Lower Bounds for the Runtime of a Global Multi-Objective Evolutionary Algorithm

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June 21, 2013

Introduction

Runtimes of multi-objective EAs are less understood:

Lower bounds are rare!

These appear to be hard to prove "where the population of the multiobjective EA spreads out before the Pareto set is reached."

[Laumanns et al., Evolutionary Computation 2004]

Our contribution:

a lower bound for the

Global Simple Evolutionary Multi-Objective Optimizer

[Giel CEC 2003]

on the

LEADINGONESTRAILINGZEROS test function

for small mutation rates.

The two-objective test function LEADINGONESTRAILINGZEROS (LOTZ)

[Laumanns et al., PPSN 2002]

Individuals are bit strings of length n.

The fitness of individual $x = 1111010010110 0$

is the pair $\big(\textsc{lo}(x), \textsc{tz}(x)\big) = (\ 4\ ,\ 2\ )$.

Weak domination:

$y \preceq x$ iff $\textsc{lo}(y) \leq \textsc{lo}(x)$ and $\textsc{tz}(y) \leq \textsc{tz}(x)$.

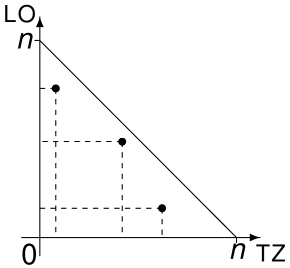
Domination:

$y \prec x$ iff at least one of the above inequalities is strict.

x and y are *incomparable* iff neither of them weakly dominates the other.

The search space of LOTZ

A *population* is a nonempty set
of pairwise incomparable individuals.



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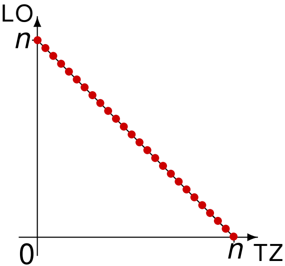


A *population* is a nonempty set of pairwise incomparable individuals.

Pareto optimality:

Individual x is *Pareto-optimal* iff there is no y s.t. $x \prec y$.

Goal: find the *Pareto front*

$$\mathcal{F} := \Big\{ \big(\text{LO}(x), \text{TZ}(x)\big) \;\Big|\; \underbrace{x \text{ is Pareto-optimal}}_{\text{LO}(x) + \text{TZ}(x) = n} \Big\}.$$

The Global Simple Evolutionary Multi-Objective Optimizer (GSEMO) [Giel, CEC 2003]

Initialization: Choose x^* uniformly at random.
Initial population $P := \{x^*\}$.

Loop:
- Pick an x from the current P uniformly at random.
- Flip each of x's bits independently with probability p to obtain the offspring x'.
- If x' is weakly dominated by any $y \in P$, discard x'.
- Otherwise, add x' to P and remove all individuals dominated by x'.

Our main result

Theorem:

GSEMO on LOTZ needs at least $\Omega(n^2/p)$ iterations to discover the complete Pareto front

- with high probability $1 - o(1)$,
- for any mutation probability $p < n^{-7/4}$.

Our lower bound matches Giel's (CEC 2003) upper bound.

Ideas: two generations of individuals

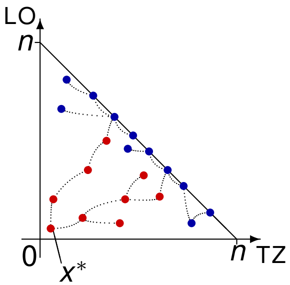
We classify all individuals that appear during a GSEMO run on LOTZ as follows.

First Generation $\mathcal{G}_1$:

all individuals that are neither Pareto-optimal nor descendants of Pareto-optimal individuals.

Second Generation $\mathcal{G}_2$:

all Pareto-optimal individuals and descendants of second-generation individuals.



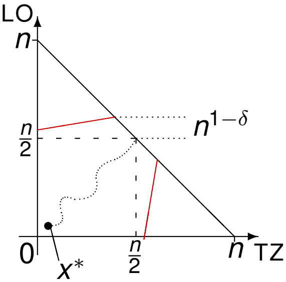
Ideas: the confinement of the first generation

Artificial fitness measure $q(x) := |x|_1$.

Rough intuitive justification:

We would expect $x = 11110100 | 101100$
to step-by-step mutate towards $11111111 | 000000$.

Lemma:
With prob. $1 - o(1)$
all $x \in \mathcal{G}_1$ are
confined to a small area
with respect to $q(\cdot)$.

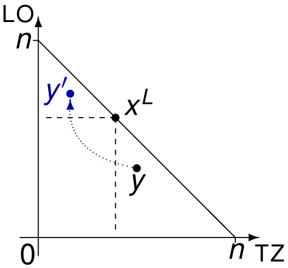


Ideas: the development of the second generation

Suppose we have touched the Pareto front.

The *population head* x^L is the Pareto-optimal individual with maximal LO value in P.

Lemma:
The prob. that a jump over x^L occurs is negligibly small: $\mathcal{O}(1/\sqrt{n})$.

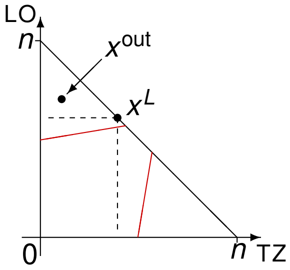


Ideas: the development of the second generation

Suppose x^L has left the confinement area of the first generation.

An *outlier* is an individual with higher LO value than x^L.

Lemma:
With probability $1 - o(1)$ there is always at most one second-generation outlier x^{out}.



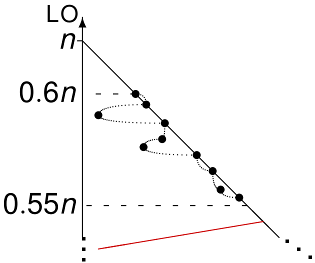
Ideas: the development of the second generation

Discovering the Pareto front requires increasing $\text{LO}_{\max}$ to n.

Recap: beyond the confinement area only x^L and possibly a single outlier x^{out} contribute to an increase in $\text{LO}_{\max}$.

We can show:

Increasing $\text{LO}_{\max}$ by $\Theta(n)$ increases the population size by $\Theta(n)$.



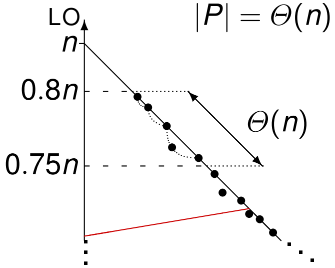
Reason:

On our way up we collect Pareto optima. These stay permanently in P.

Ideas: the development of the second generation

Given
- a population of guaranteed size $\Theta(n)$, and
- still $\Theta(n)$ to go before $\text{LO}_{\max} = n$,

the lower bound is easily derived:



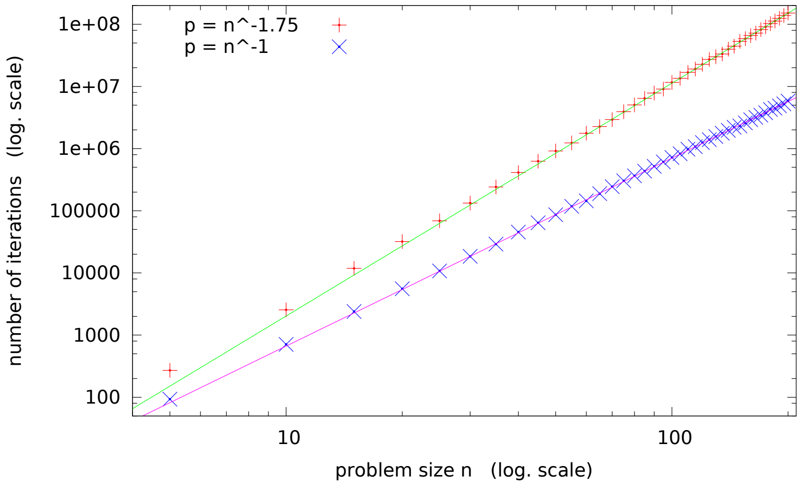
We $\Omega(n)$ times have to

(i) select x^L or x^{out} for mutation $\left(\text{prob.} \leq 2/|P|\right)$, and

(ii) flip the leftmost zero bit (prob. p).

The expected $\Omega(n) \cdot \frac{1}{2}|P| \cdot p^{-1} = \Omega(n^2/p)$ iterations are even necessary with probability $1 - o(1)$.

Experimental results (avg. #iterations, $n = 5, 10, \ldots, 200$)



$\approx 0.4n^{3.746}$ for $p = n^{-7/4}$ and $\approx 0.6n^{3.024}$ for $p = n^{-1}$.

Conclusion and future work

- GSEMO on LOTZ needs $\Theta(n^2/p)$ iterations for *small p*.

  $\rightsquigarrow$ Our lower bound matches Giel's (CEC 2003) upper bound.

- Our confinement proof for $\mathcal{G}_1$ is technically demanding.

  $\rightsquigarrow$ Dependencies!

- The applied techniques might be useful for other lower bound proofs as well.

- Extending our result to the standard case $p = 1/n$ is a natural next step.

Thank you!

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