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# On Implicit Dependence and Independence between Quantified First-Order Variables

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Focus is on relational FO with standard syntax.

The topic in a nutshell:

Inferring a certain degree of independence between quantified variables $\forall x \, \exists y$ from formula structure, in particular from the absence of atoms

$$P(\ldots x \ldots y \ldots)$$

or chains

$$P(\ldots x \ldots z \ldots), \; Q(\ldots z \ldots y \ldots)$$

and the like.

# Implicit (In)Dependence Between Variables

$$\varphi := \forall x z \exists y.\, P(x) \leftrightarrow \big(Q(x) \leftrightarrow R(y,z)\big)$$

**Claim:**
$\varphi$ and $\psi$ are equivalent

For every structure $\mathfrak{A}$ there are four possible *types* for any $\mathfrak{a} \in A$ with respect to $x$ in $\varphi$:

$$P(\mathfrak{a}) \wedge Q(\mathfrak{a}), \qquad \neg P(\mathfrak{a}) \wedge Q(\mathfrak{a}),$$
$$\boxed{P(\mathfrak{a}) \wedge \neg Q(\mathfrak{a}),} \qquad \neg P(\mathfrak{a}) \wedge \neg Q(\mathfrak{a})$$

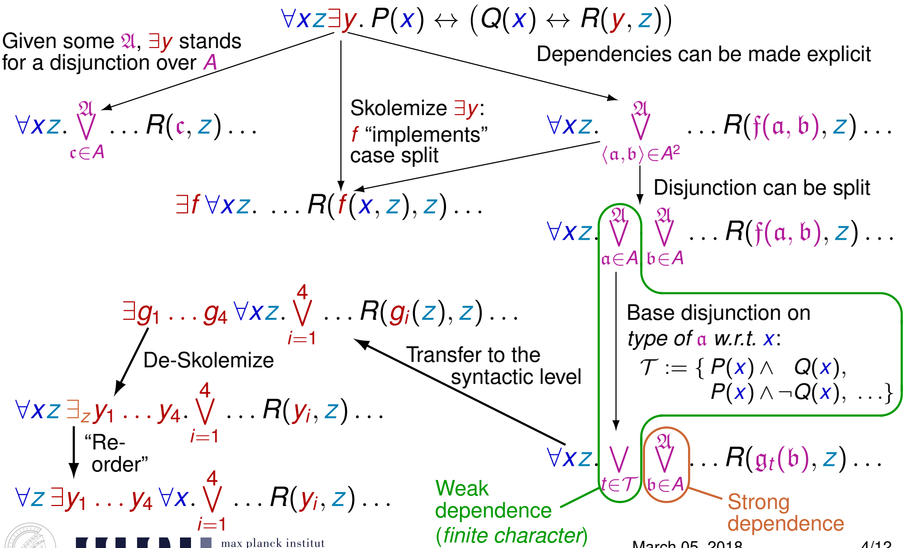
$$\mathfrak{A}, [x \mapsto \mathfrak{a}]: \qquad \forall z \exists y.\, \mathtt{true} \leftrightarrow \big(\mathtt{false} \leftrightarrow R(y,z)\big)$$
$$\mathfrak{A}, [x \mapsto \mathfrak{a}, z \mapsto \mathfrak{b}]: \qquad \exists y.\, \mathtt{true} \leftrightarrow \big(\mathtt{false} \leftrightarrow R(y,\mathfrak{b})\big)$$

$$\psi := \forall z\, \exists y_1 \ldots y_4\, \forall x.\, \bigvee_{i=1}^{4} P(x) \leftrightarrow \big(Q(x) \leftrightarrow R(y_i, z)\big)$$

Strong dependence of $y$ on $z$, due to $R(y,z)$

Weak dependence (*finite character*) of $y$ on $x$

# Implicit (In)Dependence Between Variables



$$\forall xz\exists y.\ P(x) \leftrightarrow \big(Q(x) \leftrightarrow R(y,z)\big)$$

Given some $\mathfrak{A}$, $\exists y$ stands for a disjunction over $A$

$$\forall xz.\ \bigvee^{\mathfrak{A}}_{\mathfrak{c}\in A} \dots R(\mathfrak{c}, z) \dots$$

Skolemize $\exists y$: $f$ "implements" case split

$$\exists f\, \forall xz.\ \dots R(f(x,z),z) \dots$$

Dependencies can be made explicit

$$\forall xz.\ \bigvee^{\mathfrak{A}}_{\langle\mathfrak{a},\mathfrak{b}\rangle\in A^2} \dots R(\mathfrak{f}(\mathfrak{a},\mathfrak{b}),z) \dots$$

Disjunction can be split

$$\forall xz.\ \bigvee^{\mathfrak{A}}_{\mathfrak{a}\in A} \bigvee^{\mathfrak{A}}_{\mathfrak{b}\in A} \dots R(\mathfrak{f}(\mathfrak{a},\mathfrak{b}),z) \dots$$

Base disjunction on *type of* $\mathfrak{a}$ *w.r.t.* $x$:

$$\mathcal{T} := \{\, P(x) \wedge Q(x),\ P(x) \wedge \neg Q(x),\ \dots\}$$

$$\forall xz.\ \bigvee_{t\in\mathcal{T}} \bigvee^{\mathfrak{A}}_{\mathfrak{b}\in A} \dots R(\mathfrak{g}_t(\mathfrak{b}),z) \dots$$

Weak dependence (*finite character*)

Strong dependence

Transfer to the syntactic level

$$\exists g_1 \dots g_4\, \forall xz.\ \bigvee_{i=1}^{4} \dots R(g_i(z),z) \dots$$

De-Skolemize

$$\forall xz\, \exists_z y_1 \dots y_4.\ \bigvee_{i=1}^{4} \dots R(y_i,z) \dots$$

"Re-order"

$$\forall z\, \exists y_1 \dots y_4\, \forall x.\ \bigvee_{i=1}^{4} \dots R(y_i,z) \dots$$

# How can we identify weak dependences?

One possible criterion:

Consider a relational FO sentence $\varphi := \mathcal{Q}\bar{\mathbf{x}}\bar{\mathbf{y}}.\,\psi(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ where

- $\mathcal{Q}\bar{\mathbf{x}}\bar{\mathbf{y}}$ – arbitrary quantifier prefix,
- $\bar{\mathbf{x}}$ – universally quantified variables,
- $\bar{\mathbf{y}}$ – existentially quantified variables.

Suppose we can partition $\bar{\mathbf{x}}$ into $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$ and $\bar{\mathbf{y}}$ into $\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2$ such that $\mathrm{At}_\varphi = \mathrm{At}_1(\bar{\mathbf{x}}_1, \bar{\mathbf{y}}_1) \cup \mathrm{At}_2(\bar{\mathbf{x}}_2, \bar{\mathbf{y}}_2)$ and $\mathrm{At}_1(\bar{\mathbf{x}}_1, \bar{\mathbf{y}}_1) \cap \mathrm{At}_2(\bar{\mathbf{x}}_2, \bar{\mathbf{y}}_2) = \emptyset$.

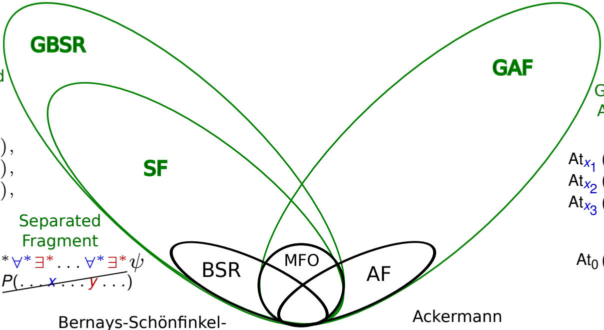
Then,

all $x_1 \in \bar{\mathbf{x}}_1$ and $y_2 \in \bar{\mathbf{y}}_2$ are either independent or weakly dependent;

all $x_2 \in \bar{\mathbf{x}}_2$ and $y_1 \in \bar{\mathbf{y}}_1$ are either independent or weakly dependent.

# Application: New decidable first-order fragments

[Sturm, V., Weidenbach, LICS'16], [V., 2018 submitted]



Generalized
BSR
Fragment

$\mathrm{At}_0(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots),$
$\mathrm{At}_1(\bar{\mathbf{y}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots),$
$\mathrm{At}_2(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{x}}_3, \ldots),$
$\vdots$

Separated
Fragment
$\exists^* \forall^* \exists^* \ldots \forall^* \exists^* \psi$
~~$P(\ldots x \ldots y \ldots)$~~

$\mathrm{At}_\forall(\bar{\mathbf{x}}, \bar{\mathbf{z}}),$
$\mathrm{At}_\exists(\bar{\mathbf{y}}, \bar{\mathbf{z}})$

Bernays-Schönfinkel-
Ramsey Fragment
$\exists^* \forall^* \psi$

Monadic
First-Order
Fragment
$\exists^* \forall^* \exists^* \ldots \forall^* \exists^* \psi$
all predicates unary
w/o equality

Ackermann
Fragment
$\exists^* \forall \exists^* \psi$
w/o equality

Generalized
Ackermann
Fragment

$\mathrm{At}_{x_1}(x_1, \bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots),$
$\mathrm{At}_{x_2}(\bar{\mathbf{u}}_1, x_2, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots),$
$\mathrm{At}_{x_3}(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, x_3, \bar{\mathbf{v}}_3, \ldots),$
$\vdots$
$\mathrm{At}_0(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots)$
w/o equality

# Generalized Bernays–Schönfinkel–Ramsey — GBSR

GBSR contains *relational* first-order sentences *with equality*

$$\forall \bar{\mathbf{x}}_1 \exists \bar{\mathbf{y}}_1 \ldots \forall \bar{\mathbf{x}}_n \exists \bar{\mathbf{y}}_n .\ \psi$$

for which the set $\mathrm{At}_\psi$ can be partitioned into

$$\begin{aligned}
\mathrm{At}_0 &(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots, \bar{\mathbf{x}}_{n-1}, \bar{\mathbf{x}}_n),\\
\mathrm{At}_1 &(\bar{\mathbf{y}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots, \bar{\mathbf{x}}_{n-1}, \bar{\mathbf{x}}_n),\\
\mathrm{At}_2 &(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{x}}_3, \ldots, \bar{\mathbf{x}}_{n-1}, \bar{\mathbf{x}}_n),\\
\mathrm{At}_3 &(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{y}}_3, \ldots, \bar{\mathbf{x}}_{n-1}, \bar{\mathbf{x}}_n),\\
&\vdots\\
\mathrm{At}_{n-1} &(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{y}}_3, \ldots, \bar{\mathbf{y}}_{n-1}, \bar{\mathbf{x}}_n),\\
\mathrm{At}_n &(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{y}}_3, \ldots, \bar{\mathbf{y}}_{n-1}, \bar{\mathbf{y}}_n),
\end{aligned}$$

⇝ all dependences are weak

⇝ there is an equivalent (much longer) $\exists^*\forall^*$-sentence

such that $\quad \mathrm{vars}(\mathrm{At}_i) \cap \mathrm{vars}(\mathrm{At}_j) \cap \bar{\mathbf{x}}_\ell = \emptyset$

for all $i \neq j$ and every $\ell$.

# GBSR: Finite model property

Given a GBSR sentence $\varphi = \forall \bar{\mathbf{x}}_1 \exists \bar{\mathbf{y}}_1 \ldots \forall \bar{\mathbf{x}}_n \exists \bar{\mathbf{y}}_n.\ \psi$

with the partition of $\mathrm{At}_\varphi$ into

$$\begin{array}{r}
\mathrm{At}_0\,(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots, \bar{\mathbf{x}}_{n-1}, \bar{\mathbf{x}}_n), \\
\mathrm{At}_1\,(\bar{\mathbf{y}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots, \bar{\mathbf{x}}_{n-1}, \bar{\mathbf{x}}_n), \\
\mathrm{At}_2\,(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{x}}_3, \ldots, \bar{\mathbf{x}}_{n-1}, \bar{\mathbf{x}}_n), \\
\vdots \\
\mathrm{At}_{n-1}\,(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{y}}_3, \ldots, \bar{\mathbf{y}}_{n-1}, \bar{\mathbf{x}}_n), \\
\mathrm{At}_n\,(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{y}}_3, \ldots, \bar{\mathbf{y}}_{n-1}, \bar{\mathbf{y}}_n),
\end{array}$$

we know that all dependences are weak.

We consider *model checking games* for players **A** and **E**:

- ⇝ For any model $\mathfrak{A} \models \varphi$ there exists a winning strategy for **E** that has a *finite image* and thus induces a *finite substructure* $\mathfrak{B} \subseteq \mathfrak{A}$ with $\mathfrak{B} \models \varphi$
- ⇝ Any model $\mathfrak{A} \models \varphi$ has a finite satisfying substructure $\mathfrak{B} \models \varphi$

# Generalized Ackermann fragment — GAF

GAF contains relational first-order sentences

$$\forall x_1\, \exists \bar{\mathbf{u}}_1 \bar{\mathbf{v}}_1\, \forall x_2\, \exists \bar{\mathbf{u}}_2 \bar{\mathbf{v}}_2 \ldots \forall x_n\, \exists \bar{\mathbf{u}}_n \bar{\mathbf{v}}_n \,.\, \psi$$

for which the set $\mathrm{At}_\psi$ can be partitioned into

$$\begin{array}{r}
\mathrm{At}_{x_1}\big(x_1, \bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots, \bar{\mathbf{v}}_{n-2}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
\mathrm{At}_{x_2}\big(\bar{\mathbf{u}}_1, x_2, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots, \bar{\mathbf{v}}_{n-2}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
\mathrm{At}_{x_3}\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, x_3, \bar{\mathbf{v}}_3, \ldots, \bar{\mathbf{v}}_{n-2}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
\vdots\\
\mathrm{At}_{x_{n-1}}\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots, x_{n-1}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
\mathrm{At}_{x_n}\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots, \bar{\mathbf{u}}_{n-1},\; x_n\;, \bar{\mathbf{v}}_n\big),\\
\mathrm{At}_0\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots, \bar{\mathbf{u}}_{n-1}, \bar{\mathbf{u}}_n\big),
\end{array}$$

such that $\mathrm{vars}(\mathrm{At}_{x_i}) \cap \mathrm{vars}(\mathrm{At}_{x_j}) \cap \bar{\mathbf{v}}_\ell = \emptyset$

for all $x_i \neq x_j$ and every $\ell$.

- $\rightsquigarrow$ strong dependences between $\forall x_i$ and $\exists \bar{\mathbf{v}}_k$
- $\rightsquigarrow$ weak dependences between $\forall x_i$ and $\exists \bar{\mathbf{u}}_k$
- $\rightsquigarrow$ equivalent to some $\exists^*\forall\exists^*$-sentence

# GAF: Finite model property

Given a GAF sentence $\varphi = \forall x_1\, \exists \bar{\mathbf{u}}_1 \bar{\mathbf{v}}_1\, \forall x_2\, \exists \bar{\mathbf{u}}_2 \bar{\mathbf{v}}_2\, \ldots\, \forall x_n\, \exists \bar{\mathbf{u}}_n \bar{\mathbf{v}}_n .\, \psi$

with the partition of $\mathrm{At}_\varphi$ into

$$\begin{aligned}
&\mathrm{At}_{x_1}\big(x_1, \bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots, \bar{\mathbf{v}}_{n-2}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
&\mathrm{At}_{x_2}\big(\bar{\mathbf{u}}_1, x_2, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots, \bar{\mathbf{v}}_{n-2}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
&\mathrm{At}_{x_3}\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, x_3, \bar{\mathbf{v}}_3, \ldots, \bar{\mathbf{v}}_{n-2}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
&\qquad\vdots\\
&\mathrm{At}_{x_{n-1}}\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots, x_{n-1}, \bar{\mathbf{v}}_{n-1}, \bar{\mathbf{v}}_n\big),\\
&\mathrm{At}_{x_n}\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots, \bar{\mathbf{u}}_{n-1}, \;x_n\;, \bar{\mathbf{v}}_n\big),\\
&\mathrm{At}_0\big(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots, \bar{\mathbf{u}}_{n-1}, \bar{\mathbf{u}}_n\big),
\end{aligned}$$

we know that dependences between $v$ and $x_i$ may be strong for *exactly one* $x_i$ ; all other existing dependences are weak.

Again, we consider *model checking games* for players **A** and **E**:

⇝ For any $\mathfrak{A} \models \varphi$ we can construct a *similar* $\mathfrak{B} \models \varphi$
for which **E** has a winning strategy with a *finite image*
and thus induces a *finite substructure* $\mathfrak{C} \subseteq \mathfrak{B}$ with $\mathfrak{C} \models \varphi$

# The bottom line is ...

Quantified first-order variables $\forall x \, \exists y$ can be:

strongly dependent        *fully* independent

weakly dependent: "*up to a finite degree*"

Reason for strong dependence:

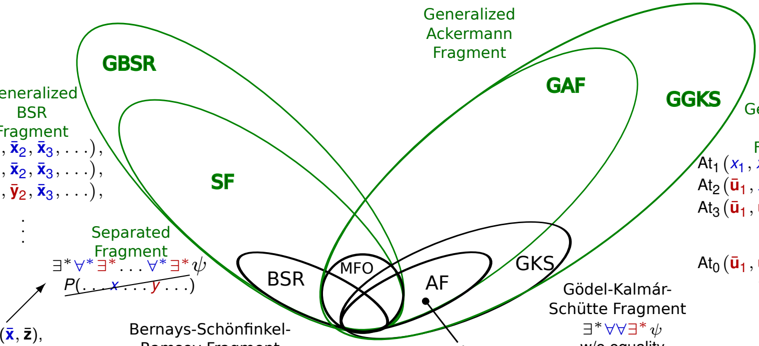
co-occurrences of variables in atoms $P(\ldots x \ldots y \ldots)$
or in chains of atoms $P(\ldots x \ldots z \ldots),\ Q(\ldots z \ldots y \ldots)$

- ⇝ Restricting co-occurrences of variables in atoms leads to decidable fragments: SF, GBSR, GAF
- ⇝ Other applications are conceivable in proof complexity, automated reasoning

Questions?

# Application: New decidable first-order fragments

[Sturm, V., Weidenbach, LICS'16], [V., 2018 submitted]



Generalized BSR Fragment

$\mathrm{At}_0(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots)$,
$\mathrm{At}_1(\bar{\mathbf{y}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \ldots)$,
$\mathrm{At}_2(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \bar{\mathbf{x}}_3, \ldots)$,
$\vdots$

Separated Fragment
$\exists^*\forall^*\exists^*\ldots\forall^*\exists^*\psi$
$\cancel{P(\ldots x \ldots y \ldots)}$

$\mathrm{At}_\forall(\bar{\mathbf{x}}, \bar{\mathbf{z}})$,
$\mathrm{At}_\exists(\bar{\mathbf{y}}, \bar{\mathbf{z}})$

Bernays-Schönfinkel-Ramsey Fragment
$\exists^*\forall^*\psi$

Monadic First-Order Fragment
$\exists^*\forall^*\exists^*\ldots\forall^*\exists^*\psi$
all predicates unary
w/o equality

Generalized Ackermann Fragment

Ackermann Fragment
$\exists^*\forall\exists^*\psi$
w/o equality

Gödel-Kalmár-Schütte Fragment
$\exists^*\forall\forall\exists^*\psi$
w/o equality

Generalized GKS Fragment

$\mathrm{At}_1(x_1, x_1', \bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots)$,
$\mathrm{At}_2(\bar{\mathbf{u}}_1, x_2, x_2', \bar{\mathbf{v}}_2, \bar{\mathbf{v}}_3, \ldots)$,
$\mathrm{At}_3(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, x_3, x_3', \bar{\mathbf{v}}_3, \ldots)$,
$\vdots$
$\mathrm{At}_0(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3, \bar{\mathbf{u}}_4, \ldots)$
w/o equality