Jacobi Curves: Computing the Exact Topology of Arrangements of Non-Singular Algebraic Curves *

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Abstract. We present an approach that extends the Bentley-Ottmann sweep-line algorithm [2] to the exact computation of the topology of arrangements induced by non-singular algebraic curves of arbitrary degree. Algebraic curves of degree greater than 1 are difficult to handle in case one is interested in exact and efficient solutions. In general, the coordinates of intersection points of two curves are not rational but algebraic numbers and this fact has a great negative impact on the efficiency of algorithms coping with them. The most serious problem when computing arrangements of non-singular algebraic curves turns out to be the detection and location of tangential intersection points of two curves. The main contribution of this paper is a solution to this problem, using only rational arithmetic. We do this by extending the concept of Jacobi curves introduced in [11]. Our algorithm is output-sensitive in the sense that the algebraic effort we need for sweeping a tangential intersection point depends on its multiplicity.

1 Introduction

Computing arrangements of curves is one of the fundamental problems in computational geometry and algebraic geometry. For arrangements of lines defined by rational numbers all computations can be done over the field of rational numbers avoiding numerical errors and leading to exact mathematical results.

As soon as higher degree algebraic curves are considered, instead of linear ones, things become more difficult. In general, the intersection points of two planar curves defined by rational polynomials have irrational coordinates. That means instead of rational numbers one now has to deal with algebraic numbers. One way to overcome this difficulty is to develop algorithms that use floating point arithmetic. These algorithms are quite fast but in degenerate situations they can lead to completely wrong results because of approximation errors, rather

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than just slightly inaccurate outputs. Assume that for two planar curves one is interested in the number of intersection points. If the curves have tangential intersection points, the slightest inaccuracy can lead to a wrong output.

A second approach besides using floating point arithmetic is to use exact algebraic computation methods like the use of the gap theorem [4] or multivariate Sturm sequences [16]. Then of course the results are correct, but the algorithms in general are very slow.

We consider arrangements of non-singular curves in the real plane defined by rational polynomials. Although the non-singularity assumption is a strong restriction on the curves we consider, this class of curves is worthwhile to be studied because of the general nature of the main problem that has to be solved.

Two algebraic curves can have tangential intersections and it is inevitable to determine them precisely in the case we are interested in exact computation. As a main tool for solving this problem we will introduce generalized Jacobi curves, for more details consider [22]. Our resulting algorithm computes the exact topology using only rational arithmetic. It is output-sensitive in the sense that the algebraic degree of the Jacobi curve that is constructed to locate a tangential intersection point depends on its multiplicity.

2 Previous work

As mentioned, methods for the calculation of arrangements of algebraic curves are an important area of research in computational geometry. A great focus is on arrangements of linear objects. Algorithms coping with linear primitives can be implemented using rational arithmetic, leading to exact mathematical results in any case. For fast filtered implementations see for example the ones in LEDA [15] and CGAL [10]. There are also some geometric methods dealing with arbitrary curves, for example [1], [7], [18], [20]. But all of them neglect the problem of exact computation in the way that they are based on an idealized real arithmetic provided by the real RAM model of computation. The assumption is that all, even irrational, numbers are representable and that one can deal with them in constant time. This postulate is not in accordance with real computers.

Recently the exact computation of arrangements of non-linear objects has come into the focus of research. Wein [21] extended the CGAL implementation of planar maps to conic arcs. Berberich et al. [3] made a similar approach for conic arcs based on the improved LEDA [15] implementation of the Bentley-Ottmann sweep-line algorithm [2]. For conic arcs the problem of tangential intersection points is not serious because the coordinates of every such point are one-root expressions of rational numbers. Eigenwillig et al. [9] extended the sweep-line approach to cubic arcs. All tangential intersection points in the arrangements of cubic arcs either have coordinates that are one-root expressions or they are of multiplicity 2 and therefore can be solved using the Jacobi curve introduced in [11].

Arrangements of quadric surfaces in $\mathbb{R}^3$ are considered by Wolpert [22], Dupont et al. [8], and Mourrain et al. [17]. By projection the first author re-
duces the spatial problem to the one of computing planar arrangements of algebraic curves of degree at most 4. The second authors directly work in space determining a parameterization of the intersection curve of two arbitrary implicit quadrics. The third approach is a space sweep. Here the main task is to maintain the planar arrangements of conics on the sweep-plane.

For computing planar arrangements of arbitrary planar curves very little is known. An exact approach using rational arithmetic to compute the topological configuration of a single curve is done by Sakalakis [19]. Hong improves this idea by using floating point interval arithmetic [13]. For computing arrangements of curves we are also interested in intersection points of two or more curves. Of course we could interpret these points as singular points of the curve that is the union of both. But this would unnecessarily increase the degree of the algebraic curves we consider and lead to slow computation.

MAPC [14] is a library for exact computation and manipulation of algebraic points. It includes a package for determining arrangements of planar curves. For degenerate situations like tangential intersections the use of the gap theorem [4] or multivariate Sturm sequences [16] is proposed. Both methods are not efficient.

3 Notation

The objects we consider and manipulate in our work are non-singular algebraic curves represented by rational polynomials. We define an algebraic curve in the following way: Let $f$ be a polynomial in $\mathbb{Q}[x,y]$. We set $\text{Zero}(f) := \{(\alpha, \beta) \in \mathbb{R}^2 \mid f(\alpha, \beta) = 0\}$ and call $\text{Zero}(f)$ the algebraic curve defined by $f$. If the context is unambiguous, we will often identify the defining polynomial of an algebraic curve with its zero set.

For an algebraic curve $f$ we define its gradient vector to be $\nabla f := (f_x, f_y) \in (\mathbb{Q}[x,y])^2$ with $f_x := \frac{\partial f}{\partial x}$. We assume the set of input curves to be non-singular, that means for every point $(\alpha, \beta) \in \mathbb{R}^2$ with $f(\alpha, \beta) = 0$ we have $\nabla f(\alpha, \beta) = (f_x(\alpha, \beta), f_y(\alpha, \beta)) \neq (0, 0)$. A point $(\alpha, \beta)$ with $(\nabla f)(\alpha, \beta) = (0, 0)$ we would call singular. The geometric interpretation is that for every point $(\alpha, \beta)$ of $f$ there exists a unique tangent line to the curve $f$. This tangent line is perpendicular to $\nabla f(\alpha, \beta)$. From now on we assume that all curves we consider are non-singular.

We call a point $(\alpha, \beta) \in \mathbb{R}^2$ of $f$ extreme if $f_y(\alpha, \beta) = 0$. Extreme points have a vertical tangent. A point $(\alpha, \beta) \in \mathbb{R}^2$ of $f$ is named a flex if the curvature of $f$ becomes zero in $(\alpha, \beta)$: $0 = (f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2)(\alpha, \beta)$.

Two curves $f$ and $g$ have a disjoint factorization if they only share a common constant factor. Without loss of generality we assume that this is the case for every pair of curves $f$ and $g$ we consider during our computation. Disjoint factorization can be easily tested and established by a bivariate gcd-computation.

For two curves $f$ and $g$ a point $(\alpha, \beta)$ in the real plane is called an intersection point if it lies on $f$ as well as on $g$. It is called a tangential intersection point of $f$ and $g$ if additionally the two gradient vectors are linearly dependend: $(f_xg_y - f_yg_x)(\alpha, \beta) = 0$. Otherwise we speak of a transversal intersection point.
Last but not least we will name some properties of curves that are, unlike the previous definitions, not intrinsic to the geometry of the curves but depend on our chosen coordinate system.

We call a single curve $f = f_n(x) \cdot y^n + f_{n-1}(x) \cdot y^{n-1} + \ldots + f_0(x) \in \mathbb{Q}[x,y]$ generally aligned if $f_n(x) = \text{constant} \neq 0$, in which case $f$ has no vertical asymptotes. Two curves $f$ and $g$ are termed to be in general relation if every two common roots $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2) \in \mathbb{C}^2$ of $f$ and of $g$ have different $x$-values $\alpha_1 \neq \alpha_2$.

Next we will introduce the notation of well-behavedness of a pair of curves. We will first give the formal definition and then describe the geometric intuition behind. We say that two pairs of curves $(f_1, g_1)$ and $(f_2, g_2)$ are separate if

1. either there are non-zero constants $c_1, c_2$ with $f_1 = c_1 \cdot f_2$ and $g_1 = c_2 \cdot g_2$
2. or the $x$-values of the complex roots of $f_1$ and $g_1$ differ pairwise from the $x$-values of the complex roots of $f_2$ and $g_2$.

We call two curves $f$ and $g$ well-behaved if

1. $f$ and $g$ are both generally aligned,
2. $f$ and $g$ are in general relation, and
3. the pairs of curves $(f, g), (f, f_y)$, and $(g, g_y)$ are pairwise separate.

![Fig. 1](image-url). In the leftmost box of the left picture the curves $f$ and $f_y$ are well-behaved, in the following three boxes they are not. In the leftmost box of the right picture the curves $f$ and $g$ are well-behaved, in the following three boxes they are not.

We will shortly give an idea of what well-behavedness of two curves means. Let $(\alpha, \beta)$ be an intersection point of two curves $f$ and $g$. We first consider the case $g = f_y$ (left picture in Figure 1). If $f$ and $f_y$ are well-behaved, there exists a vertical stripe $a \leq x \leq b$ with $a < \alpha < b$ such that $(\alpha, \beta)$ is the only extreme point of $f$ inside the stripe and the stripe contains no extreme point of $f_y$ (and no singular point of $f_y$). Especially this means that flexes of $f$ do not have a vertical tangent.

Next consider the case that neither $f$ nor $g$ is a constant multiple of the other (right picture in Figure 1): there are no constants $c_1, c_2$ with $f = c_1 \cdot g_y$ or $g = c_1 \cdot f_y$. If $f$ and $g$ are well-behaved, then there exists a vertical stripe $a \leq x \leq b$ with $a < \alpha < b$ that contains exactly one intersection
point of $f$ and $g$, namely $(\alpha, \beta)$, and there is no extreme point of $f$ or $g$ inside this stripe. Especially this means that $f$ and $g$ do not intersect in extreme points.

A random shear at the beginning will establish well-behaved input-curves with high probability. We can test whether a pair of curves is well-behaved by gcd-, resultant-, and subresultant-computation. Due to the lack of space we omit the details. If we detect during the computation that the criterion is not fulfilled for one pair of curves, then we know that we are in a degenerate situation due to the choice of our coordinate system. In this case we stop, shear the whole set of input curves by random (for a random $v \in \mathbb{Q}$ we apply the affine transformation $\psi(x, y) = (x + vy, y)$ to each input polynomial) and restart from the beginning. A shear does not change the topology of the arrangement and we end up with pairs of well-behaved curves.

4 The overall approach

We are interested in the topology of a planar arrangement of a set $F$ of $n$ non-singular input curves. The curves partition the affine space in a natural way into three different types of maximal connected regions of dimensions 2, 1, and 0 called faces, edges, and vertices, respectively.

We want to compute the arrangement with a sweep-line algorithm. At each time during the sweep the branches of the curves intersect the sweep-line in some order. While moving the sweep-line along the $x$-axis a change in the topology of the arrangement takes place if this ordering changes. This happens at intersection points of at least two different curves $f, g \in F$ and at extreme points of a curve $f \in F$. At extreme points geometrically two new branches of $f$ start or two branches of $f$ end. Extreme points of $f$ are intersection points of $f$ and $f_y$. This leads to the following definition of points on the $x$-axis that force the sweep-line to stop and to recompute the ordering of the curves:

**Definition 1** The event points of a planar arrangement induced by a set $F$ of non-singular planar curves are defined as the intersection points of each two curves $f, g \in F$ and as the intersection points of $f$ and $f_y$ for all $f \in F$.

Our main algorithmic approach follows the ideas of the Bentley-Ottmann sweep [2]. We hold up an $X-$ and a $Y$-structure. The $X$-structure contains the $x$-coordinates of event points. In the $Y$-structure we maintain the ordering of the curves along the sweep-line. At the beginning we found that for every $f \in F$ the curves $f$ and $f_y$ are well-behaved. We insert the $x$-coordinates of all extreme points into the empty $X$-structure. We shortly remark that there can be event points left to the leftmost extreme point. This can be resolved by moving the sweep-line to the left until all pairs of adjacent curves in the $Y$-structure have their intersection points to the right.

If the sweep-line reaches the next event point we stop, identify the pairs of curves that intersect, the kind of intersection they have and their involved branches, recompute the ordering of the curves along the sweep-line, and according to this we update the $Y$-structure. If two curves became adjacent that
were not adjacent in the past, we test whether they are well-behaved. If \( f \) and \( g \) are not well-behaved, we shear the whole arrangement and start from the beginning. Otherwise we compute the \( x \)-coordinates of their intersection points and insert them into the \( X \)-structure.

5 The \( X \)-structure

In order to make the overall approach compute the exact mathematical result in every case there are some problems that have to be solved. Describing the sweep we stated that one of the fundamental operations is the following: For two well-behaved curves \( f \) and \( g \) insert the \( x \)-coordinates of their intersection points into the \( X \)-structure. A well known algebraic method is the resultant computation of \( f \) and \( g \) with respect to \( y \) [6]. We can compute a polynomial \( \text{res}(f, g) \in \mathbb{Q}[x] \) of degree at most \( \text{deg}(f) \cdot \text{deg}(g) \) with the following property:

**Proposition 1** Let \( f, g \in \mathbb{Q}[x, y] \) be generally aligned curves that are in general relation. A number \( \alpha \in \mathbb{R} \) is a root of \( \text{res}(f, g) \) if and only if there exists exactly one \( \beta \in \mathbb{C} \) such that \( f(\alpha, \beta) = g(\alpha, \beta) = 0 \) and \( \beta \in \mathbb{R} \).

The \( x \)-coordinates of real intersection points of \( f \) and \( g \) are exactly the real roots of the resultant polynomial \( \text{res}(f, g) \). Unfortunately, the intersection points of algebraic curves in general have irrational coordinates. By definition, every root of \( \text{res}(f, g) \) is an algebraic number. For \( \text{deg}(\text{res}(f, g)) > 2 \) there is no general way via radicals to explicitly compute the algebraic numbers in every case. But we can determine an **isolating interval** for each real root \( \alpha \) of \( \text{res}(f, g) \), for example with the algorithm of Uspensky [5]. We compute two rational numbers \( a \) and \( b \) such that \( \alpha \) is the one and only real root of \( \text{res}(f, g) \) in \([a, b] \). The pair \( (\text{res}(f, g), [a, b]) \) yields a non-ambiguous rational representation of \( \alpha \). Of course in this representation the entry \( \text{res}(f, g) \) could be exchanged by any rational factor \( p \in \mathbb{Q}[x] \) of \( \text{res}(f, g) \) with \( p(\alpha) = 0 \). Additionally we like \( \alpha \) to remember the two curves \( f \) and \( g \) it originates from. We end up with inserting a representation \( p, [a, b], f, g \) for every event point induced by \( f \) and \( g \) into the \( X \)-structure. Remark that several pairs of curves can intersect at the event point \( x = \alpha \). In this case there are several representations of the algebraic number \( \alpha \) in the \( X \)-structure, one for each pair of intersecting curves.

During the sweep we frequently have to determine the next coming event point. In order to support this query with the help of the isolating intervals we finally have to ensure the following invariant: Every two entries in the \( X \)-structure either represent the same algebraic number, and in this case the isolating intervals in their representation are identical, or their isolating intervals are disjoint. The invariant can be easily established and maintained using \( \gcd \)-computation of the defining univariate polynomials and bisection by midpoints of the isolating intervals.
6 The Y-structure

A second problem that has to be solved is how to update the Y-structure at an event point. At an event point we have to stop with the sweep-line, identify the pairs of curves that intersect and their involved branches, and recompute the ordering of the curves along the sweep-line. As we have seen, the x-coordinate \( \alpha \) of an event point is represented by at least one entry of the form \((p, [a, b], f, g)\) in the x-structure. So we can directly determine the pairs of curves that intersect at \( x = \alpha \). For each pair \( f \) and \( g \) of intersecting curves we have to determine their involved branches. Furthermore we have to decide whether these two branches cross or just touch, but do not cross each other. As soon as we have these two information, updating the ordering of the curves along the sweep-line is easy.

In general, event points have irrational coordinates and therefore we cannot exactly stop the sweep-line at \( x = \alpha \). The only thing we can do is stopping at the rational point \( a \) to the left of \( \alpha \) and at the rational point \( b \) to the right of \( \alpha \). Using a root isolation algorithm, gcd-computation of univariate polynomials, and bisection by midpoints of the separating intervals we compute the sequence of the branches of \( f \) and \( g \) along the rational line \( x = a \). We do the same along the line \( x = b \). Finally, we compare these two orderings. In some cases this information is sufficient to determine the kind of event point and the involved branches of the curves inside the stripe \( a \leq x \leq b \). Due to our assumption of well-behavedness we can directly compute extreme points of \( f \) (consider the left picture in Figure 2):

![Figure 2](image)

**Fig. 2.** For computing extreme points it is sufficient to compare the sequence of \( f \) and \( f_y \) at \( x = a \) to the left and at \( x = b \) to the right of \( \alpha \) (left picture). The same holds for computing intersection points of odd multiplicity of two curves \( f \) and \( g \) (right picture).

**Theorem 1** Let \((\alpha, \beta) \in \mathbb{R}^2\) be an extreme point of a non-singular curve \( f \) and assume that \( f \) and \( f_y \) are well-behaved. We can compute two rational numbers \( a \leq \alpha \leq b \) with the following property: the identification of the involved branches of \( f \) is possible by just comparing the sequence of hits of \( f \) and \( f_y \) along \( x = a \) and along \( x = b \).

**Proof.** (Sketch) By assumption the curves \( f \) and \( f_y \) are well-behaved and therefore we know that \( \alpha \) is not an extreme or singular point of \( f_y \). We shrink
the isolating interval \([a, b]\) of \(\alpha\) until it contains no real root of \(\text{res}(f_y, f_{yy}, y)\). Afterwards the number and ordering of the branches of \(f_y\) does not change in the interval \([a, b]\). The number of branches of \(f\) at \(x = a\) differs by 2 from the one at \(x = b\). At \(x = a\) at least one branch of \(f_y\) lies between two branches of \(f\). The same holds at \(x = b\).

Using root isolation we compare from \(-\infty\) upwards the sequences of roots of \(f\) and \(f_y\) at \(x = a\) and at \(x = b\). The branch \(i\) of \(f\) that causes the first difference (either at \(x = a\) or at \(x = b\)) intersects the \((i + 1)\)st branch of \(f\) in an extreme point.

The same idea can be used to compute intersection points of odd multiplicity between two curves \(f\) and \(g\) where two branches of \(f\) and \(g\) cross each other (see the right picture in Figure 2) because we have an observable transposition in the sequences. Of course the test can be easily extended to arbitrary curves under the assumption that the intersection point \((\alpha, \beta)\) is not a singular point of any of the curves.

What remains to do is locating intersection points \((\alpha, \beta)\) of even multiplicity. These points are rather difficult to locate. From the information how the curves behave slightly to the left and to the right of the intersection point we cannot draw any conclusions. At \(x = a\) and at \(x = b\) the branches of \(f\) and \(g\) appear in the same order, see Figure 3. We will show in the next section how to extend the idea of Jacobi curves introduced in [11] to intersection points of arbitrary multiplicity.

\[\text{Fig. 3. Intersection points of even multiplicity lead to the same sequence of } f \text{ and } g \text{ to the left and to the right of } \alpha. \text{ Introduce an auxiliary curve } h \text{ in order to locate these intersection points.}\]

7 The Jacobi Curves

In order to locate an intersection point of even multiplicity between two curves \(f\) and \(g\) it would be helpful to know a third curve \(h\) that cuts \(f\) as well as \(g\) transversally in this point, see right picture in Figure 3. This would reduce the problem of locating the intersection point of \(f\) and \(g\) to the easy one of locating the transversal intersection point of \(f\) and \(h\) and the transversal intersection
point of $g$ and $h$. In the last section we have shown how to compute the indices $i$, $j$, and $k$ of the intersecting branches of $f$, $g$, and $h$, respectively. Once we have determined these indices we can conclude that the $i$th branch of $f$ intersects the $j$th branch of $g$.

We will give a positive answer to the existence of transversal curves with the help of the Theorem of Implicit Functions. Let $(\alpha, \beta) \in \mathbb{R}^2$ be a real intersection point of $f, g \in \mathfrak{F}[x, y]$. We will iteratively define a sequence of polynomials $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \ldots$ such that $\tilde{h}_k$ cuts transversally through $f$ in $(\alpha, \beta)$ for some index $k$. If $f$ and $g$ are well-behaved, the index $k$ is equal to the degree of $\alpha$ as a root of $\text{res}(f, g)$. The result that introducing an additional curve can solve tangential intersections is already known for $k = 2$ [11]. What is new is that this concept can be extended to every multiplicity $k > 2$. All the following results are not restricted to non-singular curves. We will show in Theorem 2 that we can determine every tangential intersection point of two arbitrary curves provided that it is not a singular point of one of the curves.

**Definition 2** Let $f$ and $g$ be two planar curves. We define generalized Jacobi curves in the following way:

\[ \tilde{h}_1 := g, \quad \tilde{h}_{i+1} := (\tilde{h}_i)xf_y - (\tilde{h}_i)yf_x. \]

**Theorem 2** Let $f$ and $g$ be two algebraic curves with disjoint factorizations. Let $(\alpha, \beta)$ be an intersection point of $f$ and $g$ that neither is a singular point of $f$ nor of $g$. There exists an index $k \geq 1$ such that $\tilde{h}_k$ cuts transversally through $f$ in $(\alpha, \beta)$.

**Proof.** In the case $g$ cuts through $f$ in the point $(\alpha, \beta)$, especially if $(\alpha, \beta)$ is a transversal intersection point of $f$ and $g$, this is of course true for $\tilde{h}_1 = g$. So assume in the following that $(g_xf_y - g_yf_x)(\alpha, \beta) = \tilde{h}_2(\alpha, \beta) = 0$. From now on we will only consider the polynomials $\tilde{h}_i$ with $i \geq 2$.

By assumption every point $(\alpha, \beta)$ is a non-singular point of $f$: $(f_x, f_y)(\alpha, \beta) \neq 0$. We only consider the case $f_y(\alpha, \beta) \neq 0$. In the case $f_x(\alpha, \beta) \neq 0$ and $f_y(\alpha, \beta) = 0$ we would proceed the same way as described in the following by just exchanging the two variables $x$ and $y$. The property $f_y(\alpha, \beta) \neq 0$ leads to $(f_xg_y)(\alpha, \beta) = g_x(\alpha, \beta)$ and because $(g_x, g_y)(\alpha, \beta) \neq (0, 0)$ we conclude $g_y(\alpha, \beta) \neq 0$. From the Theorem of Implicit Functions we derive that there are real open intervals $I_x, I_y \subseteq \mathbb{R}$ with $(\alpha, \beta) \in I_x \times I_y$ such that

1. $f_y(x_0, y_0) \neq 0$ and $g_y(x_0, y_0) \neq 0$ for all $(x_0, y_0) \in I_x \times I_y$,
2. there exist continuous functions $F, G : I_x \to I_y$ with the two properties
   
   (a) $f(x, F(x)) = g(x, G(x)) = 0$ for all $x \in I_x$,
   
   (b) $(x, y) \in I_x \times I_y$ with $f(x, y) = 0$ leads to $y = F(x)$,
   
   $(x, y) \in I_x \times I_y$ with $g(x, y) = 0$ leads to $y = G(x)$.

Locally around the point $(\alpha, \beta)$ the curve defined by the polynomial $f$ is equal to the graph of the function $F$. The same holds for $g$ and $G$. Especially we have $\beta = F(\alpha) = G(\alpha)$. Moreover, the Theorem of Implicit Holomorphic
Functions implies that $F$ as well as $G$ are holomorphic and thus developable in a Taylor series around the point $(\alpha, \beta)$ \cite{12}.

In the following we will sometimes consider the functions $h_i : I_x \times I_y \to \mathbb{R}$, $i \geq 2$, with

$$h_2 := g_{x} / g_{y} - f_{x} / f_{y} = \frac{\hat{h}_2}{g_{y} f_{y}} , \quad h_{i+1} := (h_{i})_{x} - (h_{i})_{y} \cdot \frac{f_{x}}{f_{y}}$$

instead of the polynomials $\hat{h}_i$. Each $h_i$ is well defined for $(x, y) \in I_x \times I_y$. We have the following relationship between the functions $h_i$ and the polynomials $\hat{h}_i$ defined before: For each $i \geq 2$ there exist functions $\delta_{i,1}, \delta_{i,2}, \ldots, \delta_{i,i} : I_x \times I_y \to \mathbb{R}$ such that

$$\delta_{i,i}(x, y) \neq 0 \text{ for all } (x, y) \in I_x \times I_y. \text{ For } i = 2 \text{ this is obviously true with } \delta_{2,2} = (g_{y} f_{y})^{-1}. \text{ The general case follows by induction on } i.$$

Let us assume we know the following proposition: Let $k \geq 1$. If $F^{(i)}(\alpha) = G^{(i)}(\alpha)$ for all $0 \leq i \leq k - 1$, then $h_{k+1}(\alpha, \beta) = G^{(k)}(\alpha) - F^{(k)}(\alpha)$.

We know that the two polynomials $f$ and $g$ have disjoint factorizations. That means the Taylor series of $F$ and $G$ differ in some term. Remember that we consider the case that the curves defined by $f$ and $g$ intersect tangentially in the point $(\alpha, \beta)$. So there is an index $k \geq 2$ such that $F^{(i)}(\alpha) = G^{(i)}(\alpha)$ for all $0 \leq i \leq k - 1$ and $F^{(k)}(\alpha) \neq G^{(k)}(\alpha)$. According to the proposition we have $h_{k+1}(\alpha, \beta) = G^{(k)}(\alpha) - F^{(k)}(\alpha) = 0$ for all $1 \leq i \leq k - 1$. From equation (*) we inductively obtain also $\hat{h}_{k+1}(\alpha, \beta) = 0$, $1 \leq i \leq k - 1$. Especially this means that $\hat{h}_k$ intersects $f$ and $g$ in $(\alpha, \beta)$. The intersection is transversal if and only if $((\hat{h}_k)_{x} f_{y} - (\hat{h}_k)_{y} f_{x})(\alpha, \beta) = \hat{h}_{k+1}(\alpha, \beta) \neq 0$. This follows easily from $0 \neq G^{(k)}(\alpha) - F^{(k)}(\alpha) = h_{k+1}(\alpha, \beta) = \delta_{k+1,k+1}(\alpha, \beta) \cdot \hat{h}_{k+1}(\alpha, \beta)$. \hfill \Box

It remains to state and prove the proposition:

**Proposition 2** Let $k \geq 1$. If $F^{(i)}(\alpha) = G^{(i)}(\alpha)$ for all $0 \leq i \leq k - 1$, then $h_{k+1}(\alpha, \beta) = G^{(k)}(\alpha) - F^{(k)}(\alpha)$.

**Proof.** For each $i \geq 2$ we define a function $H_i : I_x \to \mathbb{R}$ by $H_i(x) := h_i(x, F(x))$. For $x = \alpha$ we derive $H_i(\alpha) = h_i(\alpha, \beta)$. So in terms of our new function we want to prove that $H_{k+1}(\alpha) = G^{(k)}(\alpha) - F^{(k)}(\alpha)$ holds if $F^{(i)}(\alpha) = G^{(i)}(\alpha)$ for all $0 \leq i \leq k - 1$.

By definition we have $f(x, F(x)) : I_x \to \mathbb{R}$ and $f(x, F(x)) = 0$ for all $x \in I_x$. That means $f(x, F(x))$ is constant and therefore $0 = f(x, F(x))' = f_{x}(x, F(x)) + F'(x) f_{y}(x, F(x))$. We conclude $F'(x) = -f_{x}(x, F(x)) / f_{y}(x, F(x))$ and this directly leads to the equality $H'_i(x) = H_{i+1}(x)$. Inductively we obtain $H_{i+1}(x) = H_{i+1}^{(k-1)}(x)$ for all $i \geq 1$. In order to prove the proposition it is sufficient to show the following: Let $k \geq 1$. If for all $0 \leq i \leq k - 1$ we have $F^{(i)}(\alpha) = G^{(i)}(\alpha)$, then $H_{i+1}^{(k-1)}(\alpha) = (G' - F')^{(k-1)}(\alpha)$. 


1. Let \( k = 1 \). Our assumption is \( F(\alpha) = G(\alpha) \) and we have to show \( H_2(\alpha) = (G' - F')(\alpha) \). We have
\[
(*) \quad H_2(x) = h_2(x, F(x)) = \frac{g_x(x, F(x))}{g_y(x, F(x))} = \frac{f_x(x, F(x))}{f_y(x, F(x))}
\]
and both functions just differ in the functions that are substituted for \( y \) in \( \frac{g_x(x, y)}{g_y(x, y)} \). In the equality of \( H_2(x) \) we substitute \( F(x) \), whereas in the one of \( (G' - F') \) we substitute \( G(x) \). But of course \( F(x) = G(x) \) leads to \( H_2(\alpha) = (G' - F')(\alpha) \).

2. Let \( k > 1 \). We know that \( F^{(i)}(\alpha) = G^{(i)}(\alpha) \) for all \( 0 \leq i < k - 1 \). We again use the equations \( (*) \) and the fact that \( H_2(x) \) and \( (G' - F') \) only differ in the functions that are substituted for \( y \) in \( \frac{g_x(x, y)}{g_y(x, y)} \).

By taking \( (k - 1) \) times the derivative of \( H_2(x) \) and \( (G' - F') \), we structurally obtain the same result for both functions. The only difference is that some of the terms \( F^{(i)}(x) \), \( 0 \leq i < k - 1 \), in \( H_2 \) are exchanged by \( G^{(i)}(x) \) in \( (G' - F') \). But due to our assumption we have \( F^{(i)}(\alpha) = G^{(i)}(\alpha) \) for all \( 0 \leq i < k - 1 \) and we obtain \( H_2^{(k - 1)}(\alpha) = (G' - F')^{(k - 1)}(\alpha) \).

We have proven that for a non-singular tangential intersection point of \( f \) and \( g \) there exists a curve \( h_k \) that cuts both curves transversally at this point. The index \( k \) depends on the degree of similarity of the functions that describe both polynomials in a small area around the given point. The degree of similarity is measured by the number of successive matching derivatives in this point. An immediate consequence of the previous theorem, together with the well known fact that the resultant of two univariate polynomials equals the product of the differences of their roots [6], is that we can obtain the index \( k \) by just looking at the resultant of \( f \) and \( g \):

**Corollary 1** Let \( f, g \in \Phi[x, y] \) be two polynomials in general relation and let \( (\alpha, \beta) \) be a non-singular intersection point of the curves defined by \( f \) and \( g \). If \( k \) is the degree of \( \alpha \) as a root of the resultant \( \text{res}(f, g, y) \), then \( h_k \) cuts transversally through \( f \). (proof omitted)

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**References**