# On Generalizing Decidable Standard Prefix Classes of First-Order Logic

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## Generalized BSR fragment:

relational sentences

$\forall \vec{x}_1 \exists \vec{y}_1 \ldots \forall \vec{x}_n \exists \vec{y}_n .\, \psi$

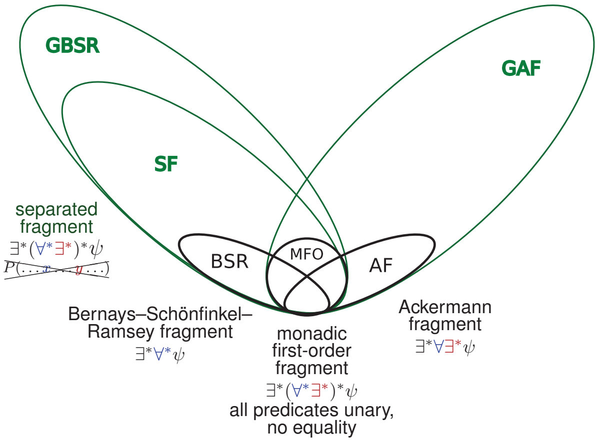
where
the set of atoms in $\psi$
can be partitioned into

$$\begin{array}{r}
\mathrm{At}_0\big(\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_{n-1}, \vec{x}_n\big),\\
\mathrm{At}_1\big(\vec{y}_1, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_{n-1}, \vec{x}_n\big),\\
\mathrm{At}_2\big(\vec{y}_1, \vec{y}_2, \vec{x}_3, \ldots, \vec{x}_{n-1}, \vec{x}_n\big),\\
\mathrm{At}_3\big(\vec{y}_1, \vec{y}_2, \vec{y}_3, \ldots, \vec{x}_{n-1}, \vec{x}_n\big),\\
\vdots\\
\mathrm{At}_{n-1}\big(\vec{y}_1, \vec{y}_2, \vec{y}_3, \ldots, \vec{y}_{n-1}, \vec{x}_n\big),\\
\mathrm{At}_n\big(\vec{y}_1, \vec{y}_2, \vec{y}_3, \ldots, \vec{y}_{n-1}, \vec{y}_n\big),
\end{array}$$

such that

$\mathrm{vars}(\mathrm{At}_i) \cap \mathrm{vars}(\mathrm{At}_j) \cap \vec{x}_\ell = \emptyset$

for all $i \neq j$ and every $\ell$.




## Generalized Ackermann fragment:

relational sentences

$\forall \mathbf{x}_1 \exists \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1 \ldots \forall \mathbf{x}_n \exists \vec{\mathbf{u}}_n \vec{\mathbf{v}}_n .\, \psi$

where
the set of atoms in $\psi$
can be partitioned into

$$\begin{array}{r}
\mathrm{At}_{\mathbf{x}_1}\big(\mathbf{x}_1, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_{n-2}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big),\\
\mathrm{At}_{\mathbf{x}_2}\big(\vec{\mathbf{u}}_1, \mathbf{x}_2, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_{n-2}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big),\\
\mathrm{At}_{\mathbf{x}_3}\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \mathbf{x}_3, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_{n-2}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big),\\
\vdots\\
\mathrm{At}_{\mathbf{x}_{n-1}}\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3, \vec{\mathbf{u}}_4, \ldots, \mathbf{x}_{n-1}, \vec{\mathbf{v}}_{n-1}, \vec{\mathbf{v}}_n\big),\\
\mathrm{At}_{\mathbf{x}_n}\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3, \vec{\mathbf{u}}_4, \ldots, \vec{\mathbf{u}}_{n-1}, \mathbf{x}_n, \vec{\mathbf{v}}_n\big),\\
\mathrm{At}_0\big(\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3, \vec{\mathbf{u}}_4, \ldots, \vec{\mathbf{u}}_{n-1}, \vec{\mathbf{u}}_n\big),
\end{array}$$

such that

$\mathrm{vars}(\mathrm{At}_{\mathbf{x}}) \cap \mathrm{vars}(\mathrm{At}_{\mathbf{x}'}) \cap \vec{\mathbf{v}}_\ell = \emptyset$

for every $\mathbf{x}' \neq \mathbf{x}$ and every $\ell$.

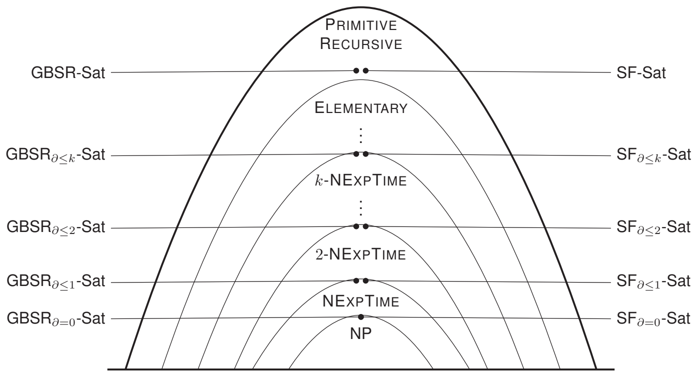
Example from GBSR ∩ GAF: $\exists \mathbf{u} \forall \mathbf{x} \exists \mathbf{y} \forall \mathbf{z}.\, \big(P(\mathbf{u}, \mathbf{z}) \wedge \neg Q(\mathbf{u}, \mathbf{x}, \mathbf{x})\big) \vee \big(P(\mathbf{y}, \mathbf{u}) \wedge Q(\mathbf{u}, \mathbf{y}, \mathbf{z})\big)$

## Computational complexity:

The satisfiability problem for GBSR / SF sentences induces a hierarchy of $k$-NExpTime-complete problems:

The **degree** $\partial$ of a GBSR sentence measures how many $\exists$-blocks interact at the level of atoms:

if variables $\mathbf{y}_1, \ldots, \mathbf{y}_\ell$
from $k$ distinct $\exists$-blocks
are interlinked via atoms
$P_1(\mathbf{y}_1, \mathbf{y}_2), P_2(\mathbf{y}_2, \mathbf{y}_3), \ldots, P_n(\mathbf{y}_{\ell-1}, \mathbf{y}_\ell)$,
then $\partial \geq k$.




## More properties:

(1) GBSR and GAF possess the **finite model property**:

- Satisfiable GAF sentences have models over finite domains.
- Every model of a GBSR sentence contains a finite substructure that is a model, too.

(2) GBSR and BSR are **equivalent**: Every GBSR sentence can be transformed into an equivalent BSR sentence. In the worst case, this transformation incurs a non-elementary blowup.

(3) GBSR and SF without equality are **closed under Craig-Lyndon interpolation**:

$\varphi \models \psi$ — exists $\chi$ — $\varphi \models \chi$ and $\chi \models \psi$

- $\varphi, \psi$ are GBSR sentences without equality.
- $\chi$ is a BSR sentence over $\varphi$'s and $\psi$'s joint voc.
- Every $P$ with a positive (negative) occurrence in $\chi$ occurs positively (negatively) in $\varphi$ and in $\psi$.

## References

Sturm, Voigt, Weidenbach. *Deciding Satisfiability when Universal and Existential Variables are Separated.* LICS 2016.
Voigt. *A Fine-Grained Hierarchy of Hard Problems in the Separated Fragment.* LICS 2017.
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