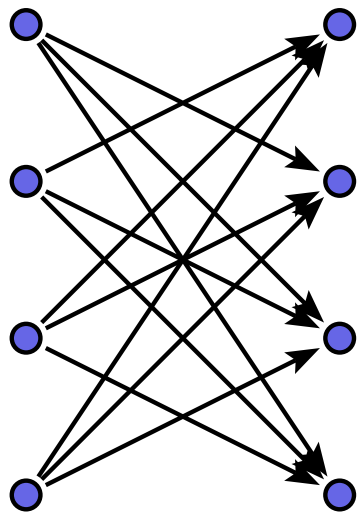
Def. Two digraphs $G = (V, E)$ and $G' = (V', E')$ are reachability substitutes for each other (w.r.t. U) if for all $u, v \in U \subseteq V, V'$:

$$u \overset{G}{\rightsquigarrow} v \quad \text{iff} \quad u \overset{G'}{\rightsquigarrow} v$$



Bad News

Theorem. Almost all digraphs with k interesting vertices have only RSs of size $\Omega(k^2/\log k)$.



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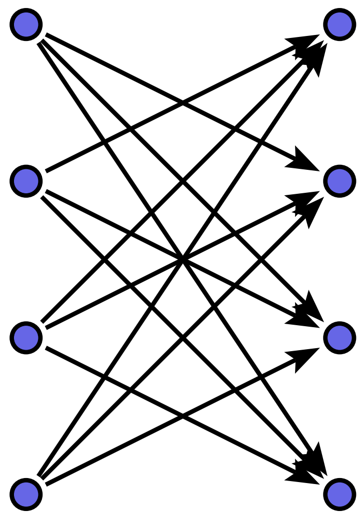

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Theorem. Finding a minimum RS (size = $|V| + |E|$) for a given digraph is NP-hard.

Planar Digraphs

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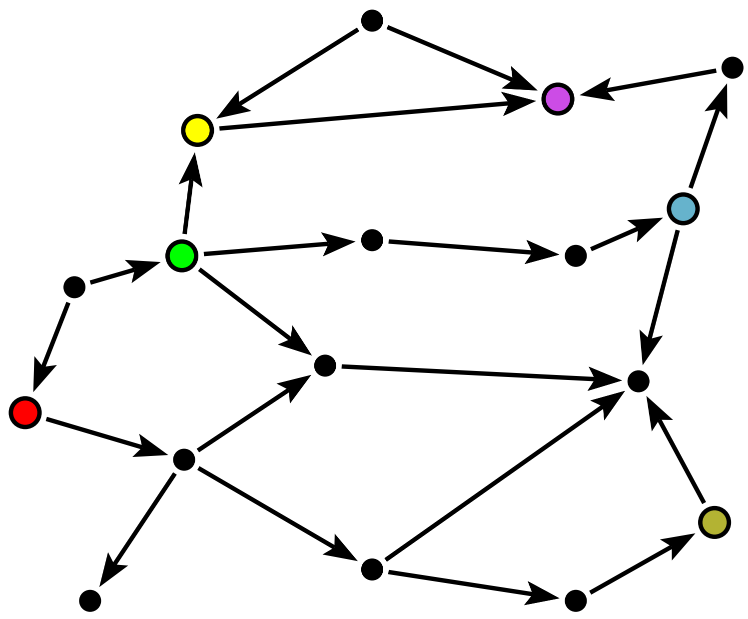
Previous result [Subramanian, 1993]: If all interesting vertices lie on a constant number of faces then there is a substitute of size $O(k \log k)$.

Tools & Techniques

- separation (balanced directed cuts)



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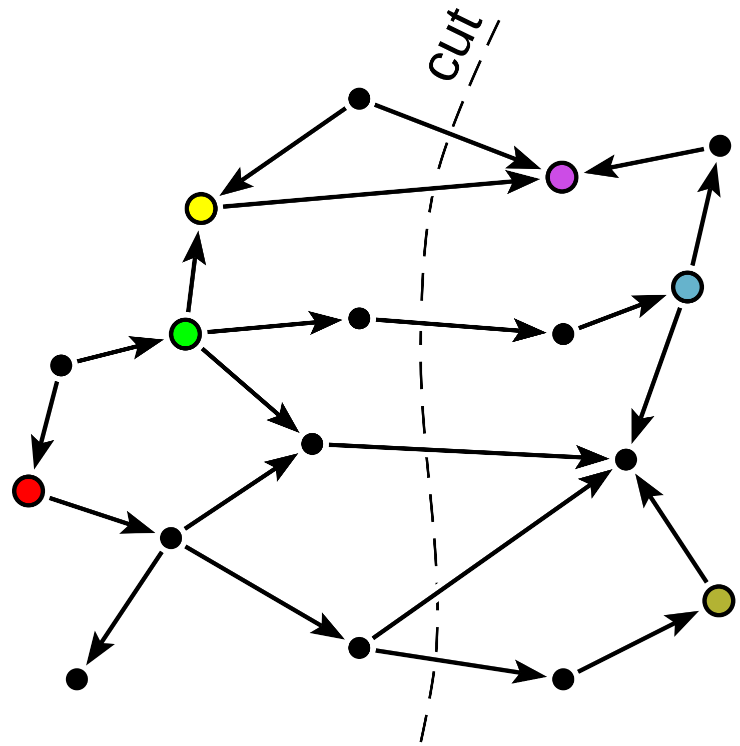

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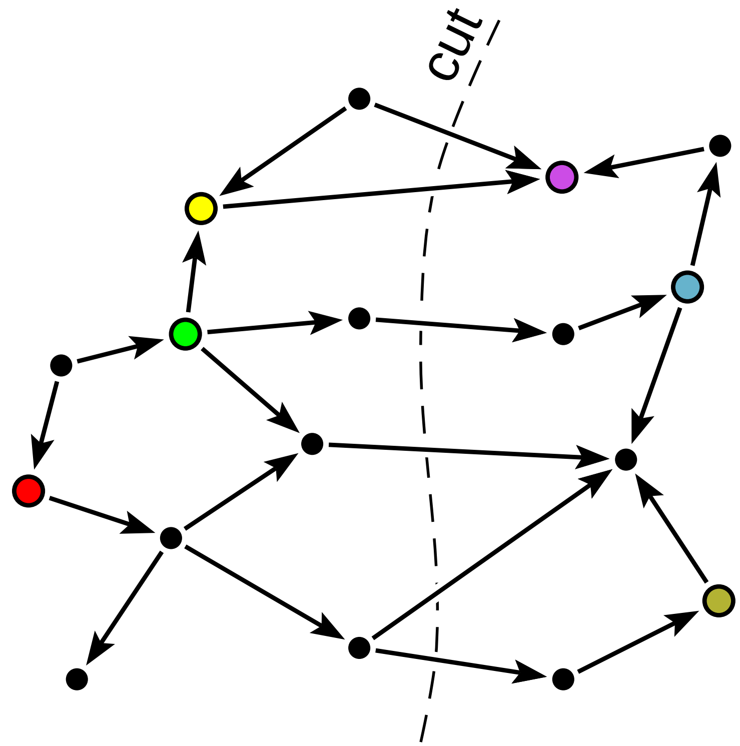

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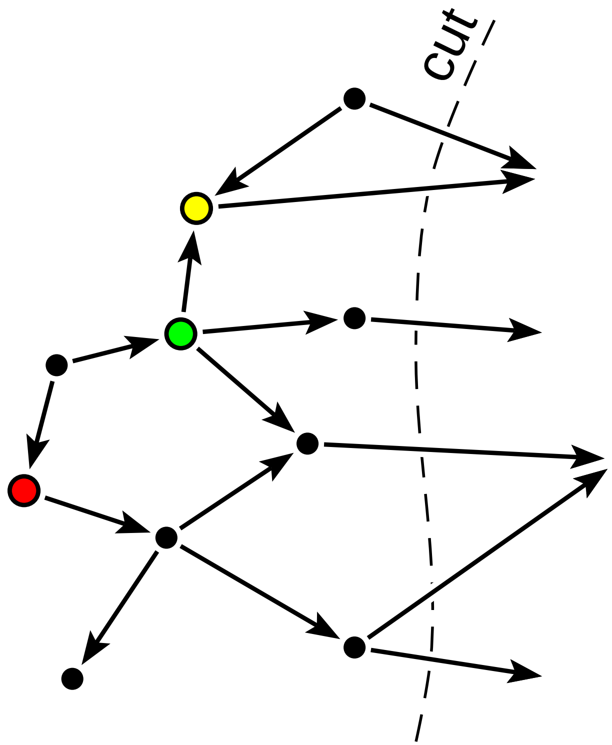

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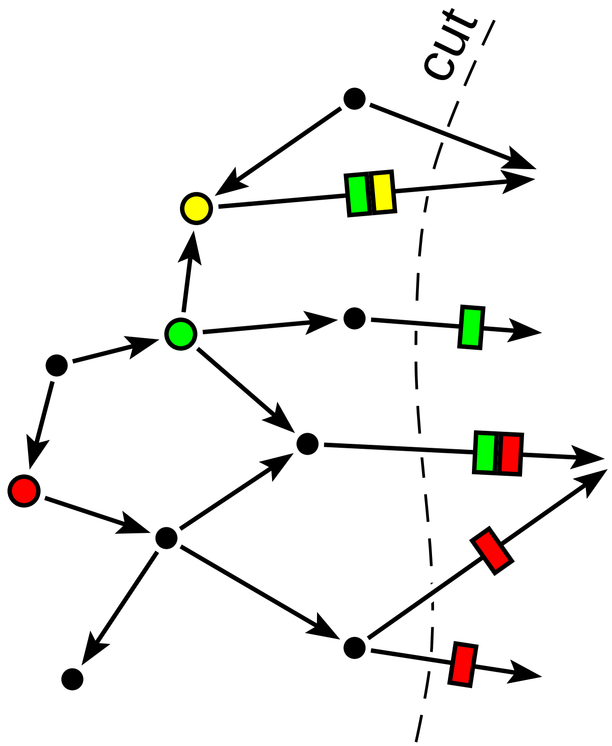

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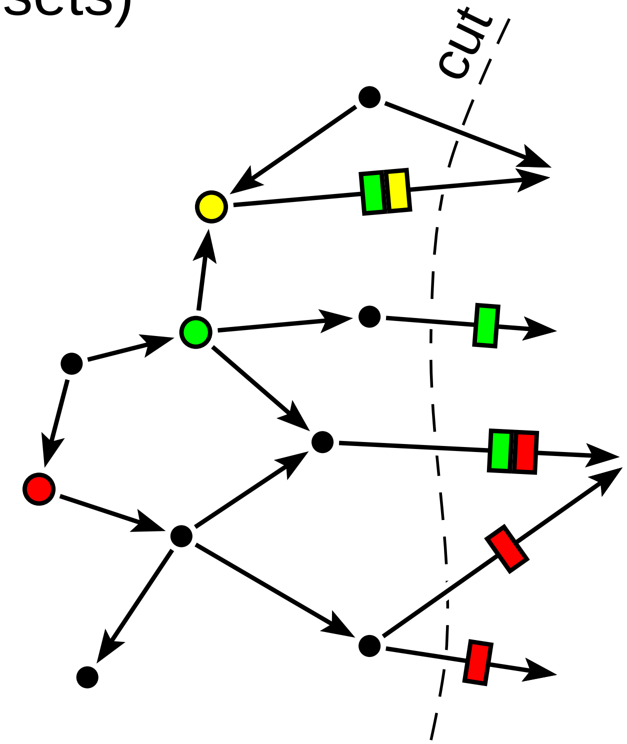

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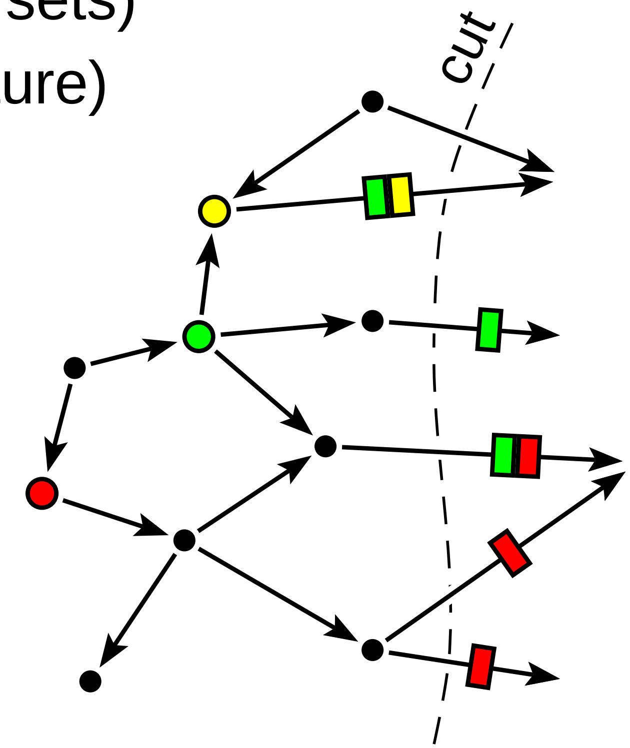

Tools & Techniques

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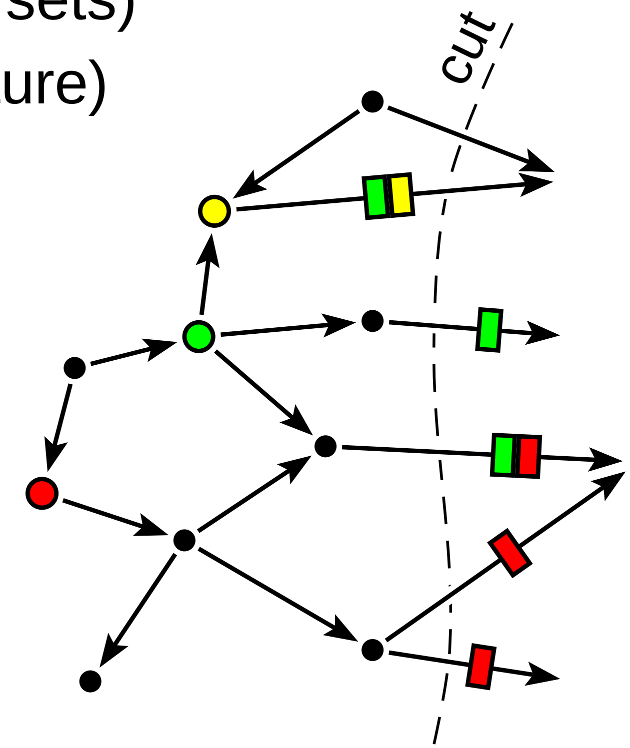
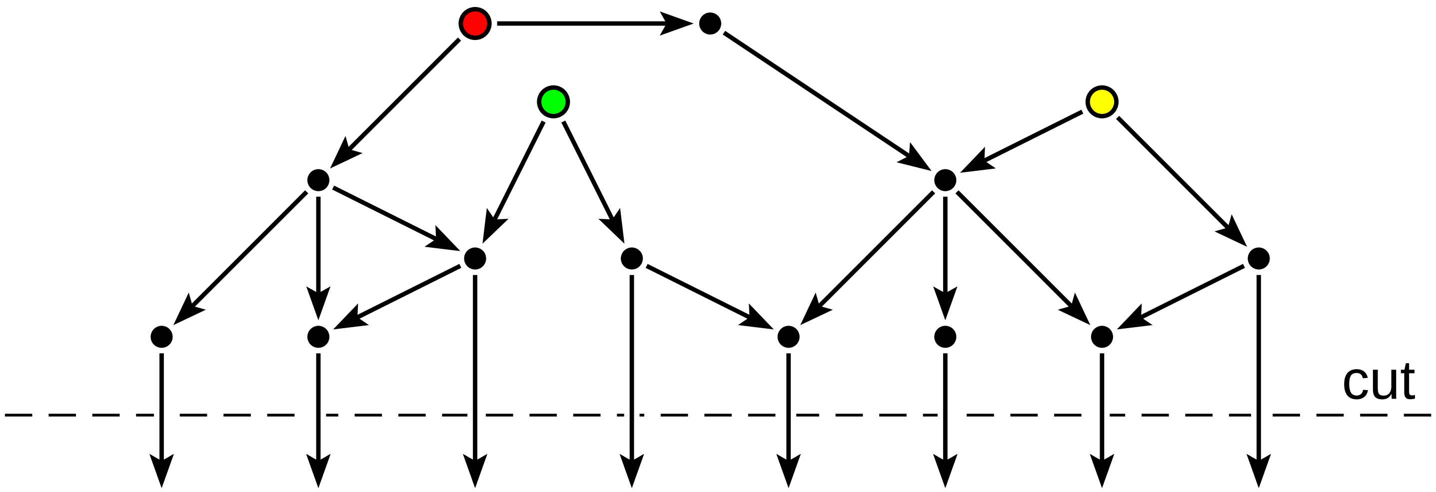
- representing reachabilities to / from the cut
  - type bound (how many color sets)
  - new encoding (interval structure)
- recurse



For simplicity, we consider only dags.

Types Along the Cut



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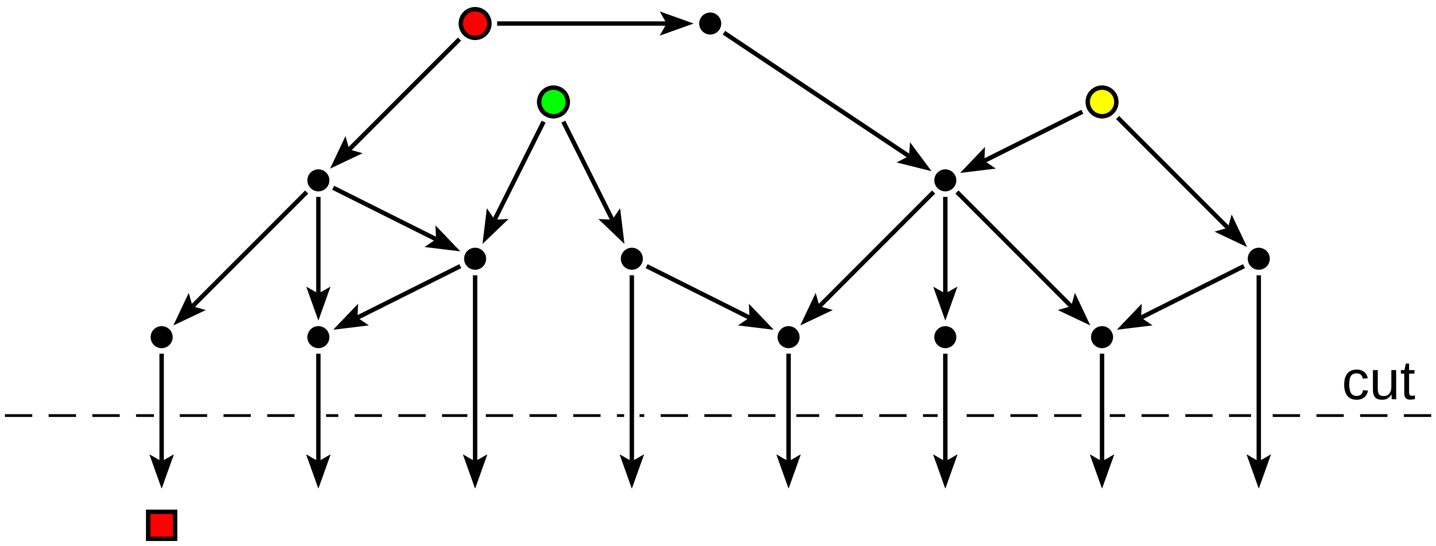

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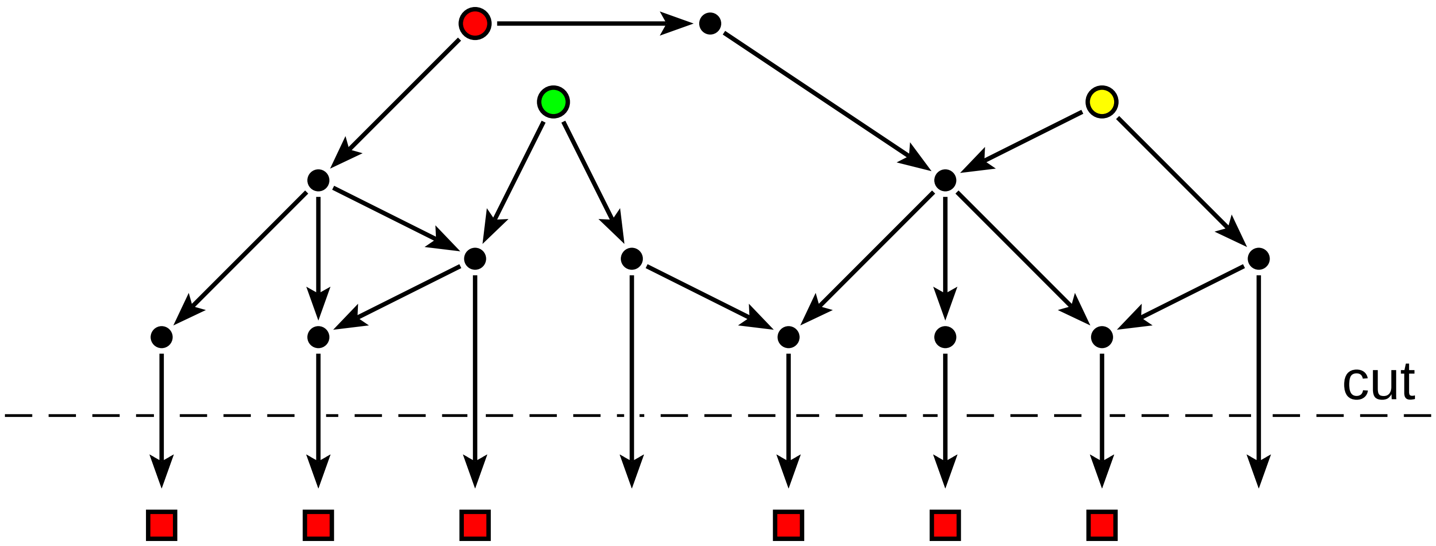

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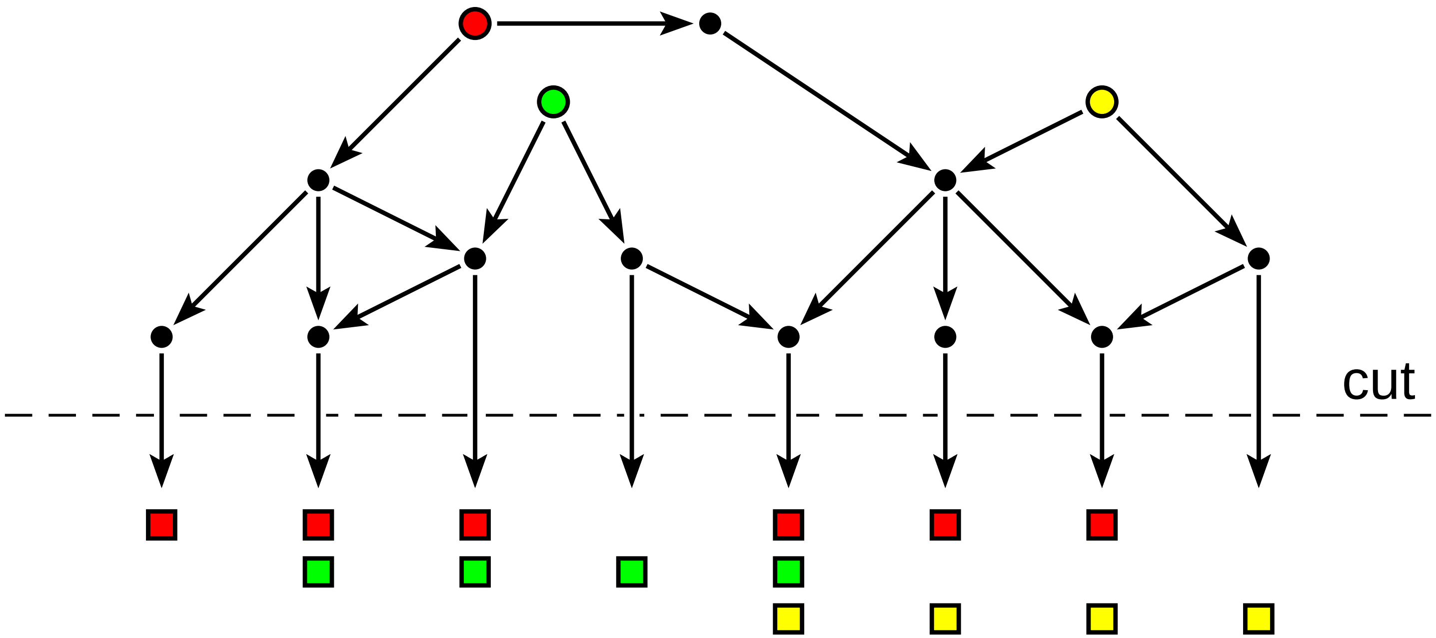

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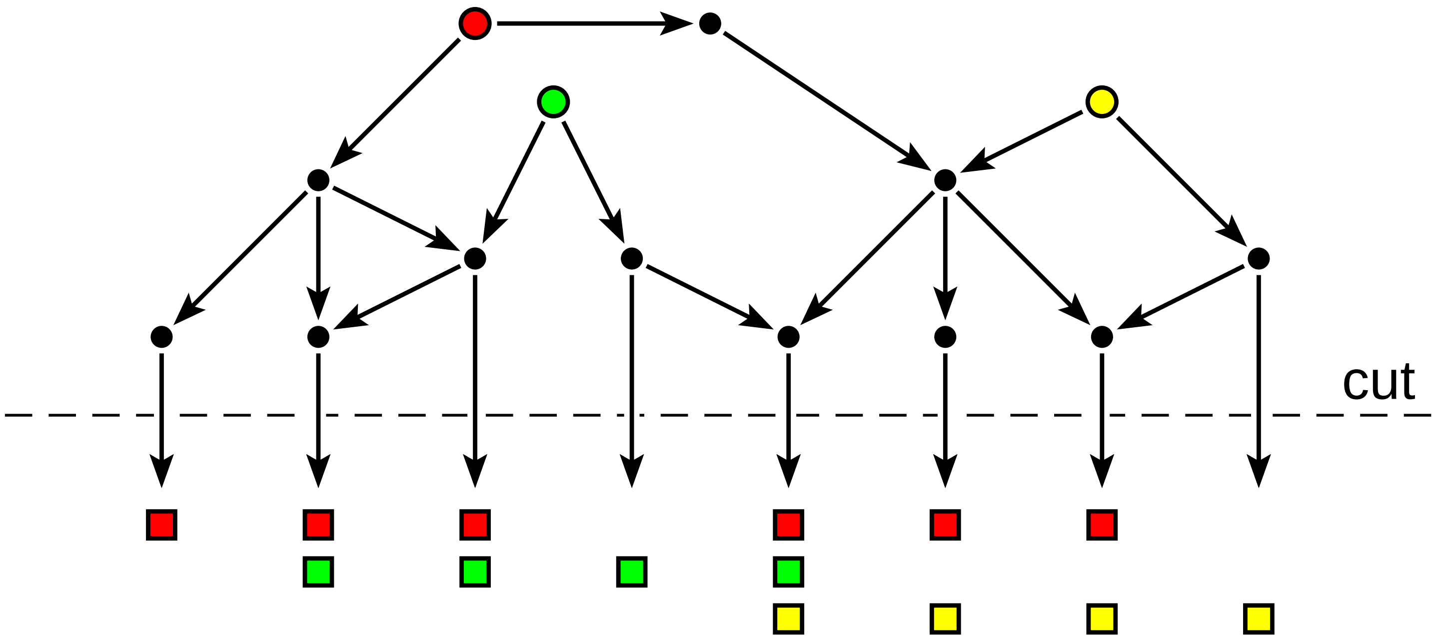

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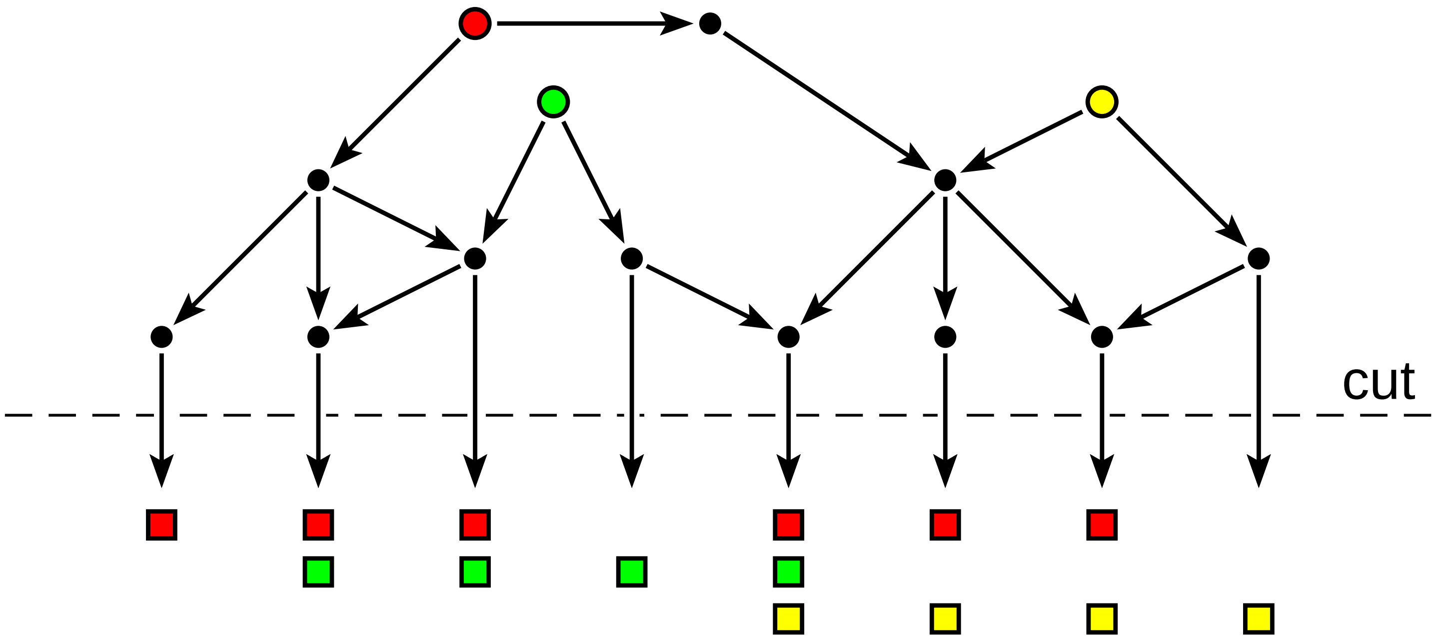

Lemma. **(The Type Bound)** The number of different types (and also of type changes!) is linear in the number of colors.

Lemma. There exists a dag of size $O(k \log k)$ that encodes all reachabilities from the k colors down to the cut line.

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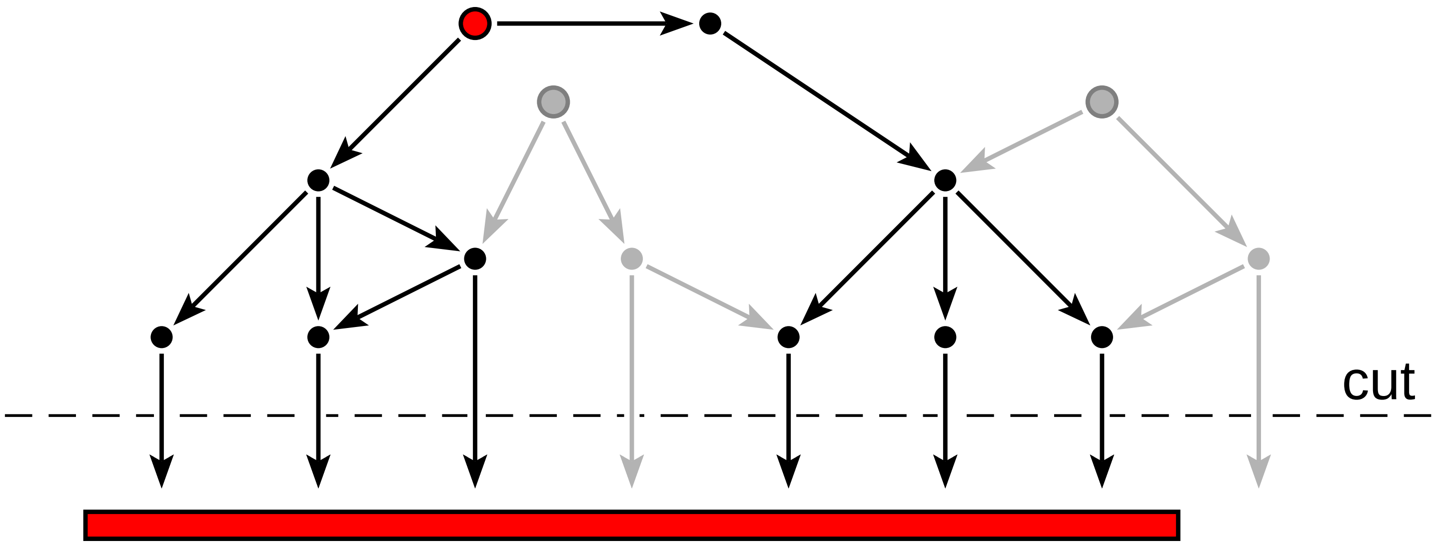

proof idea: *nested intervals*

- insert one interesting vertex after another, each together with all vertices reachable from it
- every interesting vertex must appear before all interesting vertices in its “shadow”

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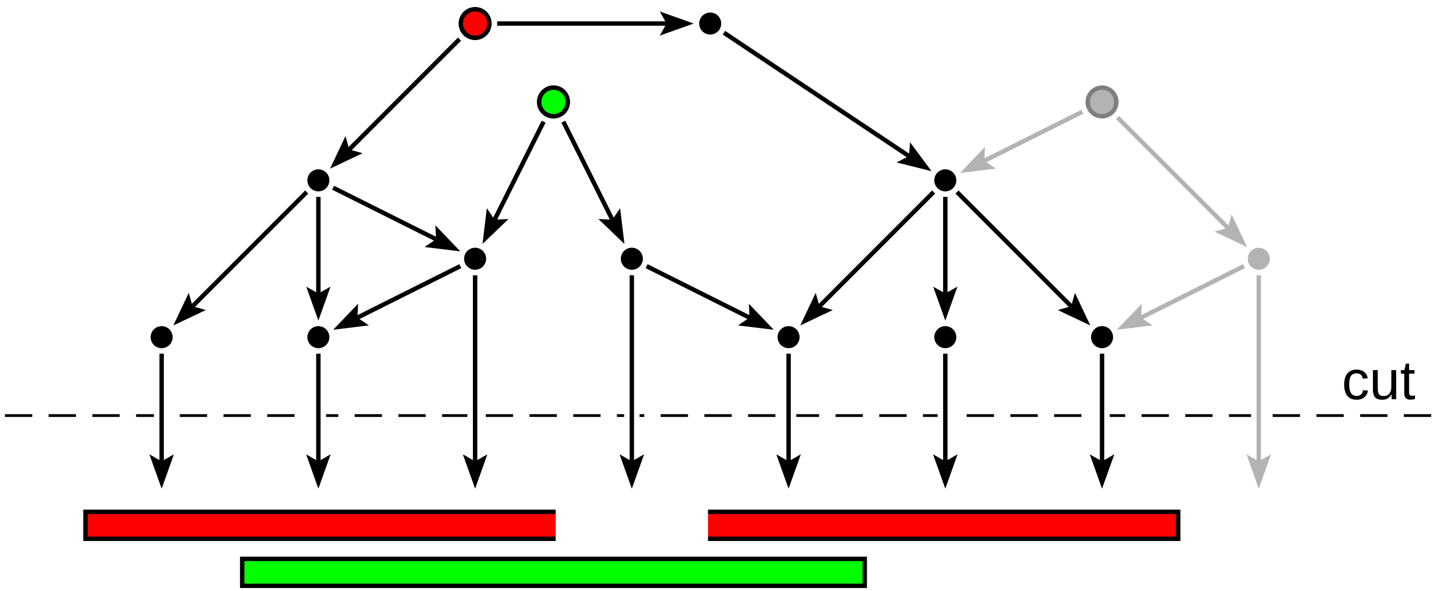

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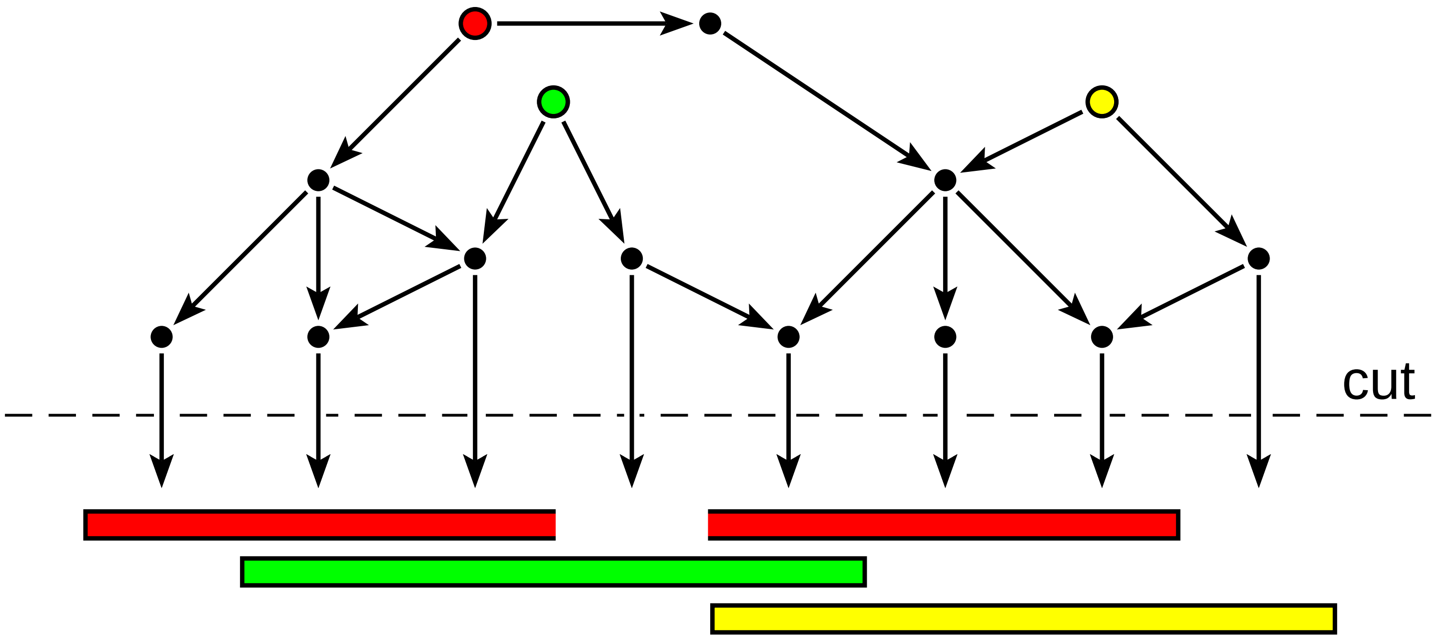

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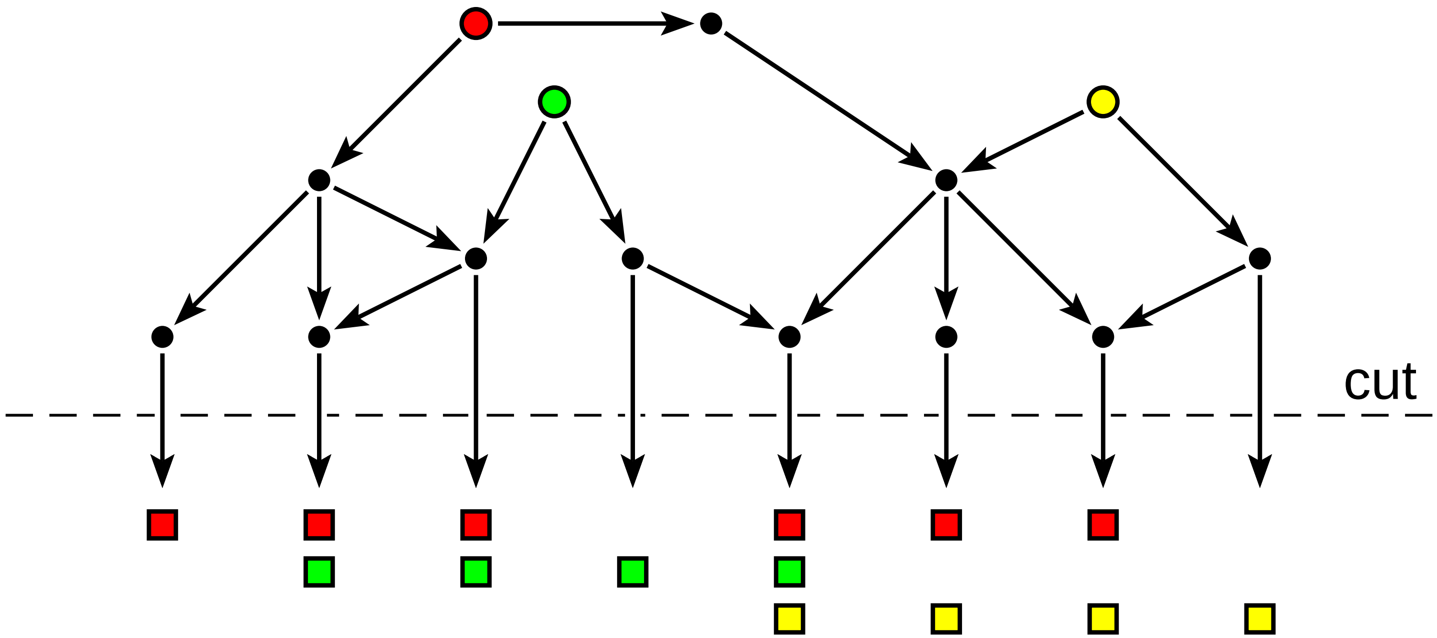

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- wanted: two simply-connected regions separated by a closed Jordan curve
- *directed* needed for cross-cut interval structure to work
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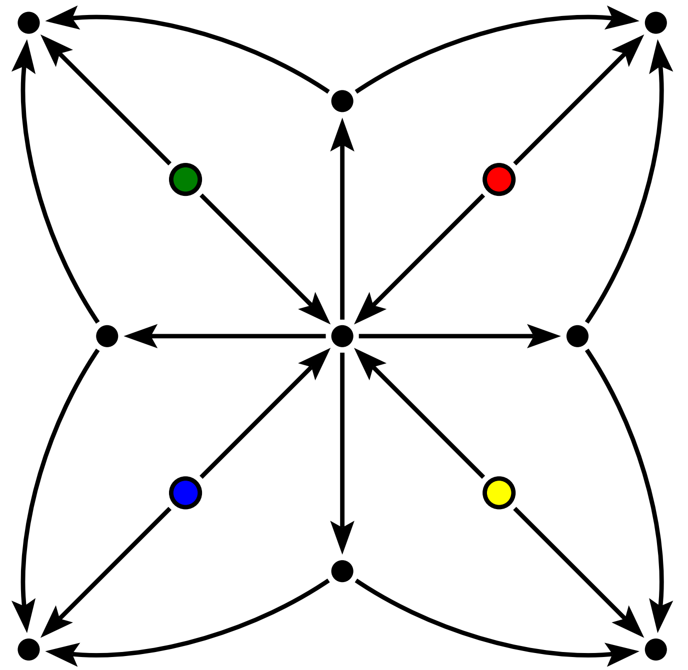

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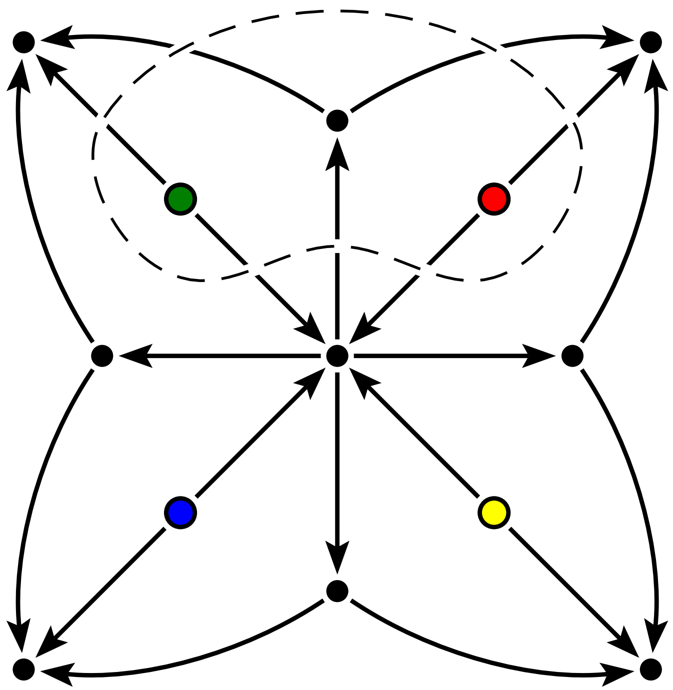

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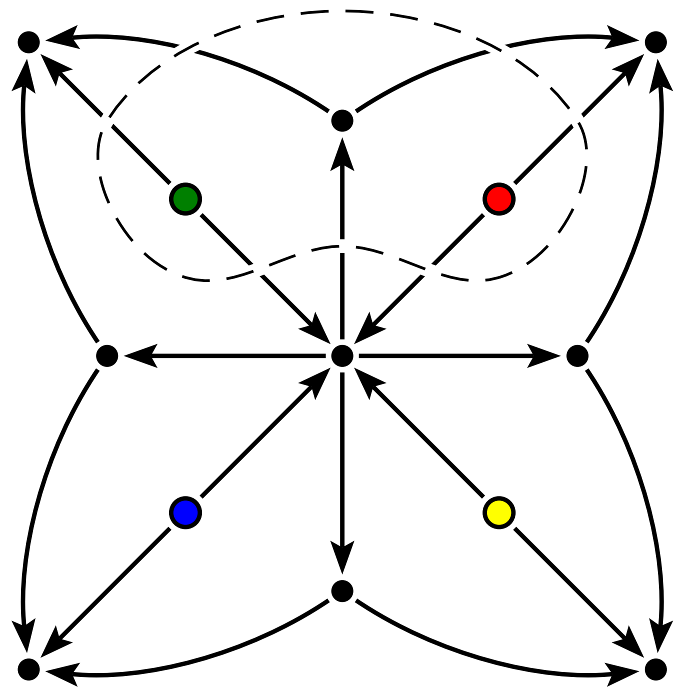

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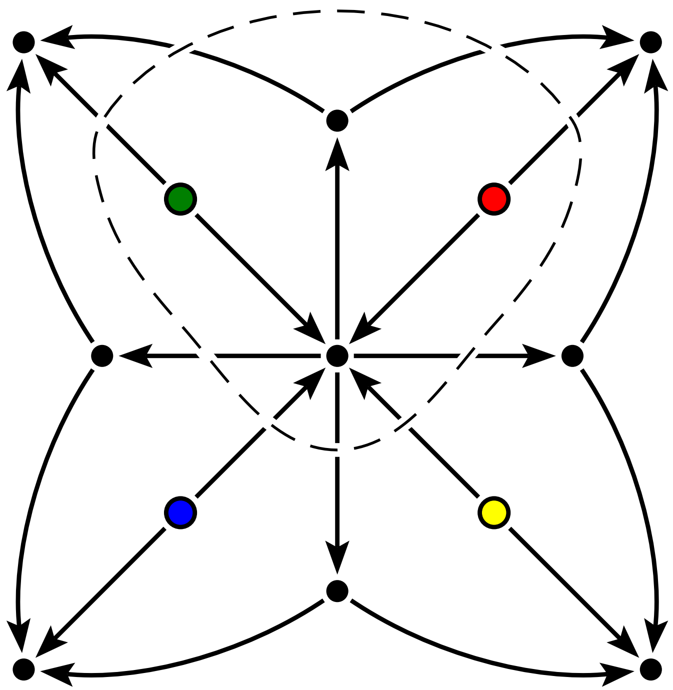

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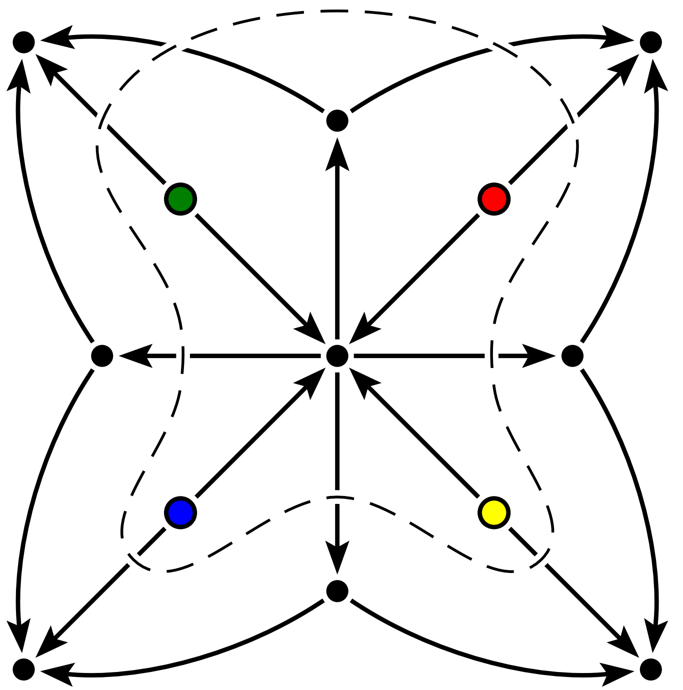

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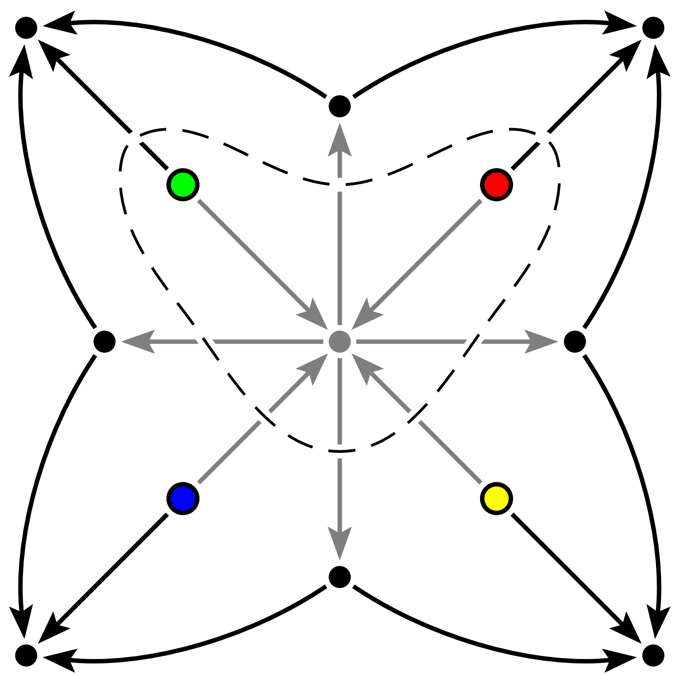

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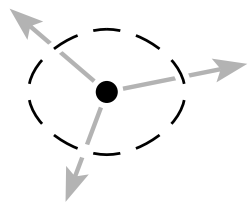

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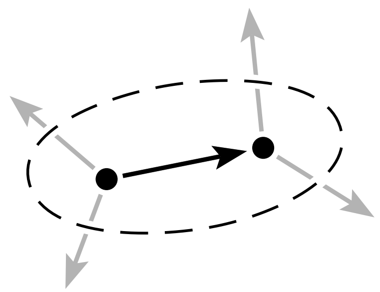

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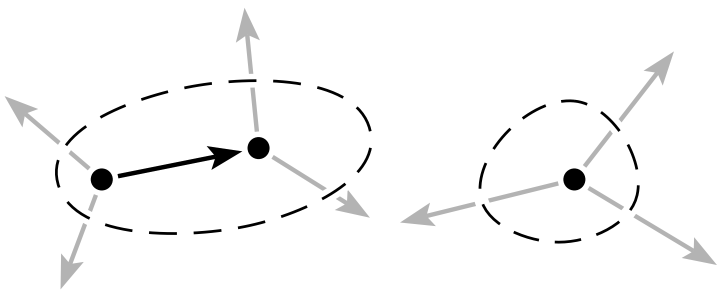

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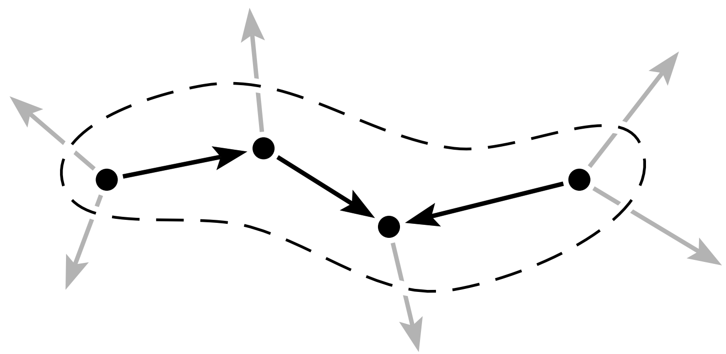

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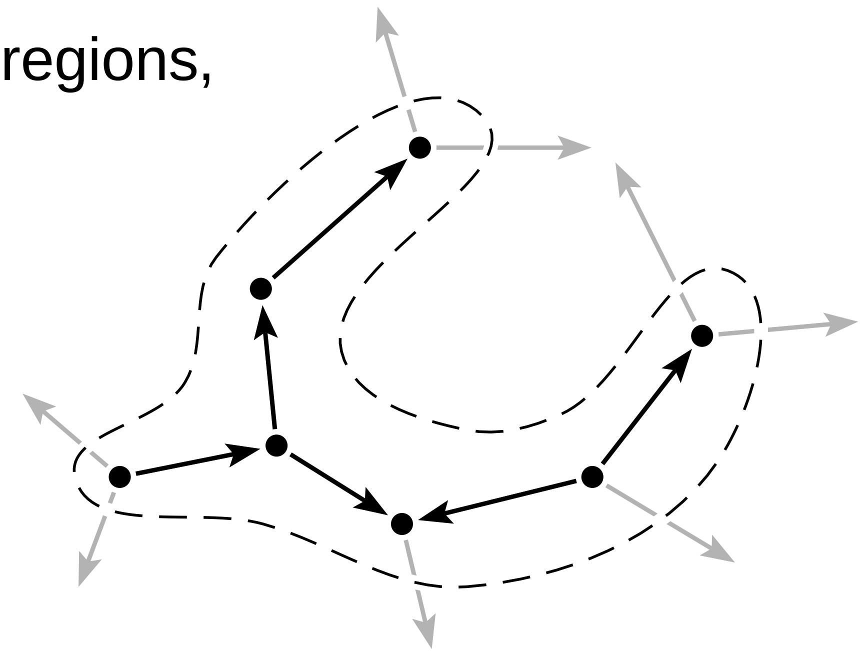

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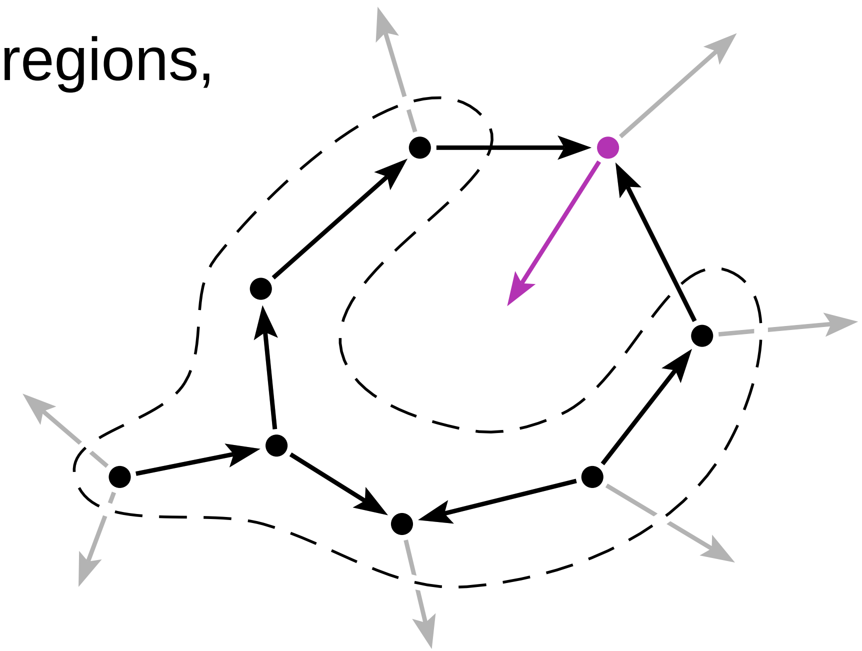

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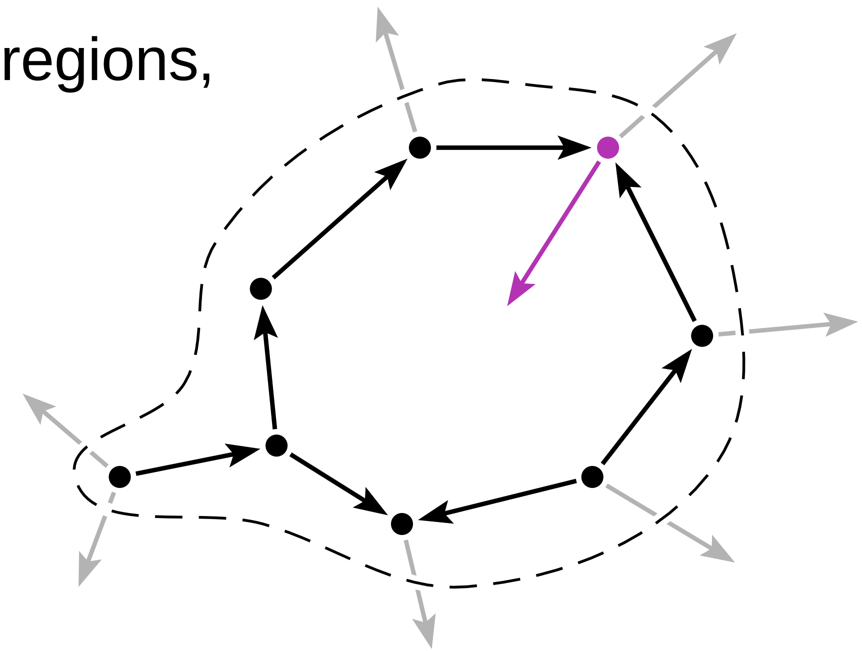

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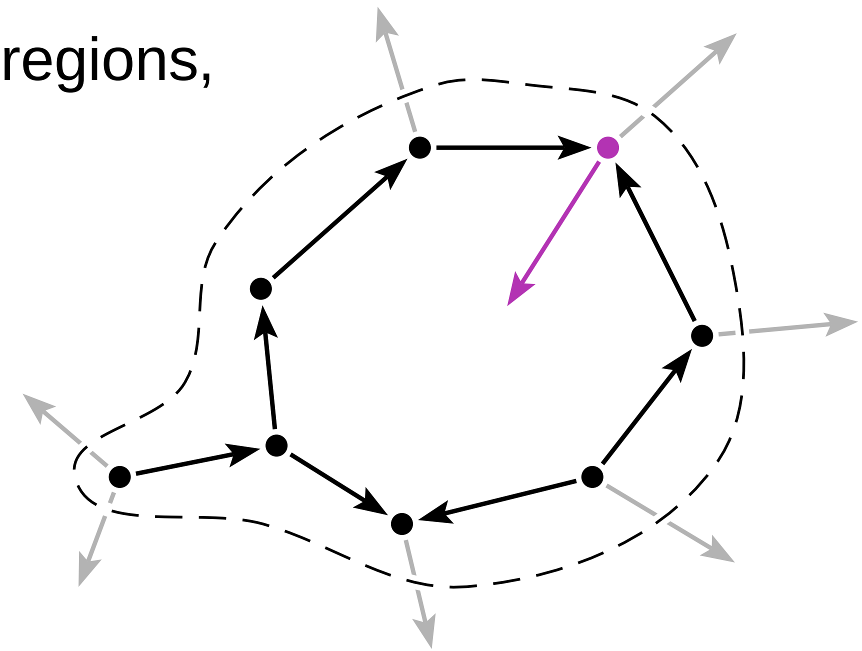
- but only one will persist



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- directed cycles can be taken care of separately in advance
  (they cut the plane into well-separated areas)

Outlook / Open Problems

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General Problems:

- Prove a super-linear lower bound on the size of general (non-planar) reachability substitutes.
- Are the two log-factors in our construction really necessary?