Lower Bounds for the Runtime of a Global Multi-Objective Evolutionary Algorithm

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Introduction

Runtimes of multi-objective EAs are less understood:
Lower bounds are rare!

These appear to be hard to prove "where the population of the multiobjective EA spreads out before the Pareto set is reached."

[Laumanns et al., Evolutionary Computation 2004]

Our contribution:
a lower bound for the

Global Simple Evolutionary Multi-Objective Optimizer

[Giel CEC 2003]
on the

LEADINGONESTRAILINGZEROS test function
for small mutation rates.
The two-objective test function **LEADINGONESTRAILINGZEROS (LOTZ)**

[Laumanns et al., PPSN 2002]

*Individuals* are bit strings of length \( n \).

The fitness of individual \( x = 11110100101100 \) is the pair \((\text{LO}(x), \text{TZ}(x)) = (4, 2)\).

**Weak domination:**

\( y \preceq x \) iff \( \text{LO}(y) \leq \text{LO}(x) \) and \( \text{TZ}(y) \leq \text{TZ}(x) \).

**Domination:**

\( y \prec x \) iff at least one of the above inequalities is strict.

\( x \) and \( y \) are *incomparable* iff neither of them weakly dominates the other.
The search space of LOTZ

A *population* is a nonempty set of pairwise incomparable individuals.
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Pareto optimality:

Individual $x$ is Pareto-optimal iff there is no $y$ s.t. $x \prec y$.

Goal: find the Pareto front

$$
\mathcal{F} := \left\{ (\text{LO}(x), \text{TZ}(x)) \mid x \text{ is Pareto-optimal} \right\}.
$$

$$
\text{LO}(x) + \text{TZ}(x) = n
$$
The **Global Simple Evolutionary Multi-Objective Optimizer (GSEMO)** [Giel, CEC 2003]

**Initialization:** Choose $x^*$ uniformly at random.
Initial population $P := \{x^*\}$.

**Loop:**
- Pick an $x$ from the current $P$ uniformly at random.
- Flip each of $x$’s bits independently with probability $p$ to obtain the offspring $x'$.
- If $x'$ is weakly dominated by any $y \in P$, discard $x'$.
- Otherwise, add $x'$ to $P$ and remove all individuals dominated by $x'$. 
Our main result

Theorem:

GSEMO on LOTZ needs at least $\Omega(n^2/p)$ iterations to discover the complete Pareto front

- with high probability $1 - o(1)$,
- for any mutation probability $p < n^{-7/4}$.

Our lower bound matches Giel’s (CEC 2003) upper bound.
Ideas: two generations of individuals

We classify all individuals that appear during a GSEMO run on LOTZ as follows.

**First Generation** $G_1$:
all individuals that are neither Pareto-optimal nor descendants of Pareto-optimal individuals.

**Second Generation** $G_2$:
all Pareto-optimal individuals and descendants of second-generation individuals.
Ideas: the confinement of the first generation

Artificial fitness measure \( q(x) := |x|_1 \).

Rough intuitive justification:
We would expect \( x = 11110100 | 101100 \)
to step-by-step mutate towards \( 11111111 | 000000 \).

Lemma:
With prob. \( 1 - o(1) \)
all \( x \in G_1 \) are
confined to a small area
with respect to \( q(\cdot) \).
Ideas: the development of the second generation

Suppose we have touched the Pareto front.

The *population head* $x^L$ is the Pareto-optimal individual with maximal $L_0$ value in $P$.

Lemma:
The prob. that a jump over $x^L$ occurs is negligibly small: $O(1/\sqrt{n})$. 

![Diagram showing the population head $x^L$, a point $y'$, and a point $y$ with annotations for the calculation involving $L_0$, $n$, and $\sqrt{n}$.](image-url)
Ideas: the development of the second generation

Suppose $x^L$ has left the confinement area of the first generation. An outlier is an individual with higher LO value than $x^L$.

Lemma:
With probability $1 - o(1)$ there is always at most one second-generation outlier $x^{\text{out}}$. 
Ideas: the development of the second generation

Discovering the Pareto front requires increasing $LO_{\text{max}}$ to $n$.

Recap: beyond the confinement area only $x^L$ and possibly a single outlier $x^{\text{out}}$ contribute to an increase in $LO_{\text{max}}$.

We can show:

Increasing $LO_{\text{max}}$ by $\Theta(n)$ increases the population size by $\Theta(n)$.

Reason:

On our way up we collect Pareto optima. These stay permanently in $P$. 

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Ideas: the development of the second generation

Given
- a population of guaranteed size $\Theta(n)$, and
- still $\Theta(n)$ to go before $LO_{\text{max}} = n$,

the lower bound is easily derived:

We $\Omega(n)$ times have to
(i) select $x^L$ or $x^{\text{out}}$ for mutation (prob. $\leq 2/|P|$), and
(ii) flip the leftmost zero bit (prob. $p$).

The expected $\Omega(n) \cdot \frac{1}{2}|P| \cdot p^{-1} = \Omega(n^2/p)$ iterations are even necessary with probability $1 - o(1)$. 

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Experimental results (avg. #iterations, $n = 5, 10, \ldots, 200$)

\[ p = n^{-1.75} \quad + \]
\[ p = n^{-1} \quad \times \]

\[ \approx 0.4n^{3.746} \quad \text{for} \quad p = n^{-7/4} \quad \text{and} \quad \approx 0.6n^{3.024} \quad \text{for} \quad p = n^{-1}. \]
Conclusion and future work

- GSEMO on LOTZ needs $\Theta(n^2/p)$ iterations for small $p$. 
  ⇒ Our lower bound matches Giel’s (CEC 2003) upper bound.

- Our confinement proof for $G_1$ is technically demanding. 
  ⇒ Dependencies!

- The applied techniques might be useful for other lower bound proofs as well.

- Extending our result to the standard case $p = 1/n$ is a natural next step.

Thank you!

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