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On Implicit Dependence and Independence between Differently Quantified First-Order Variables

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This is about concepts that have emerged in my Ph.D. research:

[Sturm, Voigt, Weidenbach, LICS'16] [Voigt, LICS'17] [Voigt, arXiv.org 2017]

Abstractly speaking:

Inferring independence between quantified variables $\forall x\, \exists y$ from formula structure, in particular from a lack of atoms

$$P(\ldots x \ldots y \ldots).$$

Focus on relational first-order logic with standard syntax

$\leadsto$ No syntactic means for explicitly marking independence

Implicit dependence between variables

$$
\begin{array}{lll}
 & \forall x_1 x_2 x_3.\ R(x_1, x_2) \wedge R(x_2, x_3) \rightarrow R(x_1, x_3) & \text{transitivity} \\
\wedge & \forall x.\ \neg R(x, x) & \text{irreflexivity} \\
\wedge & \forall x \exists y.\ \boxed{R(x, y)} & \text{existence of successors}
\end{array}
$$

cannot be swapped

causes dependence

$\rightsquigarrow$ Any model contains an infinite *R*-chain $\bullet \xrightarrow{R} \bullet \xrightarrow{R} \bullet \xrightarrow{R} \bullet \xrightarrow{R} \bullet \xrightarrow{R} \cdots$

$\rightsquigarrow$ Even stronger, there is no equivalent sentence $\exists \vec{\mathbf{y}}'\, \forall \vec{\mathbf{x}}'.\ \psi'$

$\rightsquigarrow$ *y fully depends x*

Implicit (in)dependence

$$\forall x\, \exists y.\, R(x, y) \quad \not\Leftrightarrow \quad \exists \vec{\mathbf{y}}'\, \forall \vec{\mathbf{x}}'.\, \psi'$$

⇝ *Full* dependence

$$\forall x\, \exists y.\, P(x) \rightarrow Q(y) \quad \Leftrightarrow \quad \exists y\, \forall x.\, P(x) \rightarrow Q(y)$$

⇝ *Full* independence

$$\forall x\, \exists y.\, P(x) \leftrightarrow Q(y) \quad \Leftrightarrow \quad \exists y_1\, y_2\, \forall x.\, \big(P(x) \rightarrow Q(y_1)\big) \\ \wedge \big(Q(y_2) \rightarrow P(x)\big)$$

⇝ *Finitely unfoldable* dependence

⇝ Dependences can be “transformed away”

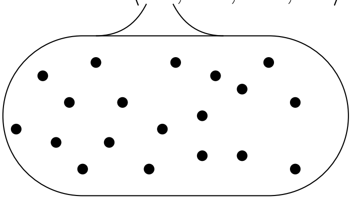
So far, we have focused on syntax...

...but there is also a semantic point of view.

Game semantics

Given: a formula $\varphi := \forall\vec{\mathbf{x}}_1\exists\vec{\mathbf{y}}_1\forall\vec{\mathbf{x}}_2\exists\vec{\mathbf{y}}_2\ldots\forall\vec{\mathbf{x}}_n\exists\vec{\mathbf{y}}_n.\ \psi$ and

a structure $\mathcal{A} := \langle\,\mathfrak{U}\,, P^{\mathcal{A}}, Q^{\mathcal{A}}, \ldots\rangle$



Game semantics

Given: a formula $\varphi := \forall\vec{\mathbf{x}}_1\exists\vec{\mathbf{y}}_1\forall\vec{\mathbf{x}}_2\exists\vec{\mathbf{y}}_2\ldots\forall\vec{\mathbf{x}}_n\exists\vec{\mathbf{y}}_n.\ \psi$ and

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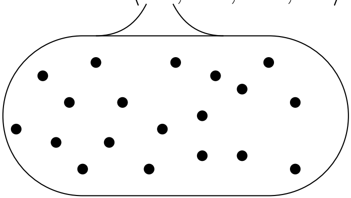

Players **A** and **E** move consecutively

$\vec{\mathbf{x}}_1 \quad \vec{\mathbf{y}}_1 \quad \vec{\mathbf{x}}_2 \quad \vec{\mathbf{y}}_2 \quad \ldots \quad \vec{\mathbf{x}}_n \quad \vec{\mathbf{y}}_n$

Game semantics

Given: a formula $\varphi := \forall\vec{\mathbf{x}}_1\exists\vec{\mathbf{y}}_1\forall\vec{\mathbf{x}}_2\exists\vec{\mathbf{y}}_2\ldots\forall\vec{\mathbf{x}}_n\exists\vec{\mathbf{y}}_n.\,\psi$ and

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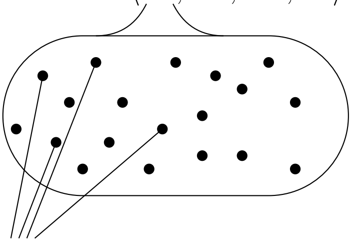

Players **A** and **E** move consecutively

$\vec{\mathbf{x}}_1$ $\vec{\mathbf{y}}_1$ $\vec{\mathbf{x}}_2$ $\vec{\mathbf{y}}_2$ $\ldots$ $\vec{\mathbf{x}}_n$ $\vec{\mathbf{y}}_n$

$\vec{\mathbf{a}}_1$

Game semantics

Given: a formula $\varphi := \forall\vec{\mathbf{x}}_1\exists\vec{\mathbf{y}}_1\forall\vec{\mathbf{x}}_2\exists\vec{\mathbf{y}}_2\ldots\forall\vec{\mathbf{x}}_n\exists\vec{\mathbf{y}}_n.\ \psi$ and

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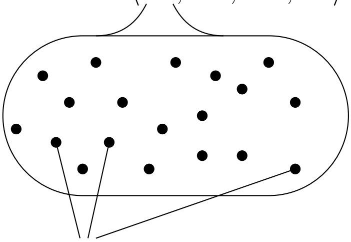


Players **A** and **E** move consecutively

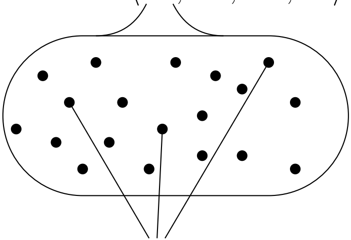
$\vec{\mathbf{x}}_1$ $\vec{\mathbf{y}}_1$ $\vec{\mathbf{x}}_2$ $\vec{\mathbf{y}}_2$ … $\vec{\mathbf{x}}_n$ $\vec{\mathbf{y}}_n$

$\vec{\mathbf{a}}_1$ $\vec{\mathbf{b}}_1$

Game semantics

Given: a formula $\varphi := \forall\vec{\mathbf{x}}_1\exists\vec{\mathbf{y}}_1\forall\vec{\mathbf{x}}_2\exists\vec{\mathbf{y}}_2\ldots\forall\vec{\mathbf{x}}_n\exists\vec{\mathbf{y}}_n.\ \psi$ and

a structure $\mathcal{A} := \langle \mathfrak{U}, P^{\mathcal{A}}, Q^{\mathcal{A}}, \ldots \rangle$



Players **A** and **E** move consecutively

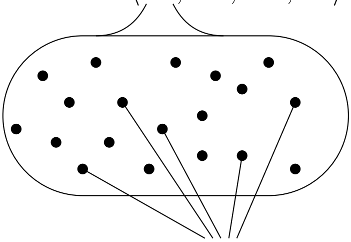
$$\vec{\mathbf{x}}_1 \quad \vec{\mathbf{y}}_1 \quad \vec{\mathbf{x}}_2 \quad \vec{\mathbf{y}}_2 \quad \ldots \quad \vec{\mathbf{x}}_n \quad \vec{\mathbf{y}}_n$$

$$\vec{\mathbf{a}}_1 \quad \vec{\mathbf{b}}_1 \quad \vec{\mathbf{a}}_2$$

Game semantics

Given: a formula $\varphi := \forall\vec{\mathbf{x}}_1\exists\vec{\mathbf{y}}_1\forall\vec{\mathbf{x}}_2\exists\vec{\mathbf{y}}_2\dots\forall\vec{\mathbf{x}}_n\exists\vec{\mathbf{y}}_n.\ \psi$ and

a structure $\mathcal{A} := \langle\, \mathfrak{U}\,, P^{\mathcal{A}}, Q^{\mathcal{A}}, \dots\rangle$



Players **A** and **E** move consecutively

$\vec{\mathbf{x}}_1 \quad \vec{\mathbf{y}}_1 \quad \vec{\mathbf{x}}_2 \quad \vec{\mathbf{y}}_2 \quad \dots \quad \vec{\mathbf{x}}_n \quad \vec{\mathbf{y}}_n$

$\vec{\mathbf{a}}_1 \quad \vec{\mathbf{b}}_1 \quad \vec{\mathbf{a}}_2 \quad \vec{\mathbf{b}}_2$

Game semantics

Given: a formula $\varphi := \forall \vec{\mathbf{x}}_1 \exists \vec{\mathbf{y}}_1 \forall \vec{\mathbf{x}}_2 \exists \vec{\mathbf{y}}_2 \ldots \forall \vec{\mathbf{x}}_n \exists \vec{\mathbf{y}}_n .\, \psi$ and

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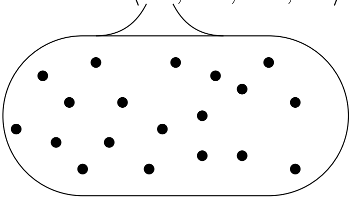


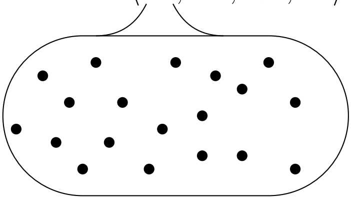
Players **A** and **E** move consecutively

$\vec{\mathbf{x}}_1$	$\vec{\mathbf{y}}_1$	$\vec{\mathbf{x}}_2$	$\vec{\mathbf{y}}_2$	$\ldots$	$\vec{\mathbf{x}}_n$	$\vec{\mathbf{y}}_n$
$\vec{\mathbf{a}}_1$	$\vec{\mathbf{b}}_1$	$\vec{\mathbf{a}}_2$	$\vec{\mathbf{b}}_2$	$\ldots$	$\vec{\mathbf{a}}_n$	$\vec{\mathbf{b}}_n$

Game semantics

Given: a formula $\varphi := \forall\vec{\mathbf{x}}_1 \exists\vec{\mathbf{y}}_1 \forall\vec{\mathbf{x}}_2 \exists\vec{\mathbf{y}}_2 \dots \forall\vec{\mathbf{x}}_n \exists\vec{\mathbf{y}}_n.\, \psi$ and

a structure $\mathcal{A} := \langle \mathfrak{U}, P^{\mathcal{A}}, Q^{\mathcal{A}}, \dots \rangle$



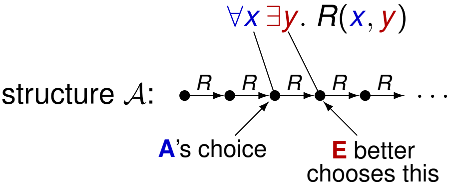
Players **A** and **E** move consecutively

$$\begin{array}{cccccccc} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{x}}_2 & \vec{\mathbf{y}}_2 & \dots & \vec{\mathbf{x}}_n & \vec{\mathbf{y}}_n \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{b}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{b}}_2 & \dots & \vec{\mathbf{a}}_n & \vec{\mathbf{b}}_n \end{array}$$

E wins iff $\mathcal{A}, [\vec{\mathbf{x}}_1 \mapsto \vec{\mathbf{a}}_1, \vec{\mathbf{y}}_1 \mapsto \vec{\mathbf{b}}_1, \dots, \vec{\mathbf{x}}_n \mapsto \vec{\mathbf{a}}_n, \vec{\mathbf{y}}_n \mapsto \vec{\mathbf{b}}_n] \models \psi$

$\rightsquigarrow$ $\mathcal{A} \models \varphi$ iff there is a *winning strategy* for **E**

Game semantics: full dependence



$$\forall x\, \exists y.\ R(x, y)$$

structure $\mathcal{A}$:

A's choice

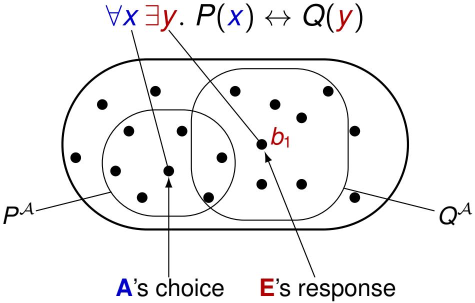
E better chooses this

$\rightsquigarrow$ Any winning strategy for **E** has to have an infinite image

Game semantics: finitely unfoldable dependence

$\forall x \, \exists y. \, P(x) \leftrightarrow Q(y)$

structure $\mathcal{A}$:

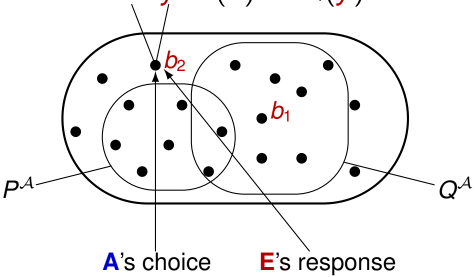


$\rightsquigarrow$ b_1 can be used as a response to any element from $P^{\mathcal{A}}$

Game semantics: finitely unfoldable dependence

$$\forall x\, \exists y.\ P(x) \leftrightarrow Q(y)$$

structure $\mathcal{A}$:



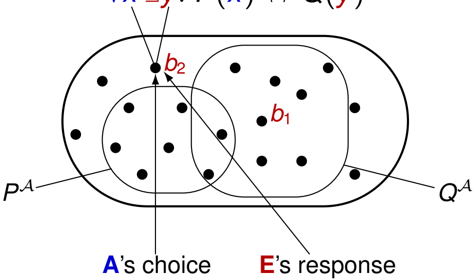
$\rightsquigarrow$ b_1 can be used as a response to any element from $P^{\mathcal{A}}$

$\rightsquigarrow$ b_2 can be used as a response to any element outside $P^{\mathcal{A}}$

Game semantics: finitely unfoldable dependence

$\forall x\, \exists y.\ P(x) \leftrightarrow Q(y)$

structure $\mathcal{A}$:



⇝ b_1 can be used as a response to any element from $P^{\mathcal{A}}$

⇝ b_2 can be used as a response to any element outside $P^{\mathcal{A}}$

⇝ A winning strategy for **E** needs only 2 domain elements

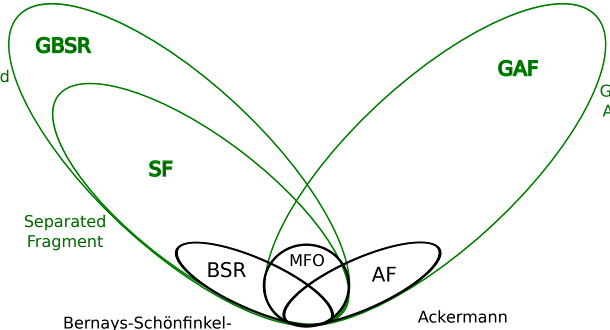
⇝ This idea can be generalized:

E needs only finitely many response options

What's this good for?

$\rightsquigarrow$ New decidable first-order fragments

New decidable first-order fragments



GBSR

Generalized BSR Fragment

SF

Separated Fragment

GAF

Generalized Ackermann Fragment

BSR

MFO

AF

Bernays-Schönfinkel-Ramsey Fragment

$\exists^*\forall^*\psi$

Monadic First-Order Fragment

$\exists^*\forall^*\exists^*\ldots\forall^*\exists^*\psi$

all predicates unary

Ackermann Fragment

$\exists^*\forall\exists^*\psi$

The separated fragment — SF [Sturm, V., Weidenbach, LICS'16]

SF contains relational first-order sentences

with equality in prenex form

$$\exists \vec{\mathbf{z}}\, \forall \vec{\mathbf{x}}_1 \exists \vec{\mathbf{y}}_1 \ldots \forall \vec{\mathbf{x}}_n \exists \vec{\mathbf{y}}_n .\ \psi$$

such that

no atom may contain blue *and* red variables.

$$\exists^* \forall^* \exists^* \ldots \forall^* \exists^* \psi$$

$$\cancel{P(\ldots x \ldots y \ldots)}$$

$\rightsquigarrow$ SF-Sat is decidable [Sturm, Voigt, Weidenbach, LICS'16]

$\rightsquigarrow$ SF-Sat is non-elementary and contains
k-NEXPTIME-complete subproblems SF$_{\partial \leq k}$-Sat [Voigt, LICS'17]

$\rightsquigarrow$ finite model property ...

... via syntactic transformations [Sturm, V., Weidenbach, LICS'16]

... via game semantics [Voigt, arXiv.org 2017]

Generalized Bernays–Schönfinkel–Ramsey — GBSR

[Voigt, arXiv.org 2017]

GBSR contains relational first-order sentences

$$\forall \vec{\mathbf{x}}_1 \exists \vec{\mathbf{y}}_1 \ldots \forall \vec{\mathbf{x}}_n \exists \vec{\mathbf{y}}_n .\ \psi$$

for which the set of atoms of ψ can be partitioned into sets

$$\begin{array}{r}
\mathrm{At}_0\big(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \ldots, \vec{\mathbf{x}}_{n-1}, \vec{\mathbf{x}}_n\big), \\
\mathrm{At}_1\big(\vec{\mathbf{y}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \ldots, \vec{\mathbf{x}}_{n-1}, \vec{\mathbf{x}}_n\big), \\
\mathrm{At}_2\big(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{x}}_3, \ldots, \vec{\mathbf{x}}_{n-1}, \vec{\mathbf{x}}_n\big), \\
\mathrm{At}_3\big(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3, \ldots, \vec{\mathbf{x}}_{n-1}, \vec{\mathbf{x}}_n\big), \\
\vdots \qquad\qquad \\
\mathrm{At}_{n-1}\big(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3, \ldots, \vec{\mathbf{y}}_{n-1}, \vec{\mathbf{x}}_n\big), \\
\mathrm{At}_n\big(\vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2, \vec{\mathbf{y}}_3, \ldots, \vec{\mathbf{y}}_{n-1}, \vec{\mathbf{y}}_n\big),
\end{array}$$

⇝ all dependences unfoldable

⇝ winning strategy over finite domain exists

such that $\mathrm{vars}(\mathrm{At}_i) \cap \mathrm{vars}(\mathrm{At}_j) \cap \vec{\mathbf{x}}_\ell = \emptyset$

for all $i \neq j$ and every ℓ.

Generalized Ackermann fragment — GAF

[Voigt, arXiv.org 2017]

GAF contains relational first-order sentences

$$\forall x_1\, \exists \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1\, \forall x_2\, \exists \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2\, \ldots\, \forall x_n\, \exists \vec{\mathbf{u}}_n \vec{\mathbf{v}}_n\, .\, \psi$$

for which the set of atoms of ψ can be partitioned into sets

$$\begin{array}{r}
\mathrm{At}_{x_1}\big(x_1, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_{n-2}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big), \\
\mathrm{At}_{x_2}\big(\vec{\mathbf{u}}_1, x_2, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_{n-2}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big), \\
\mathrm{At}_{x_3}\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, x_3, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_{n-2}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big), \\
\vdots \\
\mathrm{At}_{x_{n-1}}\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3, \vec{\mathbf{u}}_4, \ldots, x_{n-1}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big), \\
\mathrm{At}_{x_n}\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3, \vec{\mathbf{u}}_4, \ldots, \vec{\mathbf{u}}_{n-1}, \;\; x_n\; , \vec{\mathbf{v}}_n\big), \\
\mathrm{At}_0\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3, \vec{\mathbf{u}}_4, \ldots, \vec{\mathbf{u}}_{n-1}, \vec{\mathbf{u}}_n\big),
\end{array}$$

$\rightsquigarrow$ full dependences between $\forall x_i$ and $\exists \vec{\mathbf{v}}_k$

$\rightsquigarrow$ still, finite models exist

such that $\quad \mathrm{vars}(\mathrm{At}_{x_i}) \cap \mathrm{vars}(\mathrm{At}_{x_j}) \cap \vec{\mathbf{v}}_\ell = \emptyset$

for all $x_i \neq x_j$ and every ℓ.

The bottom line is ...

Quantified first-order variables $\forall x \; \exists y$ can be:

fully dependent *fully* independent

dependent *up to finite degree*

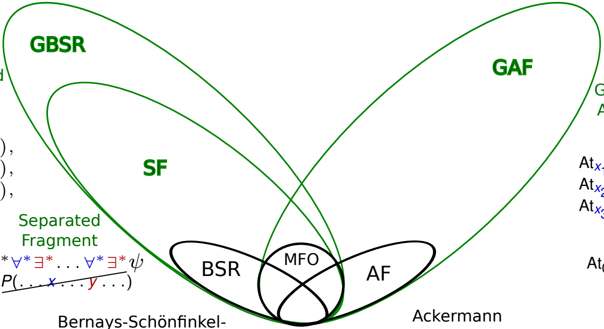
Cause for dependence:

interaction of variables in atoms $P(\ldots x \ldots y \ldots)$

Restricting joint occurrences of variables in atoms
leads to decidable fragments: *SF*, *GBSR*, *GAF*

Other applications are conceivable... ⇝ Ideas?

Application: new decidable fragments...



Generalized
BSR
Fragment

$\mathrm{At}_0(\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots),$
$\mathrm{At}_1(\vec{y}_1, \vec{x}_2, \vec{x}_3, \ldots),$
$\mathrm{At}_2(\vec{y}_1, \vec{y}_2, \vec{x}_3, \ldots),$
$\vdots$

Separated
Fragment

$\exists^*\forall^*\exists^*\ldots\forall^*\exists^*\psi$
$\cancel{P(\ldots x \ldots y \ldots)}$

Bernays-Schönfinkel-
Ramsey Fragment

$\exists^*\forall^*\psi$

Monadic
First-Order
Fragment

$\exists^*\forall^*\exists^*\ldots\forall^*\exists^*\psi$
all predicates unary

Ackermann
Fragment

$\exists^*\forall\exists^*\psi$

Generalized
Ackermann
Fragment

$\mathrm{At}_{x_1}(x_1, \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots),$
$\mathrm{At}_{x_2}(\vec{u}_1, x_2, \vec{v}_2, \vec{v}_3, \ldots),$
$\mathrm{At}_{x_3}(\vec{u}_1, \vec{u}_2, x_2, \vec{v}_3, \ldots),$
$\vdots$
$\mathrm{At}_0(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \ldots),$

Questions?