

No coincidence of center percolation and deconfinement in SU(4) lattice gauge theory

Michael Dirnberger^a, Christof Gattringer^a and Axel Maas^b

^aInstitute for Physics, Karl-Franzens University Graz,
Universitätsplatz 5, 8010 Graz, Austria

^bInstitute for Theoretical Physics, Friedrich-Schiller-University Jena
Max-Wien Platz 1, D-07743 Jena, Germany

Abstract

We study the behavior of center sectors in pure SU(4) lattice gauge theory at finite temperature. The center sectors are defined as spatial clusters of neighboring sites with values of their local Polyakov loops near the same center elements. We study the connectedness and percolation properties of the center clusters across the deconfinement transition. We show that for SU(4) gauge theory deconfinement cannot be described as a percolation transition of center clusters, a finding which is different from pure SU(2) or pure SU(3) Yang Mills theory, where the percolation description even allows for a continuum limit.

Introduction and outline

Understanding the high temperature transition of QCD to a phase of deconfined quarks and gluons is still an open problem. Finding a suitable description of the transition and the plasma phase is essential for understanding current and upcoming results from heavy ion experiments. An interesting approach is the idea of describing the deconfinement transition as a percolation phenomenon and to explore its phenomenological consequences.

For the simpler case of pure gauge theory the idea that percolation of clusters related to the center of the gauge group can be used to describe deconfinement goes back to [1] for the case of $SU(2)$ and to [2] for $SU(N)$ with $N > 2$. The clusters were constructed from the local Polyakov loops, i.e., they reflect the properties of static quark sources under the transformation with the center group \mathbb{Z}_N of $SU(N)$. More recently it was argued that for the cases of $SU(2)$ and $SU(3)$ even a continuum limit of the percolation description is possible for suitably defined clusters. First results for full lattice QCD were presented in [4].

With the encouraging results available for $SU(2)$ and $SU(3)$ one may ask the question whether the percolation description of deconfinement based on the center group is suitable for all gauge groups $SU(N)$. This is not a priori clear: The number of center elements is N , such that if the center sectors are occupied uniformly the probability that a site belongs to a particular sector is $1/N$. For $N > 3$, this probability $1/N$ is below the critical occupation probability $p_c \sim 0.316$ of random percolation on a three-dimensional simple cubic lattice. This implies that if the percolation picture of deconfinement were to apply also for $SU(N)$ with $N > 3$, the deconfinement transition must be accompanied by the onset of very strong correlations between the center phases of neighboring sites. The main goal of the current letter (see also [5]) is to establish or disprove the existence of such strong correlations and the corresponding percolation picture for $SU(N)$ Yang-Mills theory at $N > 3$.

It should be kept in mind that rigorous results for a complete description of thermal transitions, i.e., the same transition temperatures of the percolation and the thermal transition and matching critical exponents, are only available for continuous transitions. For first order transitions only numerical simulations or results for simple models suggest that percolation may be used to effectively describe thermal transitions in some cases (see, e.g., [6, 7]). The current paper provides a counter example.

Setting of the calculation

We work with pure SU(4) lattice gauge theory using Wilson's formulation. The fundamental degrees of freedom are the SU(4) valued link variables $U_\mu(x)$ with $\mu = 1, 2, 3, 4$. x denotes the sites of a $N_s^3 \times N_t$ lattice with periodic boundary conditions. N_t is the temporal extent of the lattice and $1/N_t$ is the temperature in lattice units. The action is the usual sum over all plaquettes and the partition sum is obtained by integrating over all gauge configurations. We work on lattices of sizes $20^3 \times 6$ to $40^3 \times 10$ and use ensembles of typically 500 configurations. The update was done using heat bath and overrelaxation steps [8], and the error bars we show are statistical errors from a single elimination Jackknife analysis corrected for autocorrelation. For the scale (from the string tension) and the deconfinement temperature we use the values from [9].

The basic observable in our study is the traced local Polyakov loop $L(\vec{x})$,

$$L(\vec{x}) = \text{Tr} \prod_{t=1}^{N_t} U_4(\vec{x}, t), \quad (1)$$

i.e., the ordered product of all temporal gauge links at a spatial lattice point \vec{x} and Tr denotes the trace over color indices. $L(\vec{x})$ is a gauge invariant object that corresponds to a gauge transporter which closes around compactified time, interpreted as a static source of color flux. We will also consider the normalized spatial average of the local Polyakov loops which we denote by P ,

$$P = \frac{1}{V} \sum_{\vec{x}} L(\vec{x}), \quad (2)$$

where V is the spatial lattice volume. Due to translational invariance P and $L(\vec{x})$ have the same vacuum expectation value.

The Polyakov loop $L(\vec{x})$ (and also its spatial average P) transform non-trivially under center transformations. For SU(4) the center group is $\mathbb{Z}_4 = \{1, i, -1, -i\}$ and in a center transformation all temporal links for some fixed time slice $t = \text{const}$ are multiplied with a center element $z \in \mathbb{Z}_4$. While the action and the path integral are invariant under center transformations, the local and averaged Polyakov loops transform as

$$L(\vec{x}) \longrightarrow z L(\vec{x}) \quad \text{and} \quad P \longrightarrow z P. \quad (3)$$

Below the deconfinement temperature the vacuum is invariant under center transformations and the expectation value of the Polyakov loops vanishes, i.e., $\langle L(\vec{x}) \rangle = \langle P \rangle = 0$. Above T_c the center symmetry is spontaneously broken

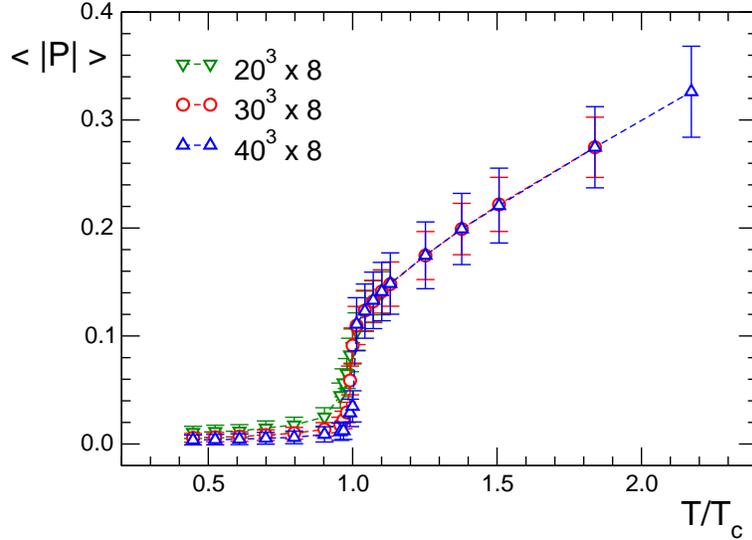


Figure 1: Expectation value $\langle |P| \rangle$ of the absolute value of the Polyakov loop as a function of the temperature. We compare the results from lattice sizes $20^3 \times 8$, $30^3 \times 8$ and $40^3 \times 8$ to illustrate the approach towards the discontinuous first order transition.

which is signaled by $\langle L(\vec{x}) \rangle = \langle P \rangle \neq 0$. In this spontaneous breaking of the center symmetry, the phase of the expectation value spontaneously selects one of the four center values. The deconfinement transition of pure $SU(4)$ lattice gauge theory is of a pronounced first order. For later comparison in Fig. 1 we show the expectation value $\langle |P| \rangle$ as a function of the temperature.

The Polyakov loop corresponds to a static color source and its vacuum expectation value is (after a suitable renormalization) related to the free energy F_q of a single quark, $\langle L(\vec{x}) \rangle = \langle P \rangle \propto \exp(-F_q/T)$, where T is the temperature (the Boltzmann constant is set to 1 in our units). Thus, when $\langle L(\vec{x}) \rangle = \langle P \rangle$ vanishes F_q is infinite and quarks are confined. On the other hand a non-zero value $\langle L(\vec{x}) \rangle = \langle P \rangle \neq 0$ implies finite F_q and quarks are deconfined. Thus, for pure gauge theory the deconfinement transition is linked to the spontaneous breaking of center symmetry.

Local Polyakov loops and cluster definition

So far we have only considered expectation values of the averaged Polyakov loop P without looking at the distribution of $L(\vec{x})$ at different spatial lattice

sites \vec{x} . Now we study this local behavior. The local Polyakov loop $L(\vec{x})$ is a complex number which we decompose into modulus and phase,

$$L(\vec{x}) = \rho(\vec{x}) e^{i\varphi(\vec{x})}. \quad (4)$$

While the distribution of the modulus is rather insensitive to the temperature (it almost perfectly follows the corresponding Haar measure distribution – as for SU(3) [3]), the distribution of the phase $\varphi(\vec{x})$ strongly depends on the temperature. In Fig. 2 we show histograms for the values of the angles $\varphi(\vec{x})$. For the top row of plots we applied center rotations of the individual configurations such that the dominant sector is always the one with real and positive phase ($\varphi \sim 0$). In the bottom row of plots, for comparison we show the distribution without rotating the dominant sector to $\varphi \sim 0$, and the random center rotation in our Monte Carlo update averages over all four sectors.

The histograms for $T < T_c$ nicely illustrate the center symmetry: The phases are distributed such that they have maxima near the center elements, which correspond to phases $\varphi = 0, \pm\pi/2$ and $\varphi = \pi$ (or $-\pi$). The maxima are of equal height and center symmetry is manifest. For $T > T_c$ in the plot with rotation of the dominant sector (rhs. top plot), we see that one of the peaks ($\varphi = 0$) is much taller than the others indicating that the corresponding center sector is considerably more populated. The figure corresponds to the situation where the spontaneous symmetry breaking has selected the $\varphi = 0$ sector. Note that the spontaneous breaking is possible only in infinite volume, and that our rotation of the dominant sector only mimicks this behavior. It is interesting to note that also above T_c the non-dominant center sectors are still populated, as there are local maxima near $\varphi = \pm\pi/2, \pm\pi$. This indicates that above T_c (at least for not too high temperatures) there is still an admixture of local Polyakov loops with a phase different from the dominant one.

Based on the phases of the local loops $\varphi(\vec{x})$ we now assign local sector numbers $n(\vec{x})$ to sites \vec{x} as follows:

$$n(\vec{x}) = 0 \Leftrightarrow \varphi(\vec{x}) \in [-\pi/4, \pi/4), \quad (5)$$

$$n(\vec{x}) = 1 \Leftrightarrow \varphi(\vec{x}) \in [\pi/4, 3\pi/4), \quad (6)$$

$$n(\vec{x}) = -1 \Leftrightarrow \varphi(\vec{x}) \in [-3\pi/4, -\pi/4), \quad (7)$$

$$n(\vec{x}) = 2 \Leftrightarrow \varphi(\vec{x}) \geq 3\pi/4 \text{ or } \varphi(\vec{x}) < -3\pi/4. \quad (8)$$

Using the local sector numbers $n(\vec{x})$ we can now define the center clusters by putting two neighboring sites \vec{x} and \vec{y} into the same cluster when $n(\vec{x}) = n(\vec{y})$. Usual cluster identification techniques [10] can then be used to find all the clusters on the lattice.

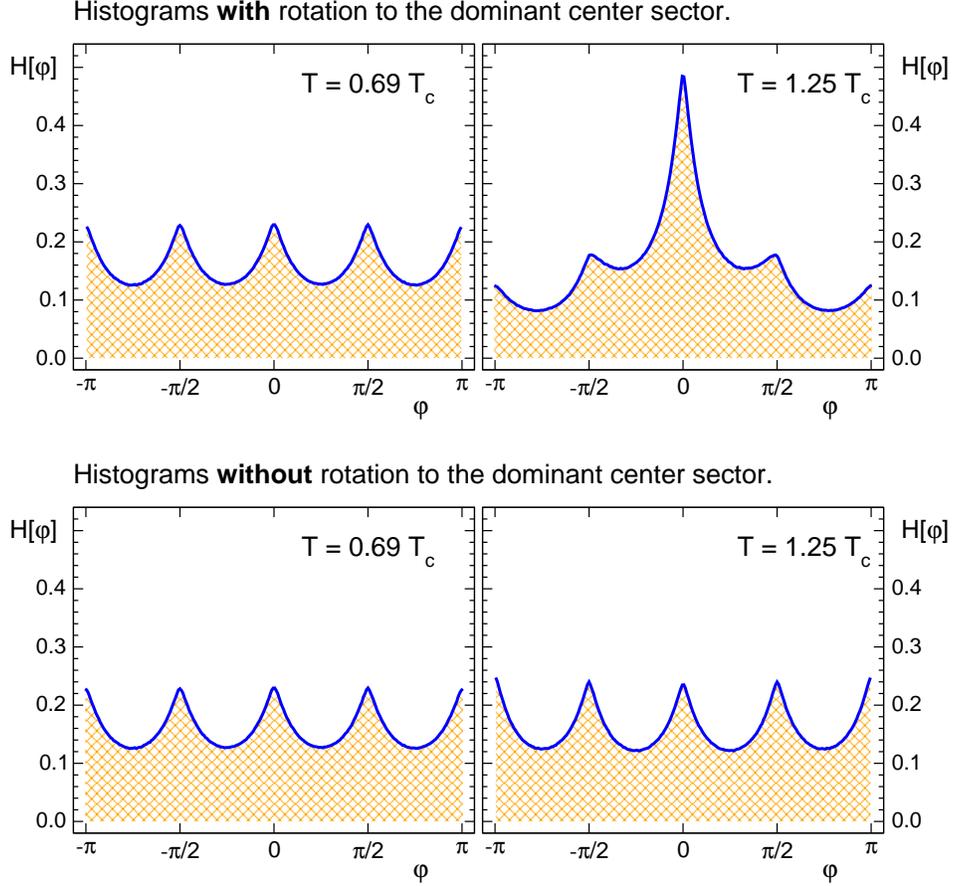


Figure 2: Histograms for the distribution of the phases $\varphi(\vec{x})$ of the local Polyakov loops $L(\vec{x})$. The results are from our $40^3 \times 6$ ensembles at temperatures $T = 0.69 T_c$ (lhs. plots) and $T = 1.25 T_c$ (rhs.). We compare the histograms with (top row of plots) and without (bottom) center rotation bringing the dominant center sector to $\varphi \sim 0$.

Properties of center clusters

Having analyzed the distribution of the phases of the local Polyakov loops and constructed the center clusters we can now study their properties as a function of the temperature. The simplest quantity is the weight W of a cluster which is simply defined as the number of sites in the cluster. In particular by W_{max} we denote the weight of the largest cluster. In Fig. 3 we show the behavior of the expectation value $\langle W_{max} \rangle / V$ of the weight of the largest cluster normalized

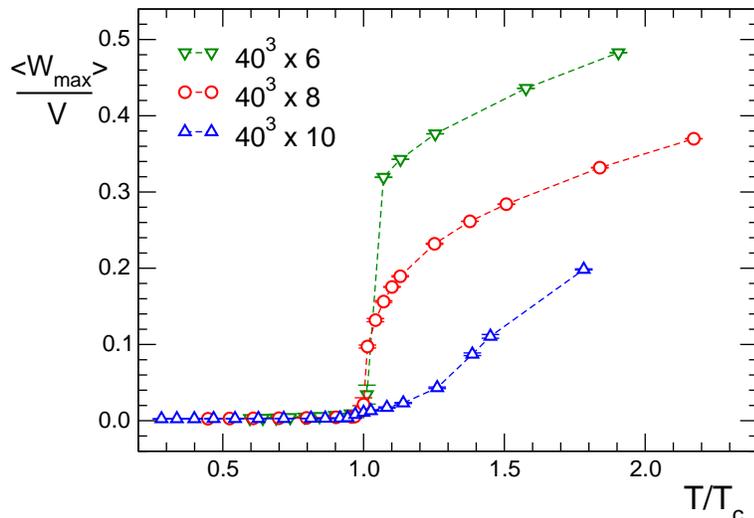


Figure 3: Weight of the largest cluster normalized by the spatial volume as a function of temperature.

by the spatial volume as a function of the temperature. For fixed $N_s = 40$ we compare the results from calculations at three different values of N_t . As N_t increases, the lattice spacing a has to be decreased in order to stay at a fixed value of the temperature $T = (aN_t)^{-1}$. Thus as one increases N_t at fixed T the continuum limit is approached, and comparing the curves for $\langle W_{max} \rangle$ at different N_t allows one to study their behavior in the continuum limit.

Below T_c the three curves for $\langle W_{max} \rangle / V$ in Fig. 3 fall on top of each other. For all three values of N_t the expectation value $\langle W_{max} \rangle$ is small compared to the volume V such that the ratio is close to 0. At T_c the largest cluster starts to grow quickly and, e.g., for the $N_t = 6$ ensembles reaches at $2 T_c$ a weight which is already half of the volume. However, it is obvious that the growth rate above T_c strongly depends on the temporal extent N_t . For larger values of N_t , i.e., closer to the continuum limit, the growth rate is considerably slower, which is a first hint that in the continuum limit $a \rightarrow 0$ ($N_t \rightarrow \infty$) the clusters might not grow fast enough to give rise to percolation.

The question of percolation can of course be addressed directly by looking at the probability of having a spanning cluster. A spanning cluster is defined as a cluster that extends from one end of the (spatial) lattice to the other. Here we use periodic boundary conditions and thus we consider a cluster to be spanning if all possible x - y planes contain at least one site of the cluster. In Fig. 4 we plot the average probability p_s to have a spanning cluster (i.e., the percolation

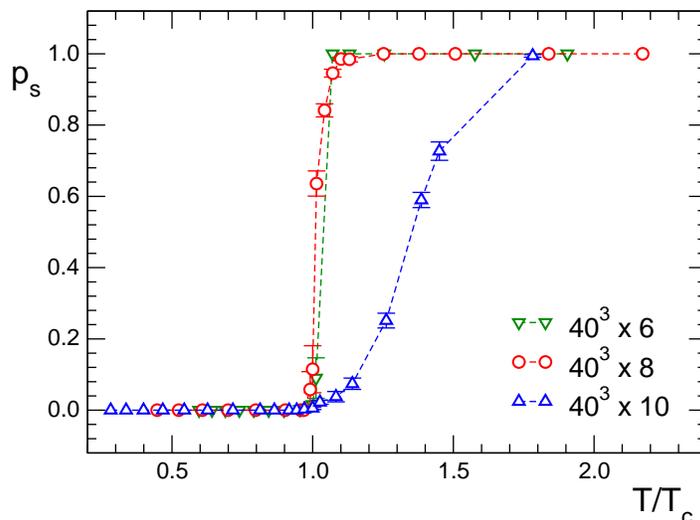


Figure 4: Probability for a spanning cluster as a function of temperature.

probability) as a function of the temperature. Again we compare the results for $N_t = 6, 8$ and $N_t = 10$ to assess the approach to the continuum limit.

The probability p_s to have a spanning cluster is essentially a step function only for $N_t = 6$. For $N_t = 8$ the probability increases somewhat slower, and for $N_t = 10$ the percolation probability reaches 1 only at $T \sim 1.8T_c$. This finding leads to the conclusion that when approaching the continuum limit (increasing N_t) the onset of full percolation (probability for spanning clusters reaches $p_s = 1$) does not coincide with the deconfinement transition at T_c .

An important question is of course whether finite volume effects could be a major effect: When increasing N_t one has to decrease a in order to keep the temperature fixed. This decreasing lattice spacing a then of course also shrinks the spatial extent $L = aN_s$ in physical units. In order to assess such possible finite size effects we compared our results for $20^3 \times 10$, $30^3 \times 10$ and $40^3 \times 10$. An example of the finite volume analysis is given in Fig. 5 where we show the weight of the largest cluster as a function of the temperature, comparing three different spatial volumes. One finds that with increasing volume, the weight of the largest cluster even decreases slightly. This decrease can be understood from the fact, that a finite hypertorus allows for additional connections of sites around the periodic boundary conditions. This is a finite volume effect that diminishes with increasing N_s giving rise to the decreasing values of W_{max}/V .

The same finite volume analysis was performed also for the probability of spanning clusters p_s and we found that the corresponding curves vary only

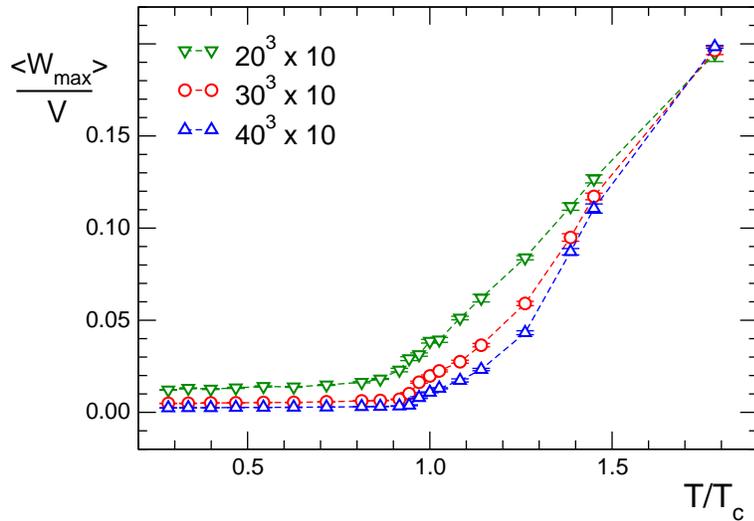


Figure 5: Finite volume study of the weight of the largest cluster plotted as a function of temperature.

slightly, although the volume changes by a factor of 8. In particular we observed that the temperature where p_s reaches $p_s = 0.5$ is at $T = 1.35 T_c$ for all three spatial volumes. We thus confirm that the discrepancy between the deconfinement temperature and the point where percolation is established is not a finite size effect.

Furthermore we have analyzed additional observable such as the gyration radius of the clusters and again found that as one approaches the continuum limit there is no coincidence of the deconfinement temperature with an onset of percolation [5].

Discussion of the results

In this paper we have analyzed the percolation properties of center clusters in pure $SU(4)$ lattice gauge theory. To construct the center clusters we consider the phases of the local Polyakov loops and use them to identify the nearest center element at each site of the spatial lattice. Neighboring sites with the same center element are then assigned to the same cluster. We studied various properties of the center clusters, and here in particular discuss our results for the weight of the largest cluster and the probability to find a spanning cluster. An assessment of these observables on lattices with different lattice spacing shows that in the continuum limit the deconfinement transition of $SU(4)$ lattice gauge

theory does not coincide with an onset of percolation for the center clusters. Cross checks with different spatial volumes rule out that finite volume effects play a major role.

The finding we present here for $SU(4)$ is different from what was established for $SU(2)$ and $SU(3)$ pure lattice gauge theory [1, 2, 3], where indeed center clusters may be constructed such that the temperature for the onset of percolation agrees with the deconfinement temperature. For these two cases a more general cluster construction is possible which does not automatically link all neighbors with equal center elements, but puts them in the same cluster only after some cutoff is applied [3], a step which corresponds to the construction of Fortuin-Kasteleyn clusters that are known to percolate at the same temperature where Potts models with continuous transitions demagnetize [11]. The construction allows one to use the cutoff parameter of the clusters to establish a continuum limit for the percolation description. For $SU(4)$ no such cutoff can be introduced as it would further weaken the percolation properties of the clusters.

For gauge groups $SU(N)$ with values of N even larger than $N = 4$, the clusters are thinned out further: The number of center elements equals N and if they are occupied with equal probability each site is in a given sector with probability $1/N$. With increasing N this probability decreases and the clusters are thinned out and shrink. We conjecture that the center percolation picture of deconfinement fails for all N larger than 3. Only the onset of very strong correlations at T_c between local Polyakov loops with phases near the same center element could give rise to percolation. Our results show that such correlations are not strong enough to enable percolation at T_c for $SU(4)$, and we believe that the even stronger correlations necessary for higher $SU(N)$ (where the probability $1/N$ is even smaller) do not exist.

One may of course speculate in what respects the groups $SU(2)$ and $SU(3)$ are different. We point out that the effective theories [12] for the center degrees of freedom, which govern the deconfinement transition of pure gauge theories are different for $SU(2)$ and $SU(3)$. In general the corresponding center symmetrical effective action has the form [12]

$$S = -\beta \sum_{\langle x,y \rangle} \left[s_x s_y^* + c.c. \right], \quad (9)$$

where the sum runs over all nearest neighbors and s_x is an element of the center group \mathbb{Z}_N , i.e., it is a phase $s_x = \exp(i2\pi k_x/N)$ with $k_x = 0, 1, \dots, N-1$. For $N = 2$ and 3 it is possible to rewrite the action to the form

$$S = -\beta \sum_{\langle x,y \rangle} A \delta_{k_x, k_y} + C, \quad (10)$$

where A and B are trivial constants and δ_{k_x, k_y} is the Kronecker delta. In other words, for $SU(2)$ and $SU(3)$ the effective theory is a 2-state (3-state) Potts model, where the agreement of the percolation of Fortuin-Kasteleyn clusters with the demagnetization transition is established (exactly for the 2-state case [11], numerically for the 3-state model [7]). For $N > 3$ the effective action (9) cannot be cast into the form (10) of the Potts model. The reason is that for $N > 3$ the contribution of two neighboring spins assumes more than two different values in the action (10). The fact that the effective theories for $N > 3$ have a form which is different from the $N = 2$ and $N = 3$ cases might explain why the center percolation description of deconfinement is not possible for $N > 3$.

A second reason might be that for the weak first order transition of $SU(3)$ the percolation description is still applicable, while for the stronger first order transition of $SU(4)$ it fails. As already pointed out in the introduction, for first order transitions the understanding of a possible relation between percolation and thermal transitions is considerably less developed, and more investigations are necessary. This work provides a particular counter example.

Finally we remark that another difficulty for a straight-forward interpretation of the results might be that $SU(4)$ is a rank three group. As a consequence, as noted above, there is no simple effective Polyakov loop model for it. Indeed, any such model contains Polyakov loops in the three fundamental representations, the 4, $\bar{4}$, and the 6. Depending on the relative weight, the effective theory can be any type of model from a 4-state Potts model to two non-interacting Ising models [13, 14]. This would correspond to regarding the center either as \mathbb{Z}_4 or as $\mathbb{Z}_2 \times \mathbb{Z}_2$. Our investigation was assuming a \mathbb{Z}_4 -like behavior, but if this would not be appropriate, one could speculate that a different definition of center clusters could still show a percolation behavior at the deconfinement transition. However, note that in three dimensions, the $SU(4)$ theory is possibly close to the \mathbb{Z}_4 option considered here [13]. This possibility does not exist for $SU(2)$ and $SU(3)$, which harbor only a single way of representing the center.

Acknowledgments: The authors thank Hubert Antlinger, Szabolcs Borsanyi, Mike Creutz, Julia Danzer, Christian Lang and Alexander Schmidt for valuable comments. The numerical calculations were done at the UNI-IT clusters of the University Graz. A. M. was supported by the FWF under grant number M1099-N16 and by the DFG under grant number MA 3935/5-1.

References

- [1] S. Fortunato, H. Satz, Phys. Lett. B **475** (2000) 311. Nucl. Phys. A **681** (2001) 466. S. Fortunato, F. Karsch, P. Petreczky, H. Satz, Phys. Lett. B **502** (2001) 321.
- [2] S. Fortunato, J. Phys. A **36** (2003) 4269.
- [3] C. Gattringer, Phys. Lett. B **690** (2010) 179.
C. Gattringer, A. Schmidt, JHEP **1101** (2011) 051.
- [4] S. Borsanyi, J. Danzer, Z. Fodor, C. Gattringer, A. Schmidt, J. Phys. Conf. Ser. **312** (2011) 012005. J. Danzer, C. Gattringer, S. Borsanyi, Z. Fodor, PoS LATTICE **2010** (2010) 176.
- [5] M. Dirnberger, *Cluster properties of local Polyakov Loops in SU(4) lattice gauge theory*, Master Thesis, Karl-Franzens University Graz, August 2011
<http://physik.uni-graz.at/itp/files/dirnberger/diplomarbeit.pdf>
- [6] A. Coniglio, F. Peruggi, J. Phys. A **15** (1982) 1873.
F. Perruggi, Physics **141A** (1987) 140.
- [7] S. Fortunato, H. Satz, Nucl. Phys. B **623** (2002) 493.
- [8] A. Maas, JHEP **1102** (2011) 076.
- [9] B. Lucini, M. Teper, U. Wenger, JHEP **0502** (2005) 033.
- [10] J. Hoshen, R. Kopelman, Phys. Rev. B **14** (1976) 3438.
- [11] C.M. Fortuin, P.W. Kasteleyn, Physica **57** (1972) 536.
A. Coniglio, W. Klein, J. Phys. A **13** (1980) 2775.
- [12] L.G. Yaffe, B. Svetitsky, Phys. Rev. D **26** (1982) 963; Nucl. Phys. B **210** (1982) 423.
- [13] N. Strodthoff, S.R. Edwards, L. von Smekal, PoS LATTICE **2010** (2010) 288.
- [14] C. Wozar, T. Kaestner, A. Wipf, T. Heinzl, B. Pozsgay, Phys. Rev. D **74** (2006) 114501.