Towards Elimination of Second-Order Quantifiers in the Separated Fragment

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Second-order quantifier elimination

Given some second-order formula $\varphi$, e.g.

$$\exists P. \forall x \exists y. P(x) \leftrightarrow \neg P(y),$$

we are interested in a formula $\psi$ such that

- $\varphi$ and $\psi$ are semantically equivalent,
- $\psi$ does not contain second-order quantifiers,
- $\psi$’s vocabulary is a subset of $\varphi$’s vocabulary, and
- all free first-order variables in $\psi$ also occur freely in $\varphi$.

Solution for example:

$$\exists y_1 y_2. y_1 \neq y_2.$$
Direct approaches to SO quantifier elimination

Two ingredients [Gabbay, Schmidt, Szałas 2008]:

(a) An elimination lemma for formulas of a certain syntactic shape.

Let $\alpha, \beta$ be FO formulas in which $P$ does not occur.

$$\exists P. \left( \forall x. \alpha \lor P(x) \right) \land \left( \forall x. \beta \lor \neg P(x) \right)$$

is equivalent to

$$\forall x. \alpha \lor \beta.$$  

(b) Effective techniques that bring SO formulas into a suitable shape.
Relational monadic second-order formulas

A formula is *relational* if it does not contain any function symbols. A formula is *monadic* if all predicate symbols in it are at most unary.

**MFO**: relational monadic first-order fragment

\[ \forall x \exists y. P(x) \leftrightarrow \neg P(y) \]

**MSO**: relational monadic second-order fragment

\[ \varphi := \forall x_1 x_2. (Q(x_1) \land Q(x_2)) \rightarrow (\forall P. P(x_1) \rightarrow P(x_2)) \]

\[ \leadsto \textrm{MSO admits elimination of second-order quantifiers via a direct approach} \quad [\text{Löwenheim 1915, Skolem 1919, Behmann 1922}] \]

\[ \leadsto \varphi \text{ is equivalent to } \forall x_1 x_2. (Q(x_1) \land Q(x_2)) \rightarrow x_1 = x_2 \]
A new fragment of first-order logic

Decidability of MFO-Sat carries over to SF-Sat.

What about SO quantifier elimination?

Be careful: restricted form of equations

\[ \exists^* \forall^* \psi \]

all predicates unary

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The separated fragment — SF [Sturm, V., Weidenbach, LICS’16]

SF contains relational first-order sentences in prenex form

$$\exists \vec{z} \ \forall \vec{x}_1 \exists \vec{y}_1 \ldots \forall \vec{x}_n \exists \vec{y}_n. \ \psi$$

such that
	no atom may contain blue and red variables.

$$\rightsquigarrow$$ SF-Sat is decidable [Sturm, V., Weidenbach, LICS’16]

$$\rightsquigarrow$$ SF-Sat is non-elementary and contains $k$-\text{NEXP TIME}-complete subproblems $\text{SF}_{\partial \leq k}$-Sat [V., LICS’17]

$$\rightsquigarrow$$ Finite model property ...

... via syntactic transformations [Sturm, V., Weidenbach, LICS’16]

... via game semantics [V., arXiv.org 2017]

$$\rightsquigarrow$$ Closed under Craig–Lyndon interpolation [V., arXiv.org 2017]
The separated fragment — SF  [Sturm, V., Weidenbach, LICS’16]

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\implies Finite model property ...

... via syntactic transformations

... via game semantics

\implies Closed under Craig–Lyndon interpolation  [V., arXiv.org 2017]

Re-usable for elimination of SO quantifiers
SF subsumes MFO

What about SO quantifier elimination for SF?

Separated Fragment
∃*∀*∃*. . . ∀*∃*ψ

Bernays-Schönfinkel-Ramsey Fragment
∃*∀*ψ

Monadic First-Order Fragment
∃*∀*∀*∃*ψ

all predicates unary

restricted form of equations
x = y

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SF does not admit SO QE in full generality

There is an SF formula $\varphi$ such that we cannot eliminate $\exists P$ in $\exists P \cdot \varphi$. [1934]

Wilhelm Ackermann

$\varphi = \exists P. P(x) \land \neg P(y) \land (\forall uv. \neg P(u) \lor P(v) \lor \neg S(u, v))$.

$\neg \varphi$ is equivalent to

$\forall P. \left( P(x) \land (\forall uv. (P(u) \land S(u, v)) \rightarrow P(v)) \right) \rightarrow P(y)$.
But, SF admits SO QE in special cases

Given a formula $\varphi$, two sets $X, Y$ of variables are *separated in $\varphi$* if no atom in $\varphi$ contains variables from $X$ and from $Y$.

**Elimination theorem:**

Given an SF sentence $\varphi := \forall \vec{x}_1 \exists \vec{y}_1 \ldots \forall \vec{x}_n \exists \vec{y}_n. \psi$, let $\vec{x} := \vec{x}_1 \cup \ldots \cup \vec{x}_n$ and $\vec{y} := \vec{y}_1 \cup \ldots \cup \vec{y}_n$.

Suppose $\varphi$ contains the unary predicate $P$ and that we can partition

$\vec{x}$ into $\vec{x}_1, \ldots, \vec{x}_{m_1}$ and $\vec{y}$ into $\vec{y}_1, \ldots, \vec{y}_{m_2}$

such that $\vec{x}_i, \vec{x}_j$ and $\vec{y}_i, \vec{y}_j$ are pairwise separated in $\varphi$ if $i \neq j$.

Suppose every $\vec{x}_i$ contains at most one variable $x_i^*$ occurring as argument of $P$ in $\varphi$. Analogously, every $\vec{y}_i$ contains at most one $y_i^*$.

Then, we can eliminate $\exists P$ from $\exists P. \varphi$. 
What is separateness good for?

If two sets $X, Y$ are separated, we can shift any FO quantifier $Qz$ inwards in such a way that

- either none of the $x \in X$ is in the scope of $Qz$
- or none of the $y \in Y$ is in the scope of $Qz$.

Example:

$$\forall x \exists y. P(x) \leftrightarrow Q(y)$$

is equivalent to

$$\left( (\forall x_1. \neg P(x_1)) \lor (\exists y_1. Q(y_1)) \right) \land \left( (\exists y_2. \neg Q(y_2)) \lor (\forall x_2. P(x_2)) \right).$$
Example: shifting quantifiers

Original: \( \forall x \exists y . P(x) \leftrightarrow Q(y) \).

Transform into DNF:
Example: shifting quantifiers

Original: \( \forall x \exists y. P(x) \leftrightarrow Q(y). \)

Shift \( \exists y \) inward:

\[
\forall x \exists y \\
\bigvee \\
\bigwedge \\
\neg P(x) \\
\neg Q(y) \\
Q(y) \\
P(x)
\]
**Example: shifting quantifiers**

Original: \( \forall x \exists y. P(x) \Leftrightarrow Q(y) \).

Shift \( \exists y \) inward:
Example: shifting quantifiers

Original: \( \forall x \exists y. P(x) \leftrightarrow Q(y) \).

Transform into “CNF”:

\[
\forall x
\]
\[
\bigvee
\]
\[
\bigwedge
\]
\[
\neg P(x) \quad \exists y_2 \neg Q(y_2) \quad \exists y_1 Q(y_1) \quad P(x)
\]
Example: shifting quantifiers

Original: \( \forall x \exists y. P(x) \iff Q(y) \).

Transform into “CNF”:

\[
\forall x \\
\land \\
\neg P(x) \quad \exists y_2 \neg Q(y_2) \quad \exists y_1 Q(y_1) \quad P(x)
\]
Example: shifting quantifiers

Original: \( \forall x \exists y . P(x) \leftrightarrow Q(y) \).

Shift \( \forall x \) inward:

\[
\forall x \\
\land \\
\lor \\
\neg P(x) \\
\lor \\
\exists y_1 Q(y_1) \\
\land \\
\exists y_2 \neg Q(y_2) \\
\land \\
P(x)
\]
Example: shifting quantifiers

Original: \( \forall x \exists y. P(x) \leftrightarrow Q(y). \)

Shift \( \forall x \) inward:
Reminder: SO QE for SF

Elimination theorem (reminder):

Given:

- SF formula $\varphi := \forall \tilde{x}_1 \exists \tilde{y}_1 \ldots \forall \tilde{x}_n \exists \tilde{y}_n. \psi$,
- unary predicate $P$,
- partition of $\tilde{x}$ into pairwise separated $\tilde{x}_1, \ldots, \tilde{x}_{m_1}$
  and of $\tilde{y}$ into pairwise separated $\tilde{y}_1, \ldots, \tilde{y}_{m_2}$,
- in every $\tilde{x}_i$ there is at most one $x_i^*$ s.t. $P(x_i^*)$ occurs in $\varphi$,
- in every $\tilde{y}_i$ there is at most one $y_i^*$ s.t. $P(y_i^*)$ occurs in $\varphi$.

Then, $\exists P$ can be eliminated from $\exists P. \varphi$. 
Effect of shifting quantifiers in $\exists P. \varphi$

$\varphi$ in CNF:

\[
\forall \vec{x}_1 \exists \vec{y}_1 \ldots \forall \vec{x}_n \exists \vec{y}_n
\]

\[
\chi_1(\vec{x}) \quad \eta_1(\vec{y})
\]

\[
\chi_k(\vec{x}) \quad \eta_k(\vec{y})
\]

\[
\eta'_1(\vec{y}_1) \lor \ldots \lor \eta'_m(\vec{y}_m)
\]

\[
\vec{x} := \bigcup_i \vec{x}_i
\]

\[
\vec{y} := \bigcup_i \vec{y}_i
\]

\[
\vec{x}_1 \quad \vec{y}_1
\]

\[
\vec{x}_m \quad \vec{y}_m
\]

\[
\vec{x}_1
\]

\[
\vec{x}_m
\]

\[
\vec{y}_1
\]

\[
\vec{y}_m
\]

\[
\exists (\vec{y}_1 \cap \vec{y}_m)
\]

\[
\exists (\vec{y}_2 \cap \vec{y}_m)
\]

\[
\exists (\vec{y}_n \cap \vec{y}_m)
\]

\[
\bigwedge \eta_i(\vec{y}_m)
\]

$\varphi'$:
Effect of shifting quantifiers in $\exists P. \varphi$

$\varphi' :$

\[
\forall \vec{x}_1 \exists \vec{y}_1 \ldots \forall \vec{x}_n \exists \vec{y}_n
\]

\[
\forall (\vec{y}_1 \cap \tilde{y}_{m_2}) \lor \ldots \lor \forall (\vec{y}_{n} \cap \tilde{y}_{m_2})
\]

\[
\eta_{i_1}(\tilde{y}_{m_2}) \ldots \eta_{i_\ell}(\tilde{y}_{m_2})
\]

$\varphi'$:

\[
\forall \vec{x}_1 \ldots \forall \vec{x}_1 \exists \vec{y}_1 \ldots \exists \vec{y}_1
\]

\[
\forall \vec{x}_1 \exists \vec{y}_1 \ldots \exists \vec{y}_1
\]

\[
\tilde{x}_1 \ldots \tilde{x}_{m_1} \tilde{y}_1 \ldots \tilde{y}_{m_2}
\]

$\vec{x} := \bigcup_i \vec{x}_i$

$\vec{y} := \bigcup_i \vec{y}_i$

\[
\eta'_1(\tilde{y}_1) \lor \ldots \lor \eta'_{m_2}(\tilde{y}_{m_2})
\]

\[
\chi_k(\vec{x}) \lor \eta_k(\vec{y})
\]

shift quantifiers inward

shift $\uparrow$

back again

shift $\uparrow$

back again

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After shifting quantifiers a little more

\begin{align*}
\exists P. \varphi \quad \text{is equivalent to} \\
\exists \exists P \\
\bigwedge \quad \ldots \\
\bigvee \\
\bigvee \\
\bigvee \\
\bigvee \\
\bigvee \\
\bigvee \\
\bigvee \\
\bigvee
\end{align*}

Each of the \( \chi_{i,j}(\tilde{u}) \) and \( \eta_{i,j}(\tilde{v}) \) is quantifier free and contains at most one literal \([\neg]P(z)\).

\( \rightsquigarrow \) Conceive these as basic formulas, build DNF.
... several syntactic transformations later ...

Eventually,

\[
\exists P \land \lor \ldots \lor \bigoplus \bigoplus \bigoplus \bigoplus \forall \tilde{u} \ldots \forall \tilde{u} \exists \tilde{v} \ldots \exists \tilde{v} \forall \tilde{u} \ldots \forall \tilde{u} \exists \tilde{v} \ldots \exists \tilde{v} \\
\chi_{1,1}(\tilde{u}) \ldots \chi_{1,k_1}(\tilde{u}) \eta_{1,1}(\tilde{v}) \ldots \eta_{1,k_1'}(\tilde{v}) \chi_{\ell,1}(\tilde{u}) \ldots \chi_{\ell,k_\ell}(\tilde{u}) \eta_{\ell,1}(\tilde{v}) \ldots \eta_{\ell,k_\ell'}(\tilde{v})
\]

can be transformed so that a simple elimination lemma is applicable.

**Elimination lemma (reminder):**

Let \( \alpha, \beta \) be FO formulas in which \( P \) does not occur.

\[
\exists P. (\forall x. \alpha \lor P(x)) \land (\forall x. \beta \lor \neg P(x))
\]

is equivalent to

\[
\forall x. \alpha \lor \beta.
\]

\( \rightsquigarrow \) GREAT, but...
... there are troubles ...

(1) \( \ldots \)the resulting formula is not necessarily an SF formula

\( \Rightarrow \) Resort to an extension of SF

\[
\begin{align*}
\text{At}_0 (\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots), \\
\text{At}_1 (\vec{y}_1, \vec{x}_2, \vec{x}_3, \ldots), \\
\text{At}_2 (\vec{y}_1, \vec{y}_2, \vec{x}_3, \ldots), \\
\ldots
\end{align*}
\]
... there are troubles ...

(2) ... with the elimination of multiple SO quantifiers $\exists R \exists P$: transforming

$$\exists P. \left( \forall x. \alpha \lor P(x) \right) \land \left( \forall x. \beta \lor \neg P(x) \right)$$

into

$$\forall x. \alpha \lor \beta$$

yields *bridging effects* caused by stretching the scope of $\forall x$ over $\alpha$ and $\beta$.

$\implies$ We might lose separateness of variable sets.

$\implies$ We might be able to eliminate one of $\exists P$, $\exists R$ but not both.
We have seen...

... the concept of the separated variable sets $X$, $Y$:
for every FO quantifier $Qz$ either all $x \in X$ or all $y \in Y$
can be moved out of the scope of $Qz$.

... the Separated Fragment (SF):
a decidable first-order fragment
that enjoys the finite model property.

... a first elimination theorem for certain second-order SF formulas:

(→) But the result is not necessarily an SF formula.
(→) How to predict eliminability of multiple SO quantifiers?

Questions?