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The Bernays–Schönfinkel–Ramsey Fragment with Bounded Difference Constraints over the Reals is Decidable

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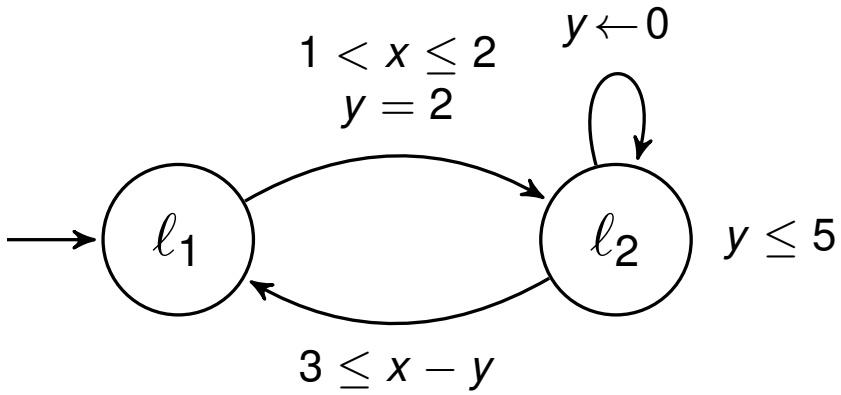
SIC Saarland Informatics Campus

International Max Planck Research School
for Computer Science

Motivation

For some formalization tasks,
we would like to have

- linear arithmetic over $\mathbb{R}$,
- uninterpreted predicates,
- universal quantification.



$\rightsquigarrow$ Convenient and powerful modeling language

$\rightsquigarrow$ Too powerful, indeed: Satisfiability is undecidable.

$\rightsquigarrow$ Remains undecidable, even when adding
only unary uninterpreted predicates.

$\rightsquigarrow$ We have to restrict the arithmetic part!

The Plan

1. Present the
   Bernays–Schönfinkel–Ramsey Fragment
   with Bounded Difference Constraints — BSR(BD)

2. Exemplary application:
   reachability for timed automata

Bernays–Schönfinkel–Ramsey with Bounded Difference Constraints — BSR(BD)

$\exists^*\forall^*$ quantifier prefix
$+$ uninterpreted predicates
$+$ linear rational arithmetic
$+$ syntactic restrictions

$=$ BSR(BD)

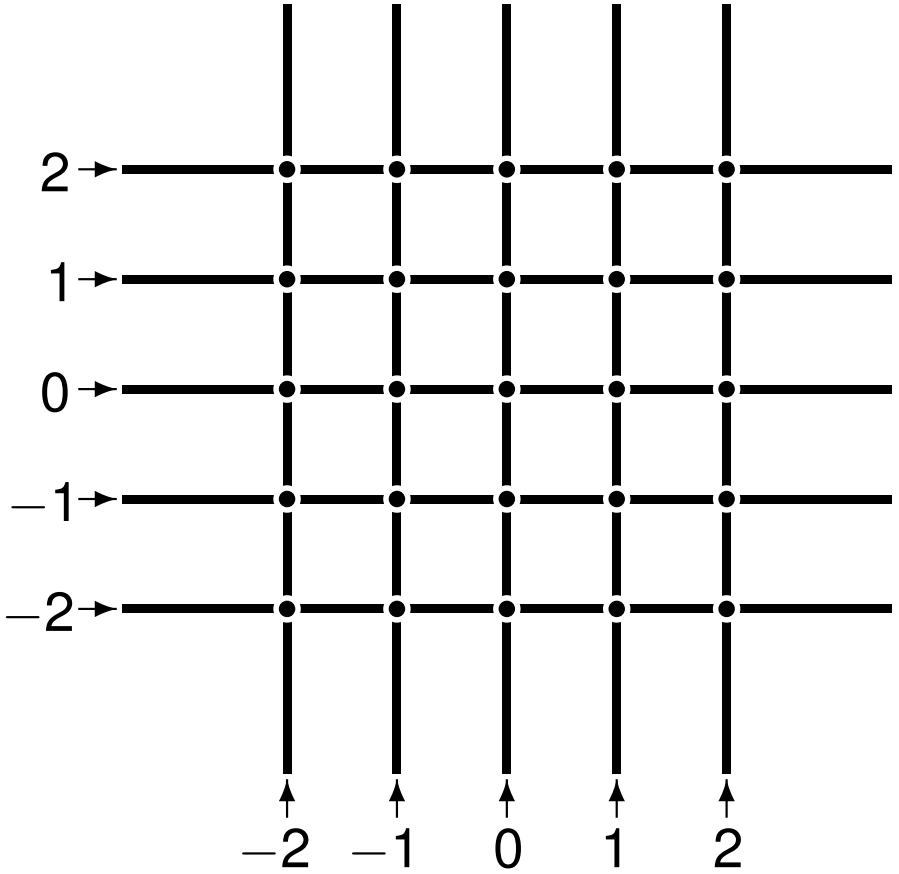
- clauses in $(\ldots) \to (\ldots)$ notation
- arithm. atoms only left of $\to$
- allowed arithm. atoms:
  - $x \triangleleft c$, $x \triangleleft y$ and
  - $x - y \triangleleft c$ conjoined with bounds $\ell \leq x, y \leq u$

where $c, \ell, u \in \mathbb{Q}$ and $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$

$$\forall xyz.\ \underbrace{x - y < \tfrac{3}{2}}_{\text{diff. constraint requires bounds}} \wedge \underbrace{-2 \leq x, y \leq 2}_{\text{diff. constraint requires bounds}} \wedge \underbrace{x < z}_{\text{no bound on } z \text{ required}} \wedge Q(x, y) \longrightarrow T(x) \vee Q(y, x)$$

$\rightsquigarrow$ Bounds are necessary for decidability.
$\rightsquigarrow$ For convenience, we only consider integer bounds here.

BSR(BD): Distinguishable Regions of $\mathbb{R}^2$



Constraints $x \triangleleft c$
with $c \in \{-2, -1, 0, 1, 2\}$
can distinguish
the grid regions.

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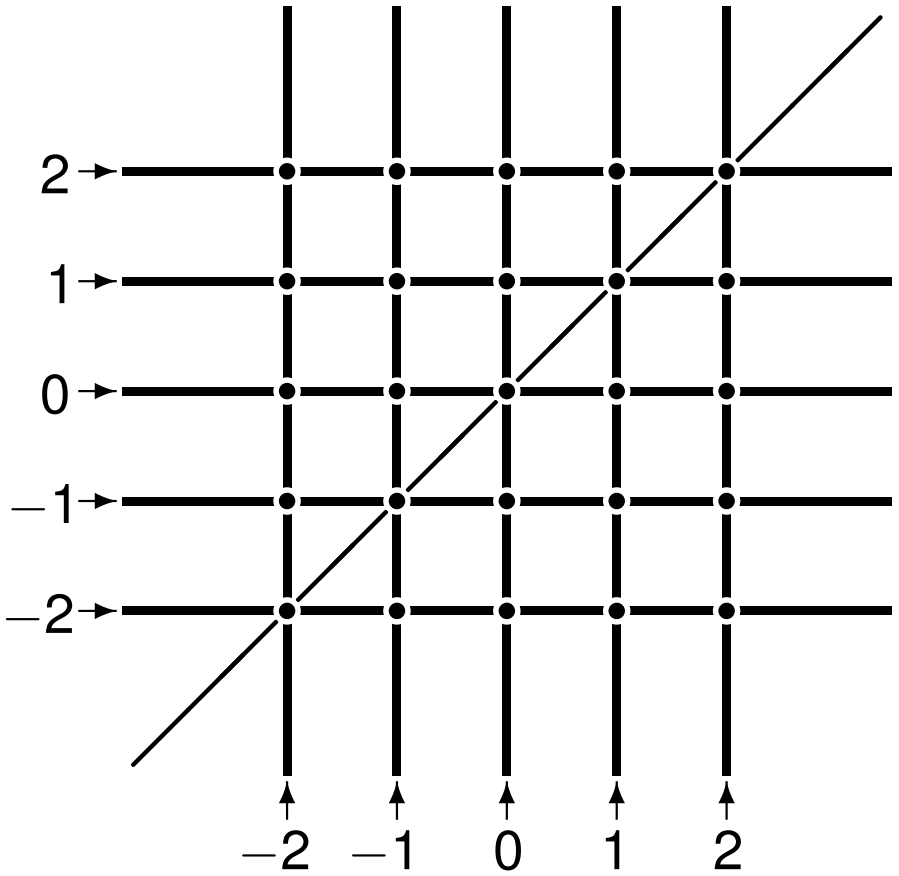

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Constraints $x \triangleleft y$ can distinguish the simple diagonal.

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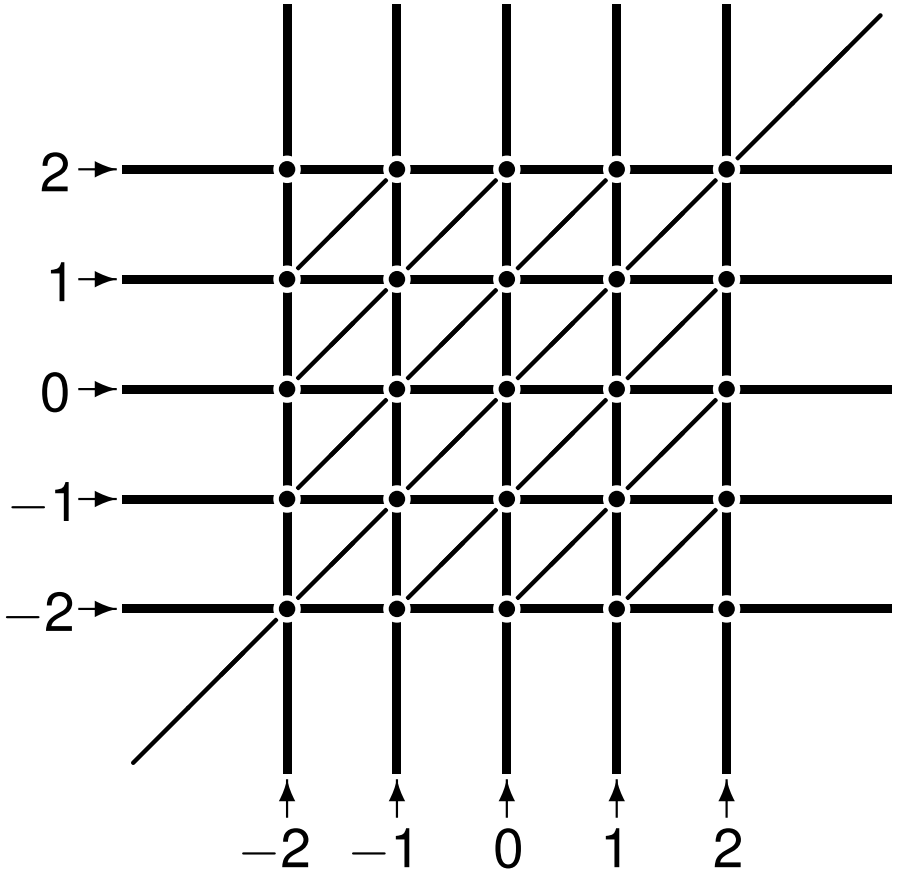

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Difference constraints
$x - y \triangleleft c$
with bounds
$-2 \leq x, y \leq 2$
can distinguish
more triangles.

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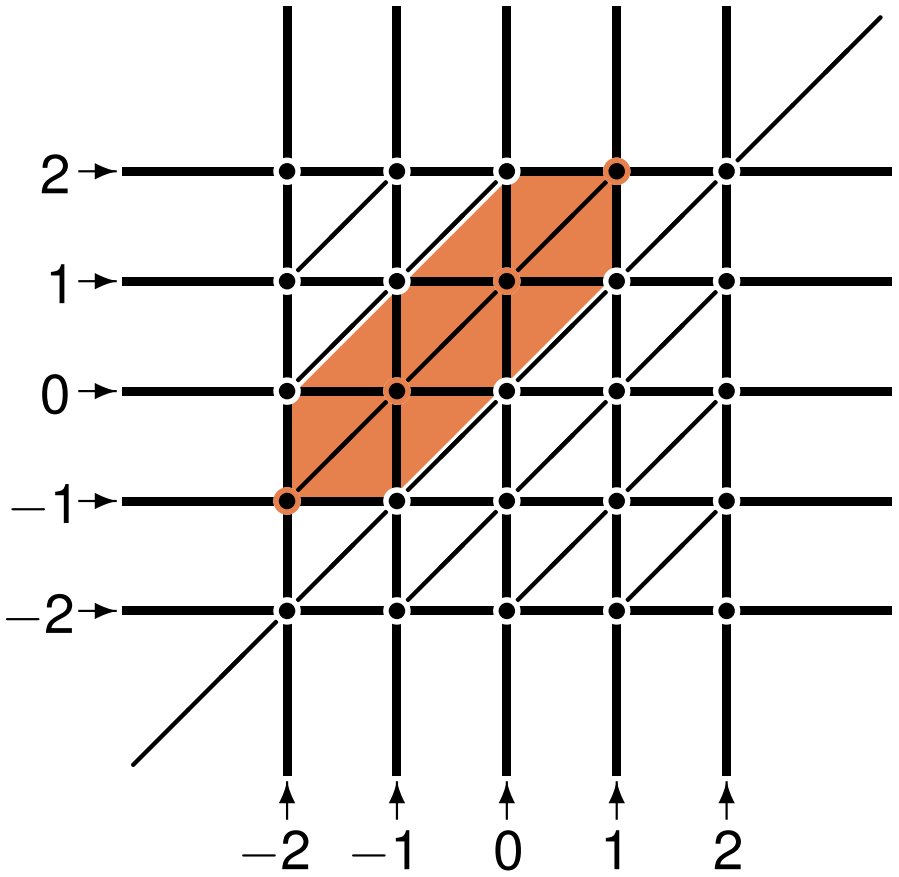


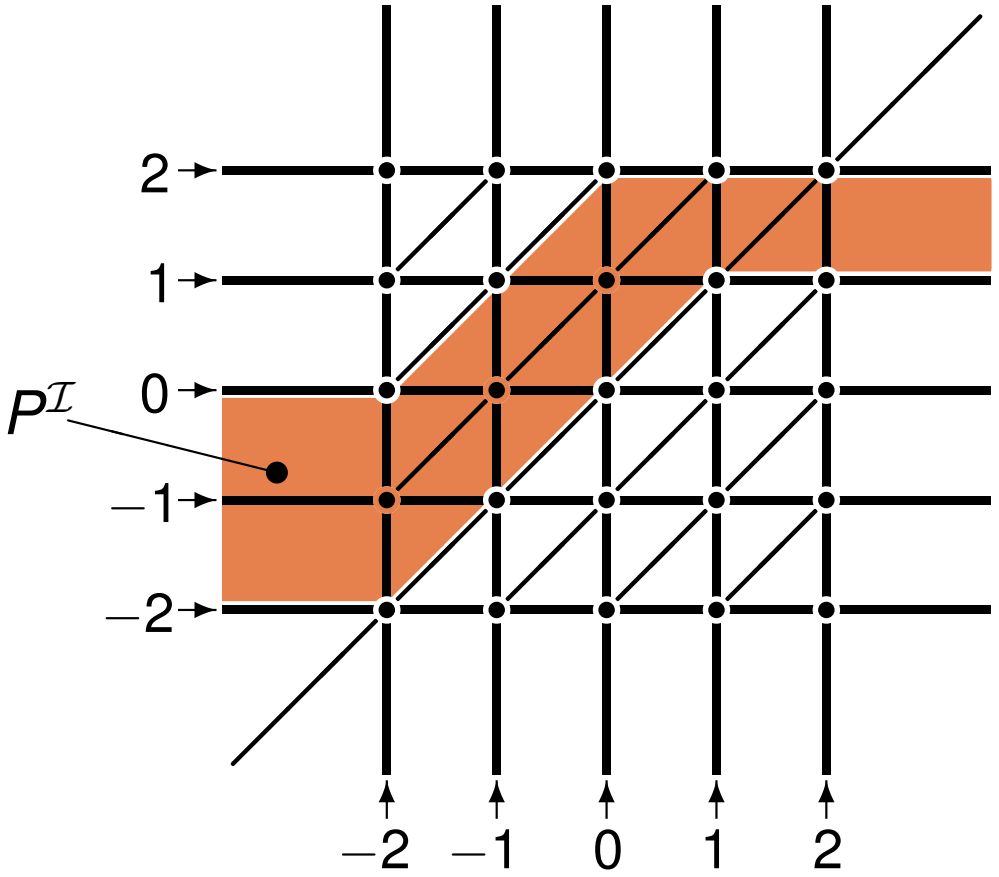
$$-2 < x - y < 0 \quad \wedge \quad -2 \leq x, y \leq 2 \quad \wedge \quad x \leq 1 \quad \wedge \quad -1 \leq y$$

Constraints $x \triangleleft c$ with $c \in \{-2, -1, 0, 1, 2\}$ can distinguish the grid regions.

Constraints $x \triangleleft y$ can distinguish the simple diagonal.

Difference constraints $x - y \triangleleft c$ with bounds $-2 \leq x, y \leq 2$ can distinguish more triangles.

BSR(BD): Interpretation of Predicates $P^{\mathcal{I}} \subseteq \mathbb{R}^2$



$$
\begin{array}{rcrcl}
-2 < x - y < 0 & \wedge & -2 \leq x, y \leq 2 & \rightarrow & P(x, y) \\
x \leq -2 & \wedge & -2 < y < 0 & \rightarrow & P(x, y) \\
1 \leq x & \wedge & 1 < y < 2 & \rightarrow & P(x, y)
\end{array}
$$

Key insight:

Predicates needn't distinguish what arithmetic constraints can't distinguish.

$\rightsquigarrow$ Only a finite number of interpretations need to be checked to decide satisfiability.

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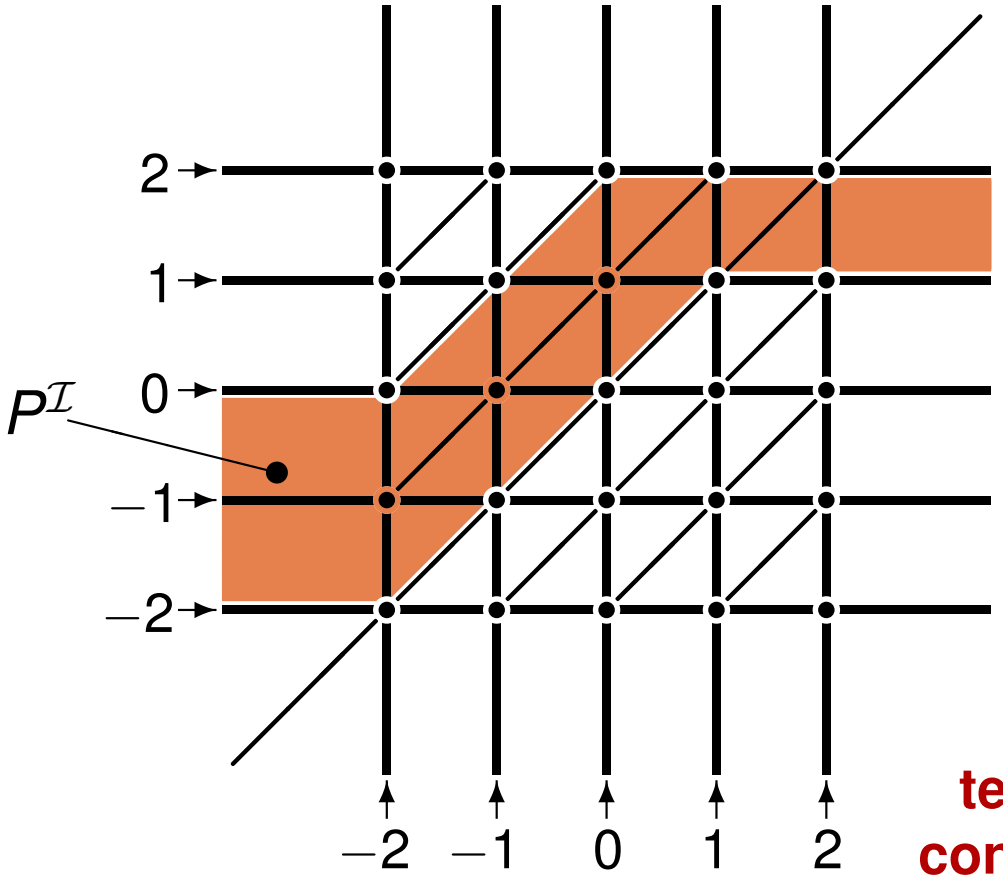

Key insight:

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main technical contribution

$\rightsquigarrow$ Only a finite number of interpretations need to be checked to decide satisfiability.

$$
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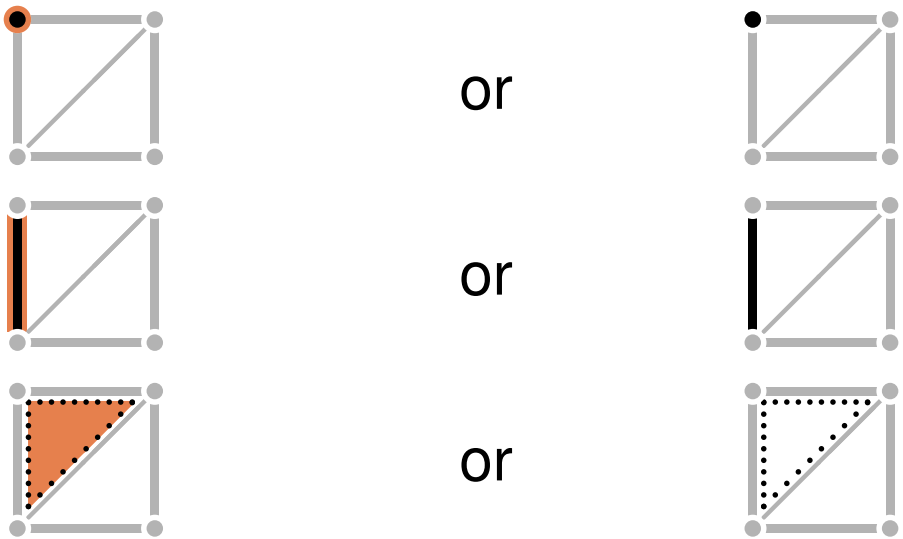
\end{array}
$$

BSR(BD): Region-Uniform Interpretations of $P^{\mathcal{I}}$

Region-uniform interpretation $P^{\mathcal{I}} \subseteq \mathbb{R}^2$:

Every region is either

fully contained in $P^{\mathcal{I}}$ or disjoint from $P^{\mathcal{I}}$.



Key lemma: For satisfiable BSR(BD) clause sets region-uniform models always exist.

BSR(BD): Region-Uniform Interpretations of $P^{\mathcal{I}}$

Given:
- a satisfiable BSR(BD) clause set N in which the arity of predicates is $\leq m$,
- some model $\mathcal{A}$ of N.

Frank P. Ramsey

There is a sufficiently large but finite subset S of $\mathbb{R}$ that covers all regions in $\mathbb{R}^m$ and over which $\mathcal{A}$ is region-uniform.

$\rightsquigarrow$ We extend $\mathcal{A}|_S$ to a model $\mathcal{B}$ of N that is region-uniform over $\mathbb{R}^m$.

BSR(BD): Region-uniformity for a finite subset of $\mathbb{R}^2$

Every model $\mathcal{A}$ is region-uniform on some sufficiently large but finite subset $S \subseteq \mathbb{R}$.

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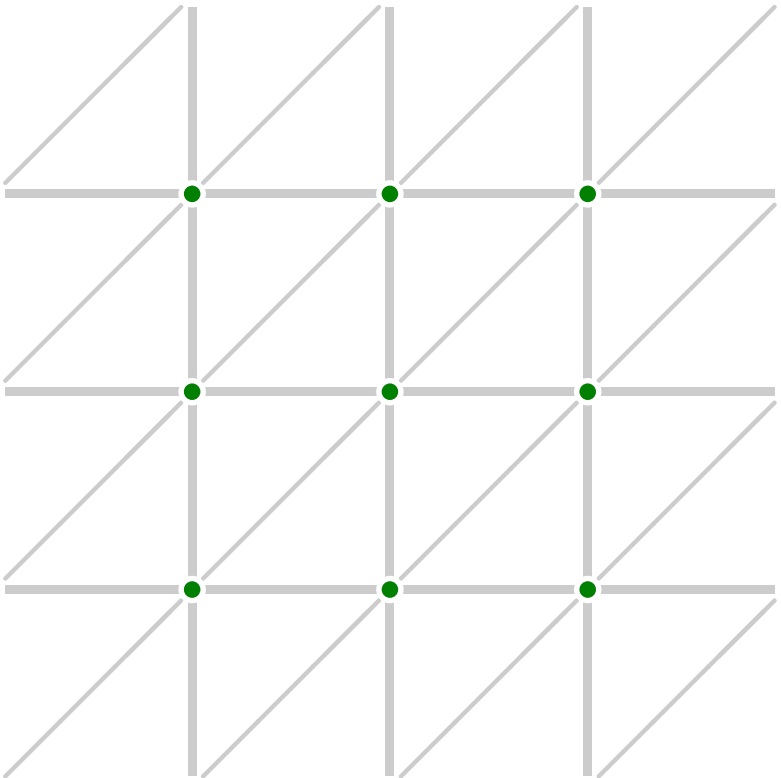

$\rightsquigarrow$ The finite subset $S^2 \subseteq \mathbb{R}^2$ must cover all regions.

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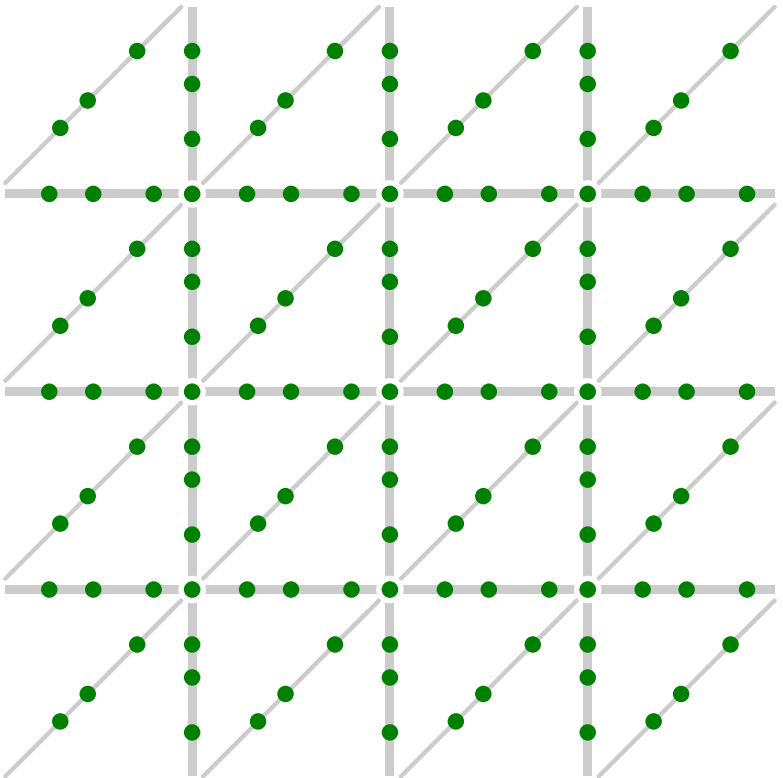

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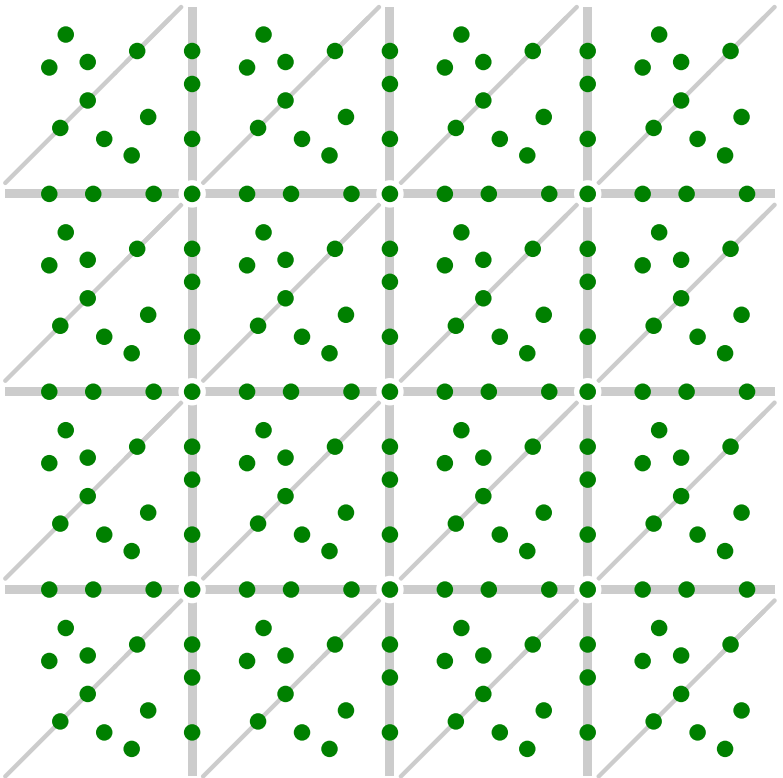

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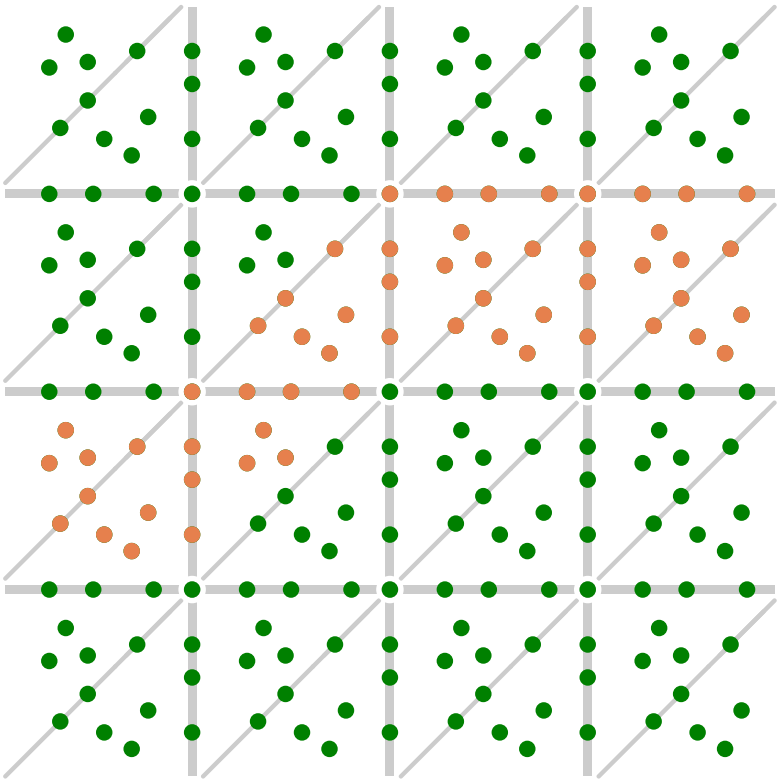

- $\rightsquigarrow$ The finite subset $S^2 \subseteq \mathbb{R}^2$ must cover all regions.
- $\rightsquigarrow$ The regions over S^2 are interpreted uniformly by $\mathcal{A}$.

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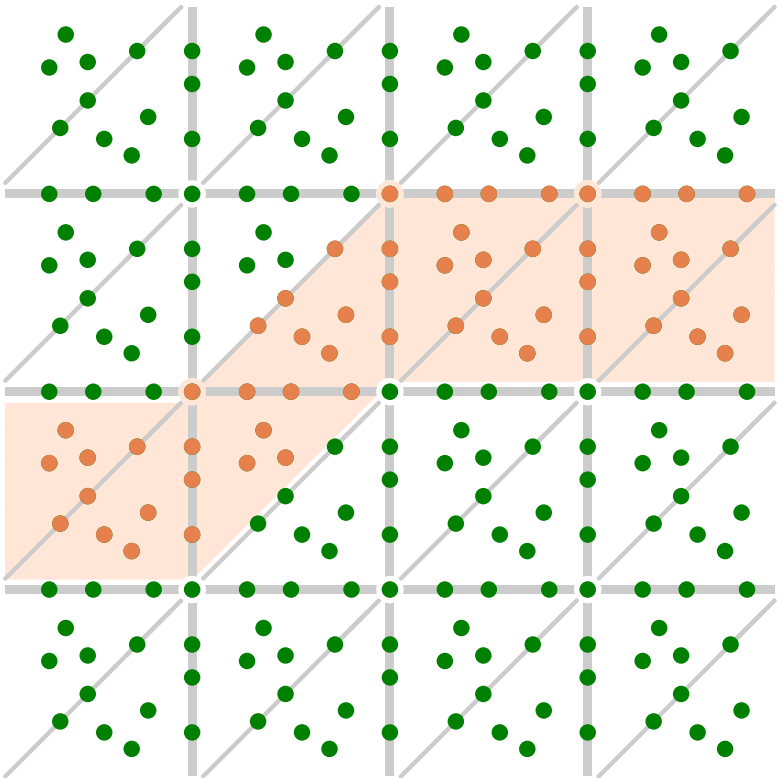

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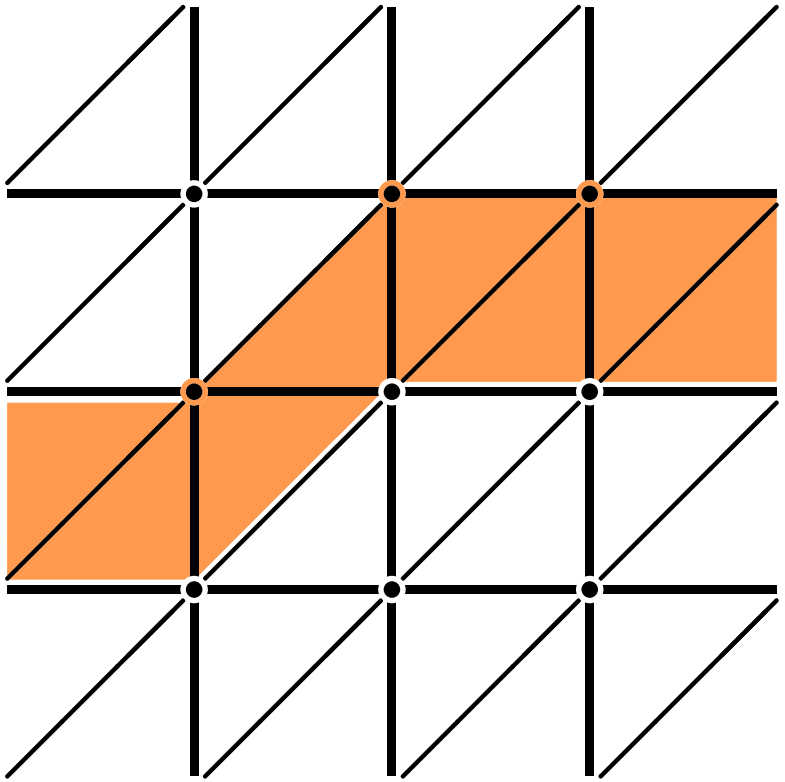

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BSR(BD): Satisfiability is Decidable

Theorem

Satisfiability for finite BSR(BD) clause sets N is decidable.

Proof:

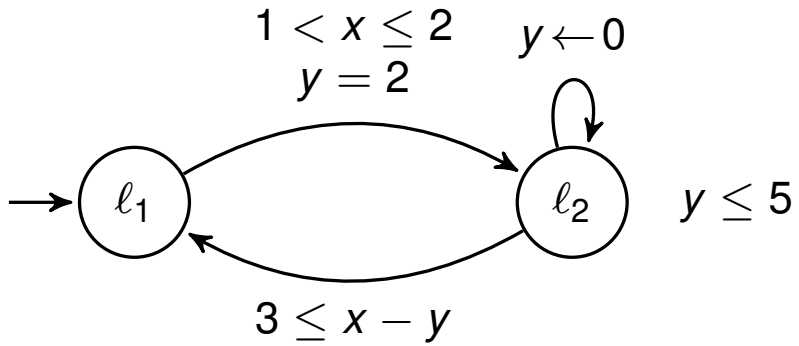
- If any model exists, then a region-uniform model exists.
- There are only finitely many region-uniform interpretations.
- We can guess one and check whether it is a model of N.

Reminder: Timed Automata [Alur&Dill 1994], [Henzinger et al. 1994]

Finite state machines equipped with *real*-valued clocks x, y, ...

Constraints: $x \triangleleft d$, $x - y \triangleleft d$, $\triangleleft \in \{<, \leq, =, \geq, >\}$, $d \in \mathbb{N}$

Operations: $x \leftarrow 0$

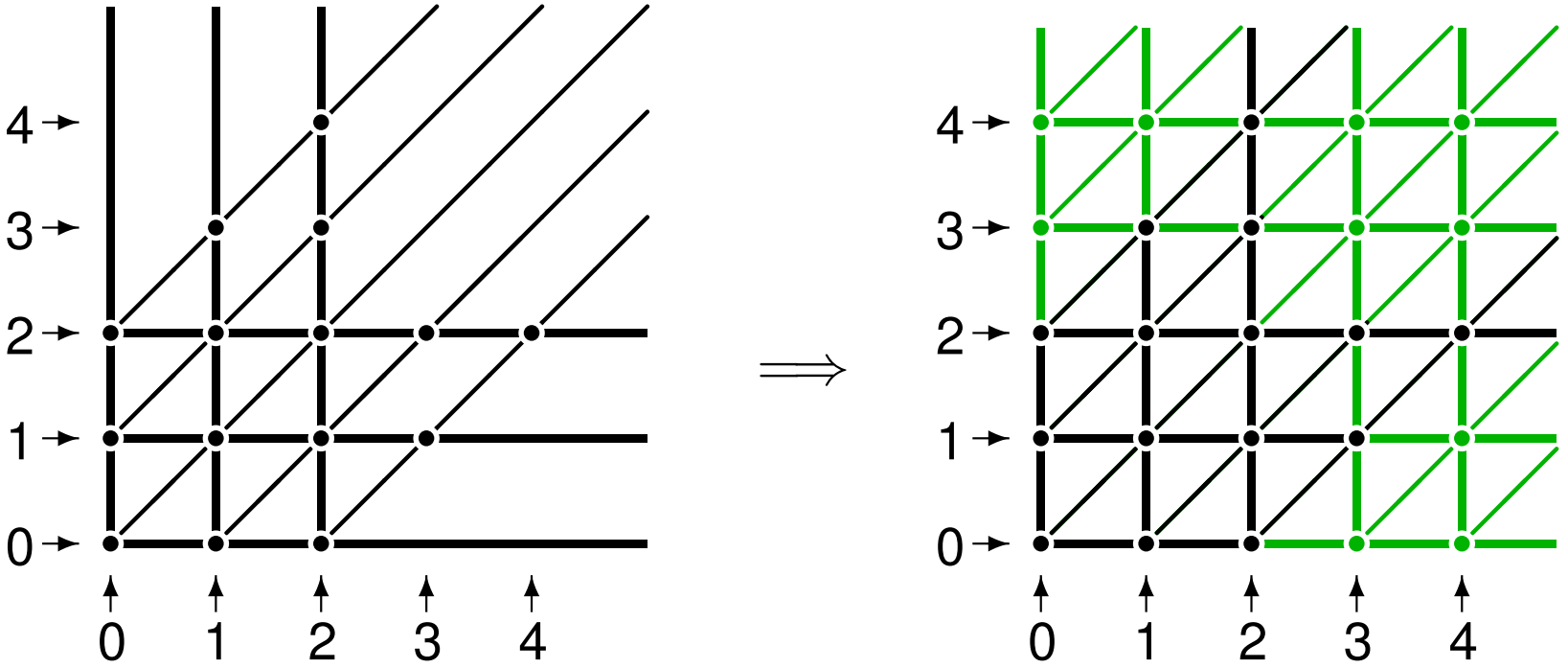


Semantics:
- states $\left\langle \ell, \begin{array}{l} x \mapsto r_1 \\ y \mapsto r_2 \end{array} \right\rangle$ with $r_1, r_2 \in \mathbb{R}$,
- transitions between locations (instantaneous), and
- progress of time (for all clocks simultaneously).

$\rightsquigarrow$ Reachability is PSPACE-complete.

TA Constraints: Distinguishable Regions of $\mathbb{R}^2$

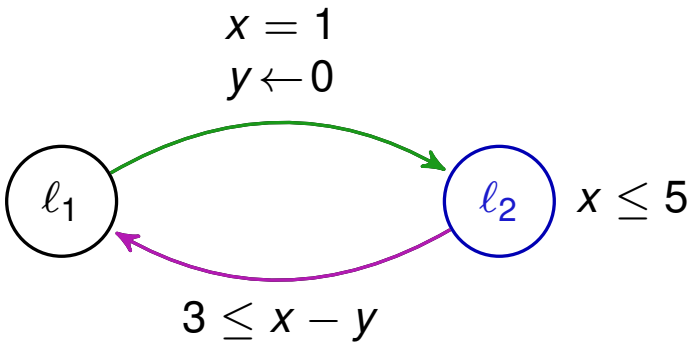
For two clocks x, y, TA constraints with constants $d \leq 2$ can distinguish regions as follows:



$\rightsquigarrow$ The BSR(BD) regions for constraints with constants from $\{-4, \ldots, 0, \ldots, 4\}$ are a refinement of the TA regions.

Encoding Reachability for a Timed Automaton

[Fietzke, Weidenbach 2012]



$$x = 1 \wedge y' = 0 \wedge x \leq 5 \wedge \mathrm{Reach}(\ell_1, x, y) \rightarrow \mathrm{Reach}(\ell_2, x, y')$$

$$(\exists t.\, t \geq 0 \wedge x' = x + t \wedge y' = y + t) \wedge x' \leq 5 \wedge \mathrm{Reach}(\ell_2, x, y) \rightarrow \mathrm{Reach}(\ell_2, x', y')$$

$$3 \leq x - y \wedge \mathrm{Reach}(\ell_2, x, y) \rightarrow \mathrm{Reach}(\ell_1, x, y)$$

Start clause: $x = 0 \wedge y = 0 \wedge \rightarrow \mathrm{Reach}(\ell_1, x, y)$

Query clause: $y = 4 \wedge \mathrm{Reach}(\ell_2, x, y) \rightarrow \square$

$\rightsquigarrow$ Saturation leads to $\square$ if the answer to the query is *YES*.

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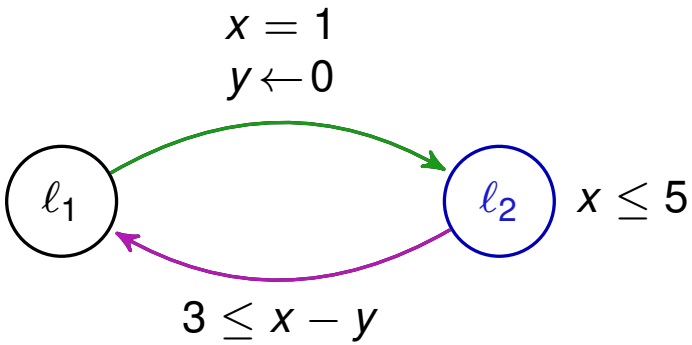

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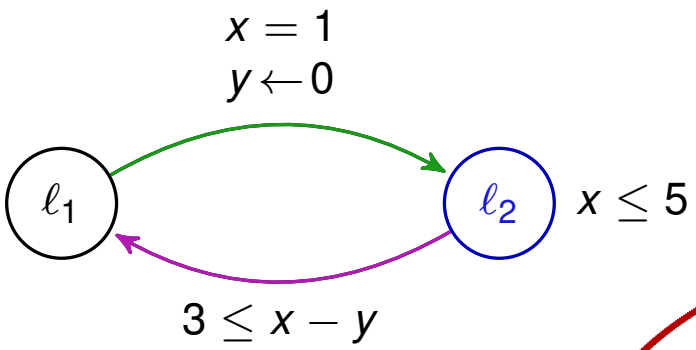

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$\exists t.\, t \geq 0 \wedge x' = x + t \wedge y' = y + t$

is equivalent to

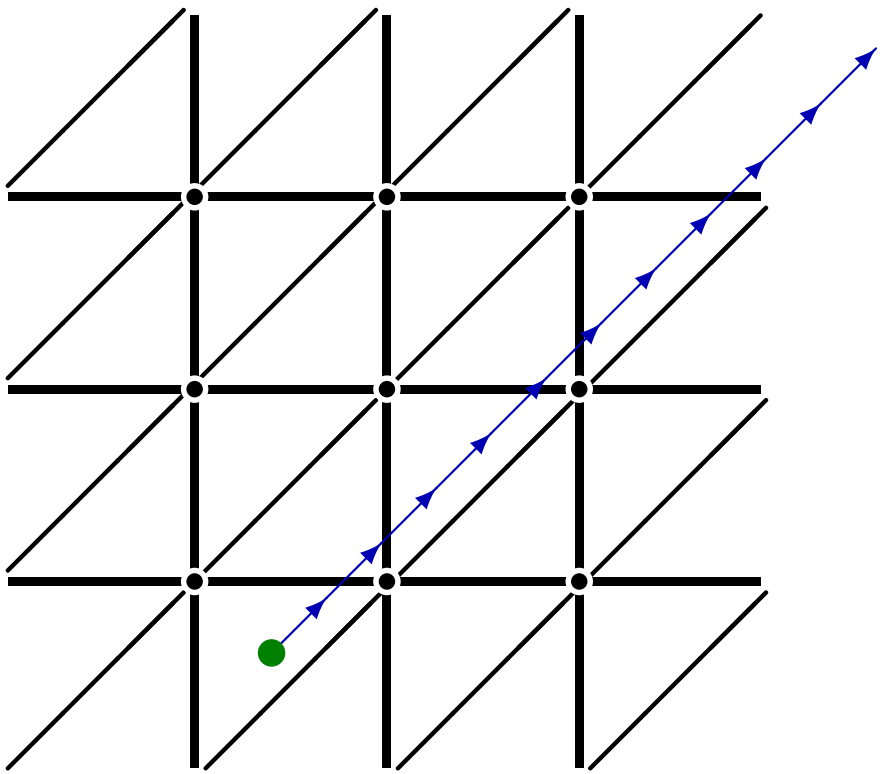
$x' \geq x \wedge y' \geq y \wedge x' - x = y' - y$

$\rightsquigarrow$ Syntax restrictions of BSR(BD) are not met.

De-Synchronizing Progression of Time

The constraint $x' \geq x \ \wedge \ y' \geq y \ \wedge \ x' - x = y' - y$

enforces *synchronous* progression of time.



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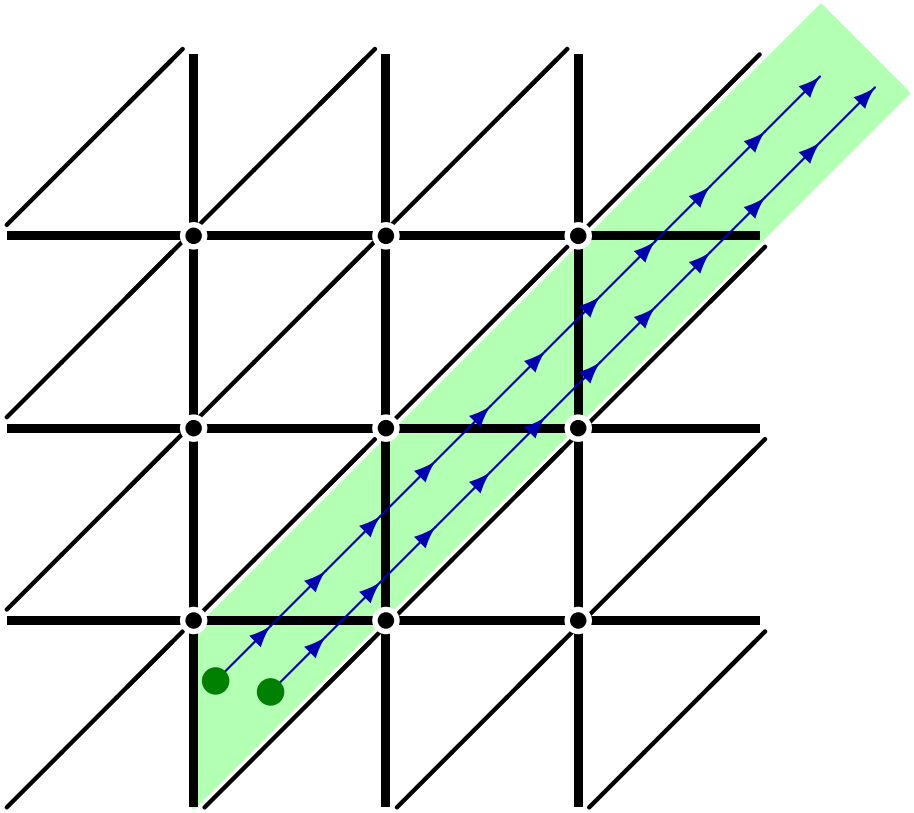

- $\rightsquigarrow$ Synchronous time progression from a reachable point yields a one-dimensional reachable area.
- $\rightsquigarrow$ Since other points in the same region are reachable, we obtain a whole reachable corridor.

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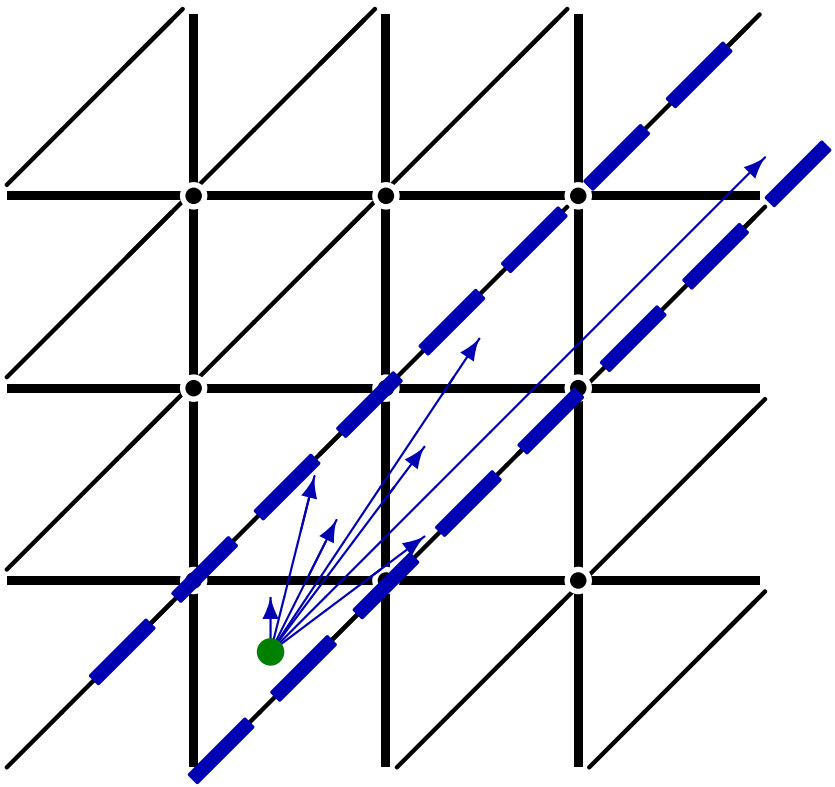

- ⇝ Synchronous time progression from a reachable point yields a one-dimensional reachable area.
- ⇝ Since other points in the same region are reachable, we obtain a whole reachable corridor.
- ⇝ We can weaken the the synchronicity requirement

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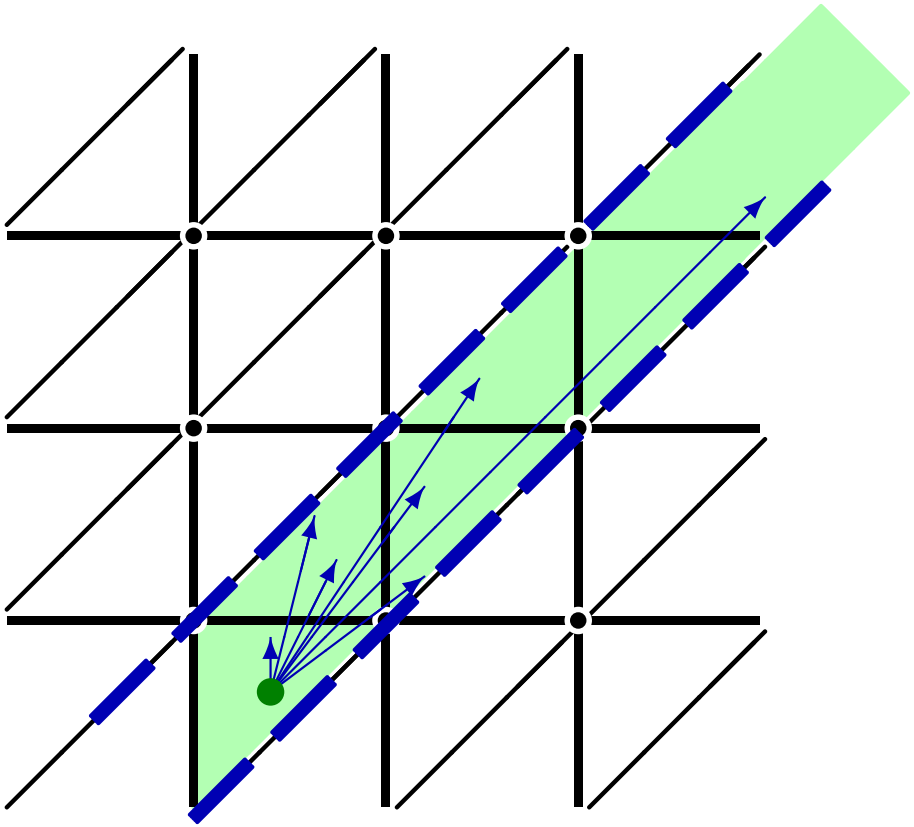
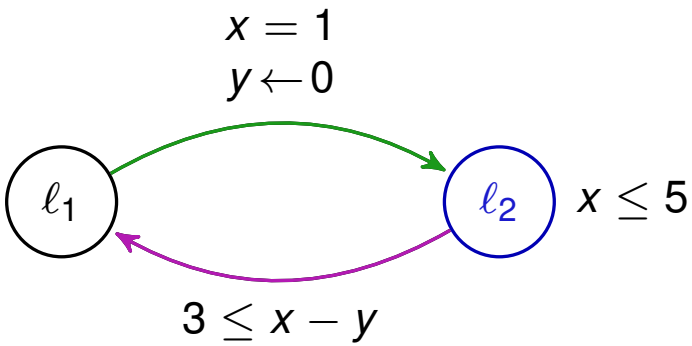


- $\rightsquigarrow$ Synchronous time progression from a reachable point yields a one-dimensional reachable area.
- $\rightsquigarrow$ Since other points in the same region are reachable, we obtain a whole reachable corridor.
- $\rightsquigarrow$ We can weaken the the synchronicity requirement to

$$\bigwedge_{k \in \{-\lambda,\ldots,\lambda\}} \big(x - y \leq k \leftrightarrow x' - y' \leq k\big) \wedge \big(x - y \geq k \leftrightarrow x' - y' \geq k\big).$$

Encoding Reachability for a Timed Automaton



$$x = 1 \wedge y' = 0 \wedge x \leq 5 \wedge 0 \leq x, y, y' < 11 \wedge \mathrm{Reach}(\ell_1, x, y) \rightarrow \mathrm{Reach}(\ell_2, x, y')$$

$$\bigwedge_{k \in \{-10,\ldots,10\}} \Big((x - y \leq k \leftrightarrow x' - y' \leq k) \wedge (x - y \geq k \leftrightarrow x' - y' \geq k) \Big) \wedge 0 \leq x, y, x', y' \leq 11 \wedge x' \leq 5 \wedge \mathrm{Reach}(\ell_2, x, y) \rightarrow \mathrm{Reach}(\ell_2, x', y')$$

$$3 \leq x - y \wedge 0 \leq x, y, y' < 11 \wedge \mathrm{Reach}(\ell_2, x, y) \rightarrow \mathrm{Reach}(\ell_1, x, y)$$

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Query clause: $y = 4 \wedge \mathrm{Reach}(\ell_2, x, y) \rightarrow \square$

⇝ Syntax restrictions of BSR(BD) are met.

⇝ Reachability for TA can be expressed with BSR(BD).

Conclusion

We have seen BSR with *bounded* difference constraints — BSR(BD).

- BSR(BD)-Sat is decidable
- BSR(BD) can express reachability for timed automata

We have not seen BSR with *simple* linear rational arithmetic.

$$\exists cd\, \forall xy.\ c \neq d \ \wedge\ x > c + 2d - 3 \ \wedge\ x < y \ \wedge Q(x, y) \longrightarrow T(x) \ \vee\ Q(y, x)$$

- Satisfiability is decidable, too.

Future work

- Instantiation methods for BSR(BD) and for BSR(SLR) as for BSR with *simple* LIA [Horbach, V., Weidenbach. CADE'17]

- More applications of BSR(BD) and BSR(SLR)

Thank You!