Curvilinear Spherical Parameterization

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Abstract

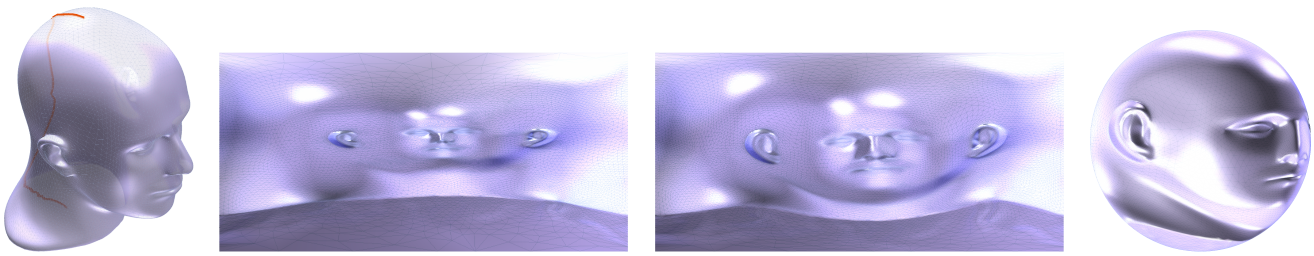
We present an efficient approach for solving the spherical parameterization problem. The essence of the approach is to look for a solution in the curvilinear coordinate system without requiring the additional spherical constraints usually needed in cartesian formulations. This setup allows us to take full advantage of some existing techniques originally developed for planar parameterization. Our results substantiate the efficiency of the method and confirm its robustness. Meshes of non-trivial geometry with tens of thousands of triangles are processed in a few seconds, always yielding bijective maps. This computational achievement bridges a so far wide gap in performance between spherical and planar parameterization.

1 Introduction

Surface parameterization establishes bijective maps from a surface onto a topologically equivalent standard domain. It serves as an essential tool in many digital geometry processing applications such as texture mapping, remeshing, shape analysis, compression, morphing, etc. Usually in these settings surfaces are represented as triangle meshes, and the maps are required to be at least piecewise linear. The most fundamental and also best-studied problem is parameterization of surface patches with disk topology (see [5] for a recent survey). In theory any higher genus model can be described as a union of patches through an atlas. The discontinuities induced by this segmentation render it unattractive, especially in the particular case of genus-zero surfaces for which the sphere is the most natural domain. The fact that many of the available geometric models are indeed homeomorphic to a sphere makes spherical parameterization an appealing geometry processing tool. The past and recent research interest in this topic echoes its relevance to applications and reflects the demand for efficient algorithms for establishing low-distortion spherical maps. Despite the recent advances in the field, the construction of spherical maps still raises theoretical and numerical challenges. The spherical setting is much more complex than the planar one, and any robust method to solve this problem cannot solely rely on simple modification or extension of traditional planar methods as pointed out in [6]. Furthermore, the additional spherical constraints and the ever increasing need for processing large input data raise challenging theoretical issues such as convergence guarantees and validity, along with practical ones, associated with finding robust numerical schemes and efficient custom solvers. Currently, methods with the most sound theoretical foundation are not in measure of addressing even moderately sized problems numerically [7].

In this work, we present a novel approach to spherical parameterization, where computation time is dominated by solving only linear systems. Our method relies on setting the problem in a curvilinear coordinates system, hence reducing it to a two-dimensional problem. The singularities of this coordinate system are effectively addressed by removing the poles and introducing a date line connecting the poles. This way an initial harmonic map can be established following the outline in [2]. In general, this initial map suffers from unacceptable distortion. As a novel contribution we undertake further steps to improve distortion. The merit of our new method is that we perform the crucial distortion improvement of the initial map in curvilinear coordinates as well, based on a variant of quasi-harmonic maps [19]. This way we benefit from guarantees on validity as well as of the availability of highly efficient and robust solving strategies, while at the same time the overall algorithm and its implementation are conceptually simple. In a final step, we apply a local distortion improvement on a small sub-patch along the date line in order to account for the unavoidable distortion induced from the Dirichlet boundary conditions.

Our results on non-trivial and considerably large input meshes show high-quality maps with a fair balance between angle and area distortion. Computation times are significantly lower than the ones of preexisting methods. The remainder of the paper is organized as follows: Section 2 overviews related work, sections 3 and 4 describe the initial and improved maps, respectively, and section 5 discusses the final local relaxation. The paper concludes with experi-



(a) (b) (c) (d)

Figure 1. Our approach in brief: (a) The poles are identified on the shape by the user (or automatically). The path from pole to pole constitutes the date line along which the mesh is cut open. (b) The poles are removed and an initial solution in curvilinear coordinates is obtained on the remaining mesh. This initial map suffers from unacceptably high distortion. (c) Therefore it is improved in a second step taking into account spherical distortion. Both, the initial and and second step operate in the curvilinear domain with fixed boundaries. (d) The secondary solution is then lifted back onto the sphere. The poles are restored, and a local spherical smoothing is performed in a region along the date line. The run time for the mannequin head model with 23KΔ is 3 seconds (3.2GHz P4 laptop).

mental results and a discussion in section 6.

2 Related work

We give a bird's eye view of spherical parameterization methods. For a more general discussion of surface parameterization we refer to the recent survey by Floater and Hormann [5]. A direct extension of harmonic planar parameterization to the spherical domain can be formulated as a minimization problem of the harmonic objective function $\int_\Omega ||\nabla \mathbf{x}||^2 dA$ over the surface subject to the constraint

$$||\mathbf{x}||^2 = 1.$$

for all vertices. The nonlinearity of the constraints makes the spherical setting more involved than its planar counterpart.

A straightforward way to solve this optimization problem is the Gauss-Seidel method where each iteration uses a local relaxation step e.g., tangential Laplacian smoothing, followed by back projection onto the sphere. This type of solution was carried out in [1, 13]. The result is used as a starting point for computing a minimal Möbius transform in [8, 9]. A principal problem is that a minimum can be reached for degenerate configurations, e.g., with vertices slipping over the sphere until triangles collapse. Heuristic solutions to this issue such as imposing stopping criteria or introducing additional boundary conditions fall short from providing satisfactory results in general. A more promising alternative is the careful analysis of the discrete objective function which tries to account for the particularities of the spherical setting and blends several measures for controlling distortion [6].

A more theoretical study of the problem was carried out in [7] and devises sufficient requirements for generating provably bijective maps. An alternative constrained minimization problem proposed in [18], adapts the planar angle based flattening method to the spherical setting. While theoretically interesting, both approaches are computationally too expensive to be of general use in practice.

Methods based on the multi-resolution paradigm which follow the original algorithmic frameworks outlined in [3, 11, 14] can also be applied to the spherical setting. In particular the choice of base mesh and objective function can be tuned for spherical parameterization as recently proposed in [12, 15].

Another breed of methods resorts to the existence of simple maps from the plane onto the sphere. In [10], a triangle is cut from the mesh and then the whole mesh is mapped into a triangular boundary. The resulting planar parameterizing is lifted to the sphere through an inverse stereographic projection. Besides the high distortion, this method generally suffers from foldovers. This is due to the fact that the boundary of the mesh in the planar domain is considered to extend implicitly to infinity. In order to overcome such limitations [17] cut the mesh into two halves and map each half to a circle. These two planar embeddings are mapped onto the sphere and serve as starting point for subsequent non-linear optimization on the sphere with an appropriate distortion measure. The latter work is motivated by [7] and yields relatively faster computation time. Most related to our approach is the method proposed in [2] which first establish an initial parameterization based on curvilinear coordinates. This solution is further improved using a non-linear optimization in the spherical domain. However, the highly non-linear nature of this optimization problem makes it un-

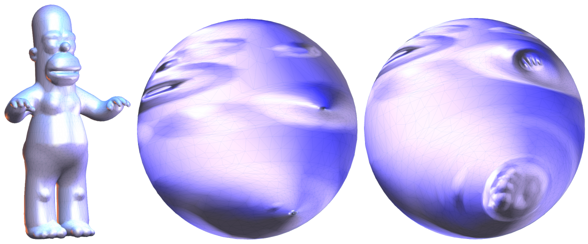


Figure 2. Comparison of the initial (center) and the secondary map (right) for the Homer model. The view puts emphasis on the extremities, only after improvement the left arm and leg develop adequately.

stable for practical use. An improvement was proposed in [16] by using a hierarchical implementation for speeding up convergence. However, the reported performance still penalizes the approach as whole.

3 Initial parameterization

In order to solve the spherical parameterization problem we need to pose the problem in a computationally tractable way. This involves adopting one of the many possible characterizations of the sphere as a mathematical object. In Euclidean space, a natural description of point locations on the sphere is achieved through the curvilinear or polar coordinates represented by the angles (θ, ϕ). In this orthogonal coordinate system only two parameters are needed to characterize point positions on the sphere. Certainly, this representation is more compact than its cartesian counterpart where three parameters are required.

In curvilinear coordinates, the spherical parameterization problem reduces to defining two appropriate scalar fields over the surface, the azimuthal angle $\theta \in (0, 2\pi)$ also known as longitude, and the polar angle or latitude $\phi \in [0, \pi]$. Although this setup simplifies the problem to a great extent, it exhibits clear limitations which are in fact, inherent to the coordinate system itself: The first one is the *pole singularity* — the longitude spans the whole range at the poles. The second one is the *periodicity* of the longitude range. The pole singularity can be addressed by first excluding the north and south poles from the problem setup and then reinserting them at the end of the optimization process. On the other hand, the periodicity of θ, requires defining a date line which connects the poles and marks the beginning and the end of the range. The date line can be efficiently setup, e.g., as shortest path between poles (see also section 6).

The mesh resulting from cutting along the date line and removing the poles is topologically equivalent to a disk, and we can readily profit from existing techniques developed for planar parameterization. A first attempt along these lines was proposed by [2]. Their method proceeds by solving the Laplace equation

$$\nabla^2 U = 0\,. \qquad (1)$$

for the pair (θ, ϕ) over the domain $[0, 2\pi] \times (0, \pi)$. The north and south pole are assigned the ϕ-values 0 and π, respectively. Technically the domain of ϕ can be represented as $[\epsilon_1, \pi - \epsilon_2]$ where ϵ_1 and ϵ_2 are very small. In our implementation we used a value of 0.02 for both, however, the method as a whole is insensitive to the chosen value as we will see in section 5. The solution to equation (1) can be efficiently carried out using either the cotangent weights or the mean value coordinates discretization of the Laplacian operator. Vertex positions on the sphere are then given by the usual polar-to-cartesian mapping

$$x = \cos\theta\ \sin\phi,\quad y = \sin\theta\ \sin\phi,\quad z = \cos\phi\,.$$

A simple computation of the first fundamental form of this mapping from polar to cartesian coordinates reveals that it is neither conformal nor equal-area. Consequently, the resulting composite map from the surface onto the sphere enjoys neither properties. This is not a limitation in itself as the results from this initial parameterization reflect well behaved maps-see figure 2, although they may suffer from unacceptable high area distortion as noted also in [2]. Furthermore, there exist no single conformal map from a finite planar domain onto the whole sphere. On the other hand aiming only for a conformal mapping may yield in general results which exhibit high area distortions-see figures 7 and 11. Our aim is to establish a mapping which fairly balances angle and area distortion, a desirable property for the spherical case as also discussed in [6]. Furthermore, we wish to be able to process large meshes efficiently. So in order to avoid a costly non-linear optimization over the sphere we wish to perform the optimization in the plane and reduce it to a linear problem.

4 Secondary parameterization

Most recently [19, 20] introduced tensorial quasi-harmonic maps for improving the distortion of planar parameterization. In the same spirit we develop a method for improving the initial spherical parameterization by incorporating a measure for spherical distortion. Arguably the most natural choice to quantify such distortion is the Jacobian of the mapping. However it is not obvious how to incorporate this 3×3 tensor into the current two dimensional curvilinear coordinate setting. Here, we restrict ourselves to the determinant of the Jacobian of the spherical mapping which

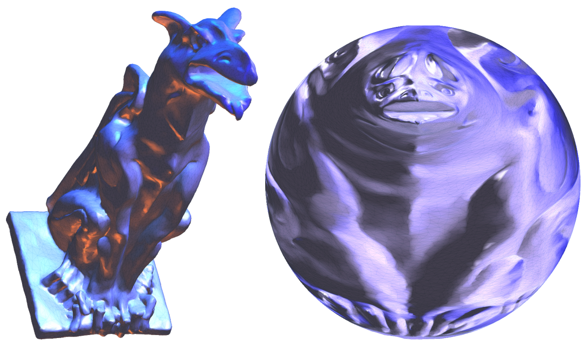


Figure 3. Spherical parameterization of the gargoyle2 model 50KΔ, runtime 7s.

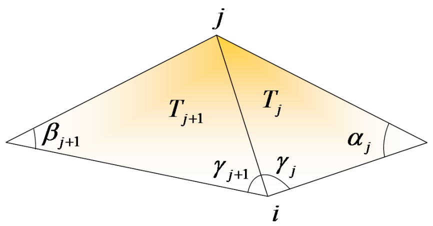


Figure 4. Local configuration for an edge (i, j). The coefficients κ_j and κ_{j+1} are associated with the triangles T_j and T_{j+1}.

quantifies area distortion. On each triangle T of the input mesh $\mathcal{M}$ we define

$$\kappa_T = \left(\frac{\sum_{T_i \in \mathcal{M}} A(T_i)}{\sum_{T_i \in \mathcal{M}} A'(T_i)} \right) \frac{A'_T}{A_T} \,,$$

where A'_T measures the area of T on the sphere, and A_T corresponds to the area on the initial surface.

The secondary mapping can be then obtained as the solution of the the scalar quasi-harmonic equation

$$\textbf{div}(\kappa\, \textbf{grad}\, U) \;=\; 0\,.$$

in terms of the pair (θ, ϕ) with similar boundary conditions as for the initial mapping. In our discrete setting, where the support of the Laplacian operator is restricted to the 1-ring of a vertex, the parameterization problem reduces to solving the following equation for all internal vertices

$$\sum_{j \in \mathcal{N}_i} w_{ij} (\mathbf{U}_j - \mathbf{U}_i) \quad = \quad 0\,.$$

A direct discretization based on defining linear basis function over the triangles yields

$$\frac{1}{2} \sum_j (\kappa_j \cot \alpha_j \;+\; \kappa_{j+1} \cot \beta_{j+1})(\mathbf{U}_j - \mathbf{U}_i) \;=\; 0\,.$$

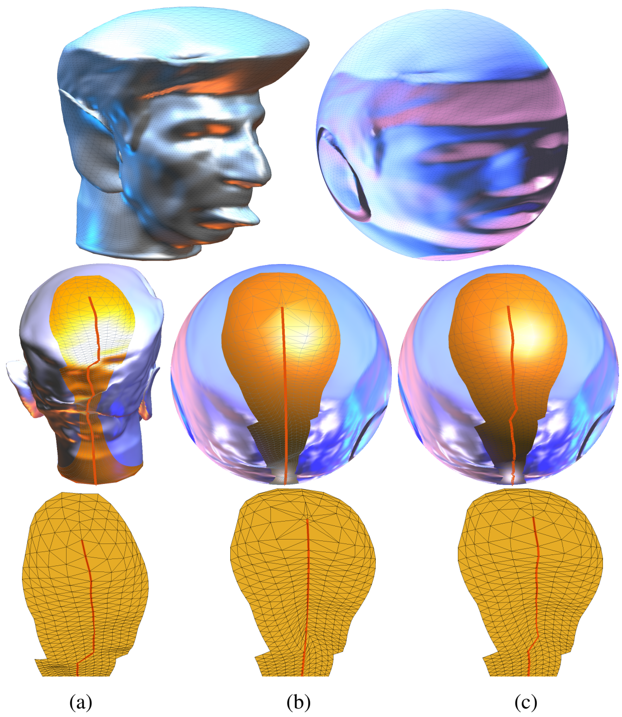


Figure 5. Effect of local distortion reduction along the date line. The highlighted region shows a five-neighborhood of the date line for the original model (a), before (b) and after (c) local improvement (close-ups in bottom row). Only this region with fixed boundary is used for relaxation.

Where scalars κ are defined per triangle, i.e., κ_j corresponds to triangle T_j. In the above expression we can recognize the cotangent weights (see, e.g., [5, 3]) generally associated with the discretization of the Laplacian when κ is constant over the mesh. Alternatively an extension of the mean value coordinates [4], yields

$$\sum_j \left[\kappa_j \,\tan \frac{\gamma_j}{2} + \kappa_{j+1} \tan \frac{\gamma_{j+1}}{2} \right] \frac{(\mathbf{U}_j - \mathbf{U}_i)}{r_j} \;=\; 0\,.$$

If the function κ is constant over the mesh, the above expression reduces to the mean value coordinates. The later coordinates are insensitive to the quality of the triangulation in the sense that they are guaranteed to be positive.

In this framework, the resulting solution can be further improved by reiterating the same process until convergence. For all our results two to three iterations revealed to be largely sufficient.

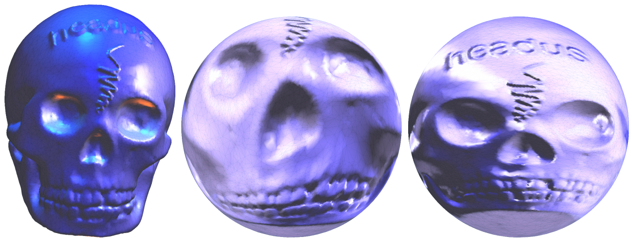


Figure 6. Comparison of spherical parameterization results for the skull model 40K△. Center: using [17]. Right: our method. Overall runtimes are 132 and 5s, respectively.

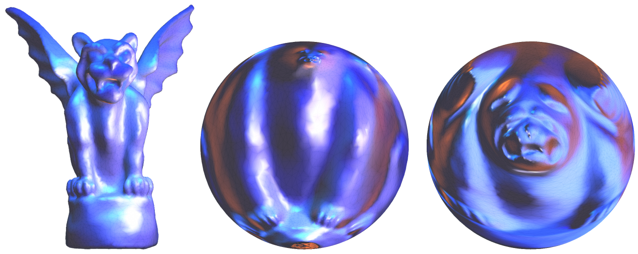


Figure 7. Comparison of spherical parameterization results for the gargoyle model 20K△. Center: [13, 1] using mean value discretization. Right: our method.

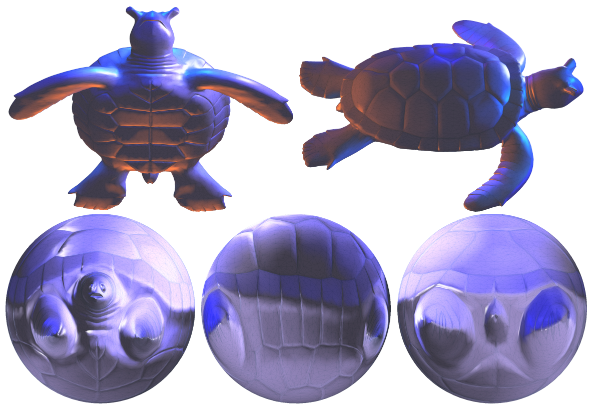


Figure 8. Different views of the turtle model 38K△, and its spherical parameterization viewed from the front, the side, and the back.

5 Local domain distortion reduction

As our spherical mapping stems from lifting a quadrilateral patch onto the sphere, the result is expected to have a higher distortion around the poles and along the date line (see figure 5). In order to overcome such artifacts, we define a sub-patch around the date line by choosing, e.g., third to fifth order neighborhood (five rings are used in our implementation), and we perform tangential Laplacian smoothing on the sub-patch. Then the algorithm reads as simple as

1. For each vertex i of the sub-patch
   - $\mathbf{x}_i := \mathbf{x}_i + [\mathcal{L}(\mathbf{x}_i) - \langle \mathcal{L}(\mathbf{x}_i), \mathbf{x}_i \rangle \mathbf{x}_i]$ (update)
   - $\mathbf{x}_i := \mathbf{x}_i / ||\mathbf{x}_i||$ (back projection)
2. Repeat step (1) until convergence.

In all our experiments three iterations revealed to be sufficient. A simple choice of $\mathcal{L}(\cdot)$ would be to use the uniform tangential Laplacian operator. However such operator ignores the geometry of the mesh and may cause feature to fade out along the sub-patch. Since the positivity of weights is crucial for the tangential Laplacian, the mean value based operator seems to be more appropriate in this step.

Furthermore, since the boundary of the sub-patch is fixed, there is *no risk* of slippage or folding and collapsing of the mesh onto one region of the sphere. Such limitations are in fact common to spherical parameterization methods based solely on the tangential Laplacian operator, and they do not apply here.

6 Results and discussion

We conducted experiments on a variety of meshes. The implementation of our method needs only a simple modification of existing planar mesh parameterization methods. All results are bijective maps and reflect a good balance between area and angle distortion as illustrated in figures 1, 3, 8, and 9. Since our method is based on a composition of several maps, the validity of each them guarantees the validity of the whole map. In theory, our extended mean value based weights are all positive and thus guarantee a bijective map. Furthermore, our tangential smoothing on the sphere is only local and thus there is no risk of mesh slippage. This way, all our intermediate maps are valid and thus the resulting map is guaranteed to be bijective. Typical timings of our method are in the order of a few seconds for meshes with tens of thousand of triangles. This confirms that our approach is significantly more efficient than preexisting methods.

We compare our results to the results of tangential smoothing methods (which lack convergence guarantees in practice) in figure 7, and to the practical approach of [17] in figure 6. We second the visual inspection of the figures with numerical charts comparing their respective area and angle distortion, see figures 10 and 11. We compute the angle distortion as the ratio of angles of the result to the input. The area distortion is computed similarly and scaled accordingly by the ratio of total areas. In the polar charts, the values are placed using the distortion value as the radius and the triangle or angle index as the polar angle.

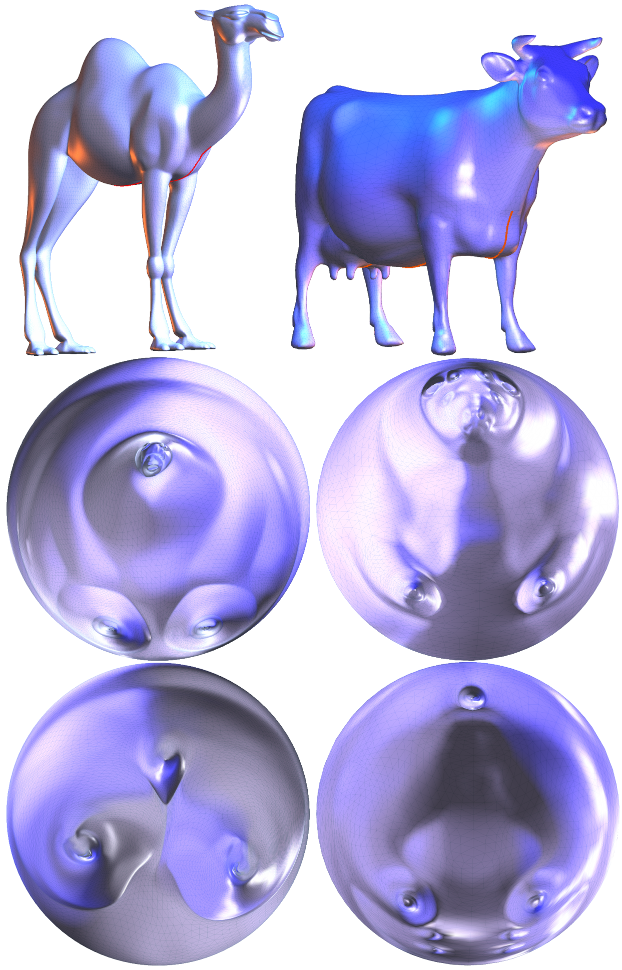


Figure 9. Features such as the long legs of the camel model 78K△ and the legs and horns of the cow model 23K△ are extreme challenges for most spherical parameterization methods. Our method efficiently embeds the models on the sphere.

At the current stage, the choice of the poles is not automatic. We do not see this as limitation as it can be useful for aligning spherical maps. In the following, we provide simple guidelines for defining the date line. The poles and the date line should in general reflect the symmetry of the models when they exhibits symmetry traits. The date line should be as straight as possible and the poles should have sufficient distance to allow the surface to evolve correctly but not too far away to cause additional stretch. For all our examples, we used the Dijkstra algorithm for the computation of shortest paths.

We plan to investigate the optimal choice of poles and date lines in a future work.

Acknowledgements

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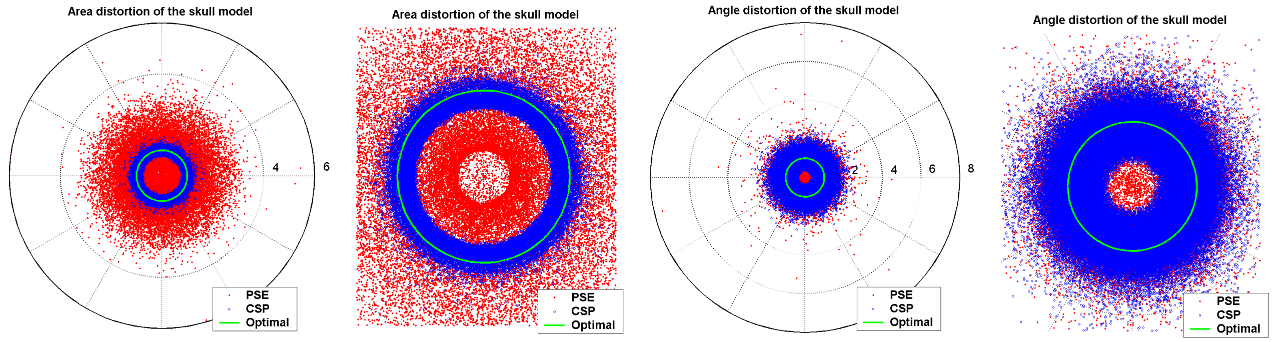

Figure 10. Comparison of the distortion induced by the method of [17] in red and our method in blue for the skull model. The quality of a map is based on how well distortion values localize around the optimal solution (green). Left: Area distortion and a zoom-in on the optimal region. Right: Angle distortion and a zoom-in.

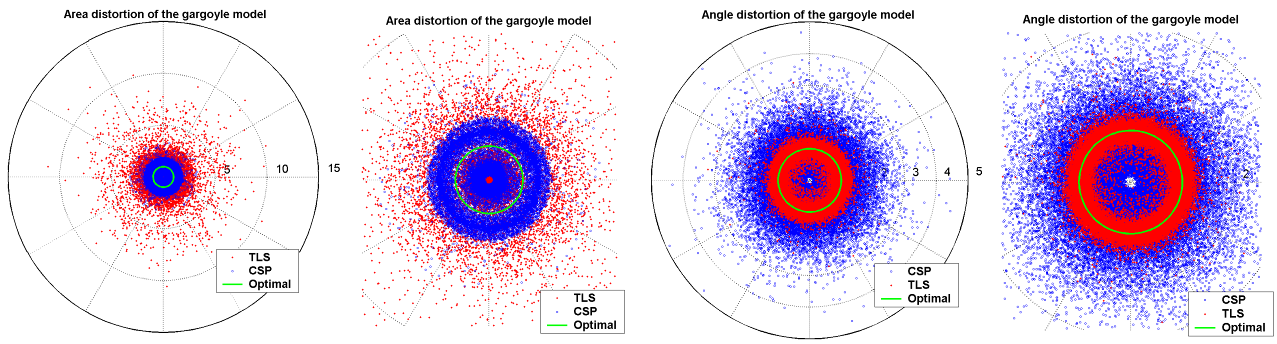

Figure 11. Comparison of the distortion induced by tangential smoothing [13, 1] in red and our method in blue for the gargoyle model. The quality of a map is based on how well distortion values localize around the optimal solution (green). Left: Area distortion and a zoom-in on the optimal region. Right: Angle distortion and a zoom-in.

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