Discrete Tensorial Quasi-Harmonic Maps

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Abstract

We introduce new linear operators for surface parameterization. Given an initial mapping from the parametric plane onto a surface mesh, we establish a secondary map from the plane onto itself that mimics the initial one. The resulting low-distortion parameterization is smooth as it stems from solving a quasi-harmonic equation. Our parameterization method is robust and independent of (the quality of) the initial map.

1. Introduction

The parameterization of surfaces is a cornerstone to various applications in computer graphics such as texture mapping, resampling, and simulation. Without loss of generality parameterization can be regarded as establishing a bijective mapping between a given surface and a parametric domain. In the following, we consider surfaces homeomorphic to a disk which are mapped onto the plane.

The main concern of surface parameterization methods is the reduction of parametric distortion as any attempt to flatten a surface onto the plane generally induces a certain amount of distortion. In fact, basic differential geometry reveals that isometric or length-preserving mappings exist only for the rare special case of surfaces having the same local intrinsic geometry as the plane. In general, a mapping is locally characterized by the first fundamental form, a 2×2 tensor which tells how distances (and hence angles and area) measured in the parametric domain are translated to distances on the surface. Hence, all information on the distortion is captured by this tensor.

Besides from providing a least-distorted mapping, a good parameterization method should be efficient for processing large surface meshes and robust, i.e., insensitive to highly irregular input. These requirements make linear parameterization methods particularly interesting. They are efficient as they only require the solution of a sparse linear system. They have proven to be robust as they guarantee a valid solution for appropriate boundary conditions. Furthermore, their relative simplicity and ease of implementation makes them highly attractive.

A major drawback of most existing linear methods is the little to no control over distortion: The generally observed scaling in triangle area when moving away from the boundary is often not acceptable as a certain amount of area preservation is crucial for many applications.

In this paper, we develop a new and general approach to efficiently reduce the parametric distortion. In brief the main contributions are

- An efficient, robust linear parameterization method.
- The construction of a quasi-harmonic map based on a tensor field, which is derived from an initial mapping.
- The discretization of the linear operator.

In contrast to most linear methods mainly relying on the Laplace equation, our work is based on the crucial observation that the general quasi-harmonic equation can account for both, area and angle distortion. In this construction a tensor field captures the mapping of each triangle and guides flattening process. Thus reducing the parametric distortion while maintaining smoothness of the solution. We derive linear operators for the discretization of the problem. To our knowledge, these operators have not been constructed before, and quasi-harmonic maps have not been taken into account in the parameterization literature. Both ingredients provide a sound theoretical framework of our approach.

The variational nature of our setting guarantees the smoothness of the parameterization. The existence and uniqueness of a solution to the arising linear system is discussed, and guarantees on the validity of the mapping are provided. Similarly to conformal maps, the quasi-harmonic map is independent of the resolution and the regularity of the input mesh.

2. Background

2.1. Overview of Related Work

The importance of parameterization techniques for computer graphics is reflected by the significant amount of work

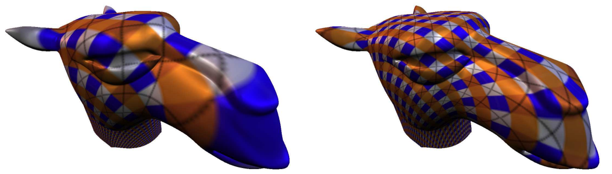


Figure 1. Texturing a camel head model ($11K\Delta$) using a discrete conformal map ([19, 6]; left) and our quasi-harmonic map (right).

on the topic in the last years, see [10] for an extensive recent survey. In the following, we briefly overview related work.

Most of the research effort in this area aims at controlling and reducing distortion by minimizing a certain deformation energy. For several approaches, the arising numerical problem is non-linear as e.g. in [5, 14, 23, 24, 29]. These methods commonly require hierarchical solvers [15, 22] even for moderately sized meshes and are computationally involved in general. A quasi-Newton type optimization to reduce parametric stretch was proposed in [28]. An extension to the shape preserving weights [8] was introduced in [12] by using an additional anisotropic stretch term. Desbrun et al. [6] simplify the problem to a univariate non-linear optimization and linearly blend base parameterizations. An alternative approach [26] avoids global optimization problems by simultaneously cutting the surface and computing the parameterization. In this paper we consider surfaces homeomorphic to a disk and do not allow additional cuts. Higher genus surfaces can be partitioned into appropriate patches which are parameterized separately, see e.g. [19].

Surface parameterization can be addressed as a linear problem by solving the Laplace equation. This leads to the discrete conformal maps [7, 13, 20] possibly with Dirichlet or Neumann boundary conditions [6, 19]. An interesting approach was proposed by Lévy [18] which allows the user to specify point constraints interactively. A method based on Laplacian smoothing of regular grid covering the planar domain of the initial mapping was introduced in [25]. Most recently, Floater introduced a convex combination mapping based on the mean value property of harmonic functions [9] as a superior alternative to the shape preserving method [8].

The discrete conformal maps attempt to preserve angles and allow scaling, i.e., the associated first fundamental form would be a scaled version of the identity matrix in an ideal setting. Several of the non-linear approaches attempt to minimize certain matrix norms ([5, 14]) or a stretch defined in terms of the eigenvalues of the metric tensor ([23]). In contrast to these, our approach takes into account and derives the tensor as a whole.

2.2. Discrete Setting

A triangular surface mesh $\mathcal{S}$ is described as the pair $(\mathcal{K}, S)$, where $\mathcal{K}$ is a simplicial complex representing the connectivity of vertices, edges and faces, and $S = (X_1, \ldots, X_n)$, where $X_k \in \mathbb{R}^3$ refer to the geometric positions of the vertices.

We represent a *parameterization* of surface as an isomorphic mesh $\mathcal{U} = (\mathcal{K}, (U_1, \cdots, U_n))$, where $U_k = (u_k, v_k)^\top \in \mathbb{R}^2$, $1 \leq k \leq n$, denotes positions in the (planar) parameter domain.

We define the *1-ring neighborhood* of a vertex $i \in \mathcal{K}$ as the set of adjacent vertices $\mathcal{N}_i = \{j | (i,j) \in \mathcal{K}\}$ directly connected to i by an edge. Linear parameterization schemes can be interpreted as solving the Laplace equation. For the mapping function g from the surface to the plane we have the two-dimensional equation

$$\nabla^2 g = 0 . \tag{1}$$

The driving principle behind this setup can be interpreted as the equilibrium state of a membrane, which conforms to solving the Laplace equation given appropriate boundary conditions. While this topic has motivated a wide range of research and applications in mathematics and physics (see e.g. [3] and the references therein), its formulation in a modern discrete form goes back at least to the famous 1928 paper of Courant, Friedrichs, and Lewy [4] on the discretization of partial differential equations. (The discretization of complex processes may date back to Bernoulli's treatment of the brachistochrone problem.) This was the origin of many modern discretization methods such as finite differences and finite elements. Some of the early work on the computation of conformal maps and the discretization of differential operators can be consulted in [2, 27].

In our context, where the support of the Laplacian operator is restricted to the 1-ring of a vertex, the parameterization problem reduces to solving the following equation for all internal vertices

$$\mathcal{L}(U_i) = \sum_{j \in \mathcal{N}_i} w_{ij}(U_i - U_j) = 0 . \tag{2}$$

Given appropriate conditions on the boundary and on the weights $w_{ij}, j \in \mathcal{N}_i$, the resulting parameterization is guaranteed to be a one-to-one mapping, see e.g. [10]. The parameterization depends closely on the choice of weights used for the discrete operator $\mathcal{L}$.

3. Motivation

Conformal maps in the strict Riemann sense preserve angle measures continuously. In a discrete setting, this property is no longer valid as non-planar 1-rings cannot be

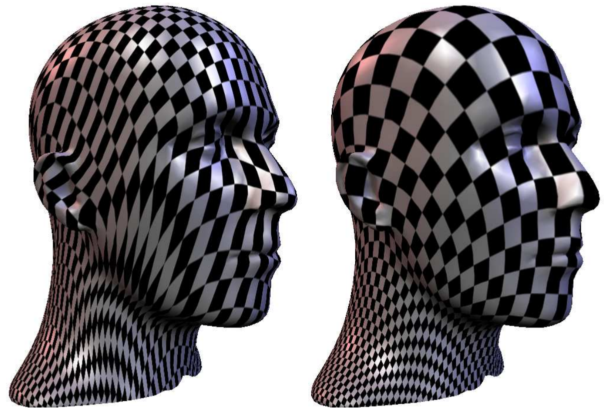


Figure 2. The figure compares results obtained for the *mannequin head* using [28] (left) with the tensorial quasi-harmonic map (right).

mapped conformally to the plane. Furthermore, a conformal map in its canonical form does not preserve measures other than angles. As a typical observation, the distortion grows as one moves away from the boundary – the more complicated the geometry of the mesh, the more significant the distortion. Linear parameterization methods establish a discrete form of conformal maps exhibiting nice mathematical properties. However, they might not be suitable for a wide range of practical applications due to the lack of control over area distortion. Fig. 1 illustrates this situation.

Given the scarcity of isometric maps which preserve angles and areas at the same time, a good direction of investigation is to establish smooth blends of both kinds – authalic and conformal parameterizations. An elegant approach is proposed in [6] yielding locally area preserving properties. In [28] an alternative approach is proposed, which uses a spring analogy to set up a weighted system of linear equations. The weights are intended to account for the distortion measure proposed in [23], and the iterative quasi-Newton type optimization method reduces the computational complexity of the original problem to a great extent. However, there is no guarantee of convergence to a minimum. In practice, divergence of the method is observed, and the results may even degenerate after a number of iterations.

Our goal is to derive the general construction of a class of maps that offer both, good area and good angle preserving properties smoothly across the mesh. Our approach to the problem stems from the following question: Given a map g from the plane to a surface mesh, can we find a mapping f from the plane onto itself that has the same properties as the initial map (see Fig (3)). Finding such a map would yield an optimal parameterization that mimics the original mapping but this time from the plane into itself. In the following we define the required properties of such maps, and we propose a method for establishing them. In brief, the basic approach can be overviewed as follows:

1. Provide an initial map g, e.g., a discrete conformal parameterization. (Any non-degenerate setting without overlapping boundaries would work, like, e.g., a reasonable projection.)
2. For every triangle, estimate the Jacobian matrix C of the map f from the first fundamental form of g. (Section 4)
3. Building upon our discretization of the quasi-harmonic equation, compute weights w_{ij} and set up a linear system. (Sections 5 and 6)
4. Solve the arising linear system. The same boundary conditions as in step (1) are used.
5. If required, iterate restarting from step 2.
(Note that our method stabilizes after few iterations and would not degenerate.)

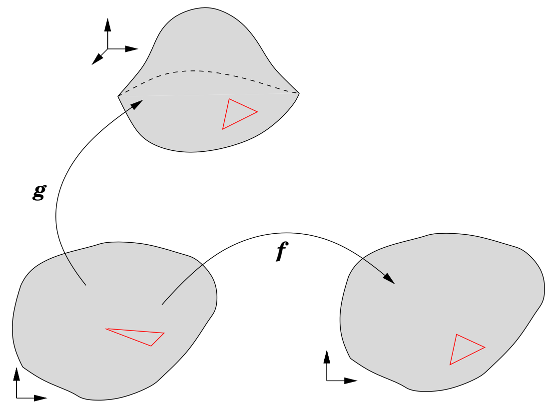


Figure 3. g maps the planar domain to the surface, and f is a mapping of the plane onto itself.

4. Quasi-Harmonic Maps

In order to motivate the ensuing discussion, we start with the following simple problem: Given a single triangle of a surface mesh and a map g that carries the triangle into a corresponding triangle in the planar domain. Solely from the properties of the map g, we can establish a second map f that reproduces exactly the initial triangle on the plane from the second triangle. Formally, the problem can be coined as follows. Recalling basic theory of differential geometry (see e.g. the introduction in the survey [10] and references therein), the deformation of the initial mapping is given by

$$dX = JdU , \tag{3}$$

where the Jacobian J is a 3x2 tensor. Taking the norm we get

$$\|dX\|^2 = dU^\top J^\top JdU . \tag{4}$$

In this expression, we already recognize the first fundamental form $\mathbf{I} = J^\top J$. An ideal mapping in the plane, which exactly mimics the behavior of the initial mapping, would induce the same distortion of lengths as this initial mapping, i.e.,

$$\|dx\|^2 = dU^\top \mathbf{I} dU , \tag{5}$$

where x denotes a displacement in the plane. Now, we would like to derive an expression similar to (3) – this time in the plane – i.e.,

$$dx = CdU . \tag{6}$$

Here, the Jacobian C of the new mapping is a 2×2 matrix which should satisfy

$$\mathbf{I} = C^\top C . \tag{7}$$

Since the first fundamental form is symmetric positive definite, the matrix C can be found as the square root of $\mathbf{I}$. Note that $\mathbf{I}$ can be computed from the Jacobian of the mapping. For a concise formulation of the square root of a 2×2 matrix we refer to [17]. This derivation of the Jacobian C is original compared to existing parameterization methods where emphasis is solely on the eigenvalues and the related norms.

So far, we illustrated the basic idea and considered only a single triangle. Given a surface mesh, immediately the question arises, whether we can compute the vertex positions of the new mesh from the Jacobian matrices of the transformations defined over each triangle. It turns out that the problem is generally overdetermined and may not have a solution at all. This does not come as a surprise due to the fact that isometric mappings exists only for the special case of developable surfaces.

Excluding a direct approach to the problem, we propose an alternative to account for the Jacobian matrix. Recalling that harmonic maps minimize the Dirichlet energy

$$\int_\Omega \|\nabla g\|^2 , \tag{8}$$

our approach attempts to recover the geometry of the original mesh by minimizing the following energy functional

$$\int_\Omega (C\nabla f) \cdot (\nabla f) . \tag{9}$$

The partial differential equation associated with this energy is the following quasi-harmonic equation

$$\mathbf{div}(C\mathbf{grad}f) = 0 . \tag{10}$$

This equation and its many variants are typically used for modeling many steady state problems in mechanics and electromagnetism (see e.g. [30]). Say, the distribution of vertices is given by f, the flux of the distribution f is given by C $\mathbf{grad}f$, and due to conservation law this type of problem conforms to solving the two-dimensional elliptic steady state equation (10). And consequently this guarantees that our solution will show the same qualitative behavior as general elliptic problems. In particular the solution will not show jumps but will vary smoothly.

Under mild conditions the solution to the problem exists and is unique, see [11] (Appendix I) for a detailed analysis of the convergence properties. In our case the symmetric nature of the tensor C and given that it does not stem from a degenerate initial parameterization (triangle collapsing to one point) guarantees that a solution to the quasi-harmonic equation exists and is unique.

Moreover, we note that our setup is a special case of the Leray-Lions equation [16]. A geometric interpretation of such maps can be found in [1].

5. Discretization

In the following we solve the variational problem associated with the quasi-harmonic equation (10). We refer to vertices and triangles as explained in Fig. 4 (a).

In dealing with equations of this type, it is much more advantageous to tackle the integral form rather then the partial differential equation itself. This is not fortuitous as the Gauss divergence theorem yields a much easier expression to handle

$$\int_A \mathbf{div}(C\mathbf{grad}f)dA = \oint_{\partial A} (C\mathbf{grad}f) \cdot n \; dl = 0 . \tag{11}$$

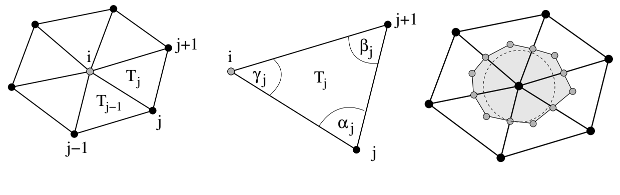
Hence, the integral form of the quasi-harmonic equation obtained by integrating over some non-overlapping regions reduces to a simple integration of the gradient over their boundaries. We define the non-overlapping areas for the integration as the dual mesh defined by the centroids of the triangles and the midpoints of the edges over every 1-ring neighborhood (see Fig. 4 (b)). We remark that this dual has exactly the same area as the mesh.

The gradient on a triangle $\{i, j, j+1\}$ associated with the edge $\{i, j+1\}$ is given by

$$\mathbf{grad}_T(f_{j,j+1}) = \frac{\mathbf{x}^\perp_{i,j+1}}{2A_T} ,$$

where $\mathbf{x}^\perp_{i,j}$ denotes the edge vector rotated by $\frac{\pi}{2}$ (see Fig. 5).

Equation (11) involves only the normal to the boundary of the ring which is just the normalized boundary edge rotated by $\frac{\pi}{2}$, so given an edge and its two adjacent triangles



(a) (b)

Figure 4. (a) Indexing of direct neighbors j and associated triangles in the 1-ring of the center vertex i, angles are indexed per triangle j. As all derivations are local to the 1-ring, we avoid double indices, so implicitly $T_j = \{i, j, j+1\}$. (b) The integration area for the discretization of (11) is given by the dual mesh. For the mean value coordinates the integration area is a circle.

$T_{j-1} = \{i, j-1, j\}$ and $T_j = \{i, j, j+1\}$, the weight associated the edge $\{i, j\}$ can be written in the following form

$$w_{ij} = \frac{\mathbf{x}^{\perp}_{j-1,i} C_{j-1} \cdot \mathbf{x}^{\perp}_{j,j-1}}{4A_{j-1}} + \frac{\mathbf{x}^{\perp}_{j+1,j} C_j \cdot \mathbf{x}^{\perp}_{i,j+1}}{4A_j}, \quad (12)$$

In the discrete setting, the tensor field C is defined piecewise per triangle and can be computed in a straightforward manner using simple algebra. From equation (12). It turns out that when the tensors C_{j-1} and C_j associated respectively with triangles T_j and T_{j-1} are both equal to the identity, this expression simplifies to the well-known cotangent weights, which are widely used for the computation of the discrete Laplacian operator. We note that in this special case the initial mapping is isometric, and hence the solution will be identical to the initial map.

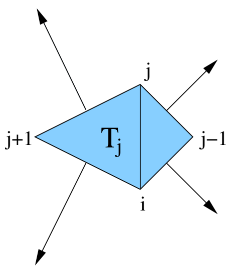


Figure 5. The two triangles adjacent to edge (i, j). The outward vectors $x^{\perp}$ correspond to respective edges rotated by $\frac{\pi}{2}$.

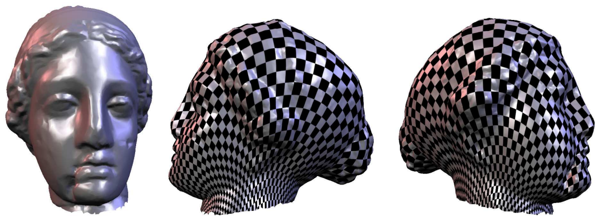


Figure 6. The tensorial discrete quasi-harmonic map of the *venus* model is visualized by applying a checker board texture.

6. Mean Value Coordinates extension

A sufficient condition for guaranteeing the validity of the approach is the positivity of the weights in equation (12), formally expressed as the discrete maximum principle. For solving the Laplace equation based on the cotangent weights, this means that the Delaunay criterion should hold for all internal vertices while all angles facing boundary edges should be acute. Given appropriate boundary conditions this condition is sufficient to guarantee the construction of a valid mapping. However, in general it is not necessary, and in many instances the arising system may have a valid solution even in the presence of negative weights. For the quasi-harmonic setting described above, the Delaunay condition is no longer sufficient. In fact, the validity is guaranteed only if *all* angles are acute.

One way to satisfy this condition (or at least the Delaunay criterion) is, for instance, to apply Rivara's edge bisection algorithm [21], which is guaranteed to keep angles away from 0 and π.

Another way is the use of alternative weights. A promising class of weights are Floater's *mean value coordinates* [9], as they guarantee positivity regardless of the interior angles of the triangle mesh.

For the sake of completeness, we will show how to derive a new class of discrete quasi-harmonic maps that are based on the mean value coordinates. Here, we note that for a general tensor C, rather complicated trigonometric expressions have to be integrated over circular regions resulting in a comparatively complicated formula for the weights. For ease of explanation and for the sake of a concise final expression, we restrict ourselves in this section to the case where C is a scaled unit tensor, formally $C = \kappa I$. (In fact, this is the special case of a scalar version of the approach, for example, κ can be taken as the per triangle ratio of the initial map and the original mesh.) Note that all our numerical examples for the mean value coordinates are computed with general tensors, and we refer to Appendix A for the general formula.

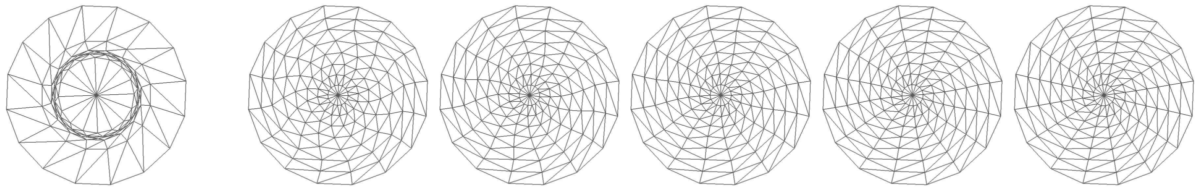


Figure 7. Self-correction and convergence for a regular half-sphere model. An ill-shaped initial map (left) is chosen. The series on the right shows the quasi-harmonic maps obtained from the 1st to 5th iteration.

We proceed similarly to [9], however, we discretize the quasi-harmonic equation again starting from the divergence theorem. For this purpose, we first compute the gradient: Given a triangle $T_j = \{i, j, j+1\}$ with vertex positions $\mathbf{u}_i$, $\mathbf{u}_j$, $\mathbf{u}_{j+1}$ and a function f defined over this triangle, the following relations hold for a point $\mathbf{u} = (u, v)$ in the triangle. Let γ_j denote the angle at u_i. In local polar coordinates

$$\mathbf{u} - \mathbf{u}_i = r\,(\cos\gamma, \sin\gamma)^\top\,, \tag{13}$$

and for f

$$f(\mathbf{u}) - f(\mathbf{u}_i) \;=\; \mathbf{grad} f \cdot (\mathbf{u} - \mathbf{u}_i)\,,$$

where the gradient is given by

$$\mathbf{grad} f \;=\; H_T(\mathbf{u} - \mathbf{u}_i)\,.$$

Here, the value of H_T can be found by applying a simple linear interpolation over the triangle T, we get

$$H_T \;=\; \frac{\sin(\gamma_j - \gamma)}{r\,r_j\,\sin\gamma_j}(f(\mathbf{u}_j) - f(\mathbf{u}_i)) \;+\; \frac{\sin(\gamma)}{r\,r_{j+1}\,\sin\gamma_j}(f(\mathbf{u}_{j+1}) - f(\mathbf{u}_i))\,.$$

Where r_j and r_{j+1} denote the length of $u_i - u_j$ and $u_i - u_{j+1}$ respectively. Now substituting this expression in (11), and keeping in mind that the gradient and the outward normal to the circle are collinear, we obtain the following expression

$$\int_0^{2\pi} \kappa\,\mathbf{grad} f \cdot n d\gamma \;=\; \sum_j \int_0^{\gamma_j} \kappa_j H_{T_j}\, r\, d\gamma\,, \tag{14}$$

and expanding the right hand side we obtain

$$\sum \kappa_j \int_0^{\gamma_j} \frac{\sin(\gamma_j - \gamma)}{r_j\,\sin\gamma_j}(f(\mathbf{u}_j) - f(\mathbf{u}_i)) \;+\; \frac{\sin(\gamma)}{r_j\,\sin\gamma_j}(f(\mathbf{u}_{j+1}) - f(\mathbf{u}_i))\,.$$

(Notice that the r term was simplified.) After integration and reassembly of terms, we get the following

$$\sum_j \left[\frac{1 - \cos\gamma_j}{\sin\gamma_j}\kappa_j + \frac{1 - \cos\gamma_{j+1}}{\sin\gamma_{j+1}}\kappa_{j+1}\right] \frac{(f(\mathbf{u}_j) - f(\mathbf{u}))}{r_j} \;=\; 0\,. \tag{15}$$

If κ is constant over the mesh, we can recognize the mean value coordinates in the above expression. We note that for this derivation, the dual mesh can be used as circles around the vertices (see Fig. 4), and since the arising equations are independent of the radii of the circles, we conjecture a possible connection to circle packing. Consequently, the integration area is smaller than the total area of the mesh. This explains the loss of conformality accumulated by this discretization method, which is here traded for robustness.

7. Discussion

Our approach explicitly accounts for directions as well as distortion. This is due to the newly introduced planar Jacobian that mimics the reference Jacobian. In fact, this means that we can start from any initial parameterization, possibly suffering from high distortion, as the use of the geometric tensor defined over the mesh implicitly yields a self-correcting scheme. This is illustrated by a small example in Fig. 7. For real models, we can even start from a projection (eventually forcing a convex boundary) onto a least-squares plane – ignoring foldovers. All the examples were computed in this manner, showing the robustness to high-distortion for complex data.

Successive solutions of weighted linear systems have been used for surface parameterization in [28]. According to the authors, their method diverges towards larger distortion after few iterations. We observe that our method does not degenerate while iterating, rather it stabilizes, and the result does not change after few (3-5 in the examples) iterations. We believe that this difference arises from the fact that although the spring analogy gives a nice interpretation of differential equations, it is not always intuitive to derive accurate discretizations, like for the discrete conformal parameterization, solely based on this analogy.

It is worthwhile to note that taking powers of the tensor C extends the range of possible mappings.

8. Results

Fig. 11 compares the distortion of angles and triangle areas for discrete conformal maps, stretch minimization using [28] and the quasi-harmonic map for the *mannequin head* model (Fig. 2). In the graphs, the distortion on the horizontal axis is plotted over the sequence of mesh elements, showing a peak for every triangle. The optimal ratio is 1. The figure shows that the quasi-harmonic map preserves conformality to a great extent while the area distortion is effectively reduced.

We applied our parameterization method to a variety of reasonably complex, non-trivial geometric models, for the planar setting. The Figures 1,2, 6, and 8-10 show examples, where the map is visualized either by mapping a regular checker texture, or the planar image is rendered with the original shading. The solution of the linear system within one iteration is in the order of seconds as it is typical for solving this class of problems on current hardware. Three to five iterations have been used as described above.

9. Conclusion

We presented a new low-distortion surface mesh parameterization method based on solving a discrete quasi-harmonic equation. For developing the method, we introduce new linear operators, which capture the metric properties of the surface. The variational nature of the approach ensures the smoothness of the resulting map. Our parameterization method is robust and insensitive to possible ill configurations of the initial map. In fact, for most cases the methods converges from a simple projection on the least squares plane even for complex models.

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A. Mean Value Coordinates

In Section 6 we simplified the discretization of the quasi-harmonic equation (10) effectively to the scalar case. Now given a general (symmetric) tensor $C = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$, the integral (14) can be written in the following form again using local polar coordinates

$$\int_0^{2\pi} C\, \mathbf{grad} f \cdot n d\gamma \;=\; \sum_j \int_0^{\gamma_j} r H_{T_j} C \begin{pmatrix} \cos(\gamma) \\ \sin(\gamma) \end{pmatrix} \cdot \begin{pmatrix} \cos(\gamma) \\ \sin(\gamma) \end{pmatrix} d\gamma\,,$$

and we obtain the weights associated with each edge j as

$$\begin{aligned} w_j \;=\; & \frac{1}{3r_j} \left[\, (a_j + 2b_j + 2c_j \sin\gamma_j) \tan\tfrac{\gamma_j}{2} \right. \\ & \qquad \left. + \, (a_j - b_j) \sin\gamma_j \, \right] \\ + \; & \frac{1}{3r_j} \left[\, (a_{j+1} + 2b_{j+1}) \tan\tfrac{\gamma_j}{2} \right. \\ & \qquad + \, 2c_{j+1} \sin\gamma_{j+1}^2 \\ & \qquad \left. + \, (a_{j+1} - b_{j+1}) \sin\gamma_{j+1} \cos\gamma_{j+1} \, \right], \end{aligned}$$

where again indices denote edges and associated triangles.

It is easy to see that the above expression simplifies to (15) if $a = b$ and $c = 0$, and consequently if in this case a is constant over the mesh, we obtain Floater's mean value coordinates.

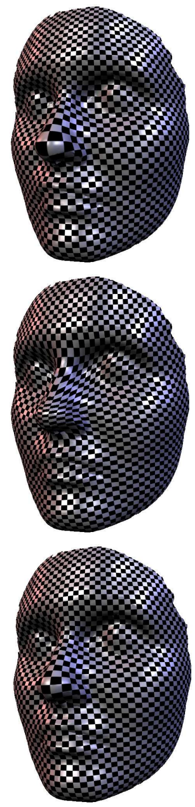


Figure 8. Mapping a face model ($28K\Delta$); from top: discrete conformal map [19, 6], stretch minimization using [28], quasi-harmonic map.

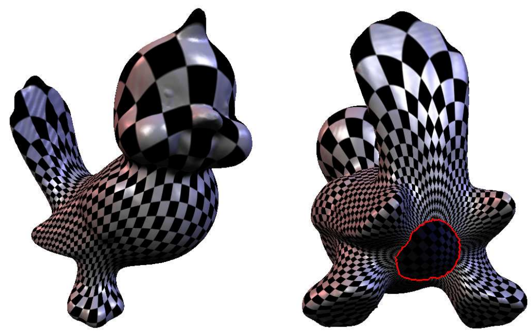


Figure 9. Quasi-harmonic map of the *tweety* ($97K\Delta$) model with a hole (red boundary) cut in its bottom.

Figure 10. Maps of a sculpture ($89K\Delta$). Left: Mean value coordinates. Right: Quasi-harmonic map. The mesh was obtained from a range scan of the wooden sculpture *Freezing Old Woman* (*Frierende Alte*, 1937) by the German expressionist sculptor Ernst Barlach (1870-1938).

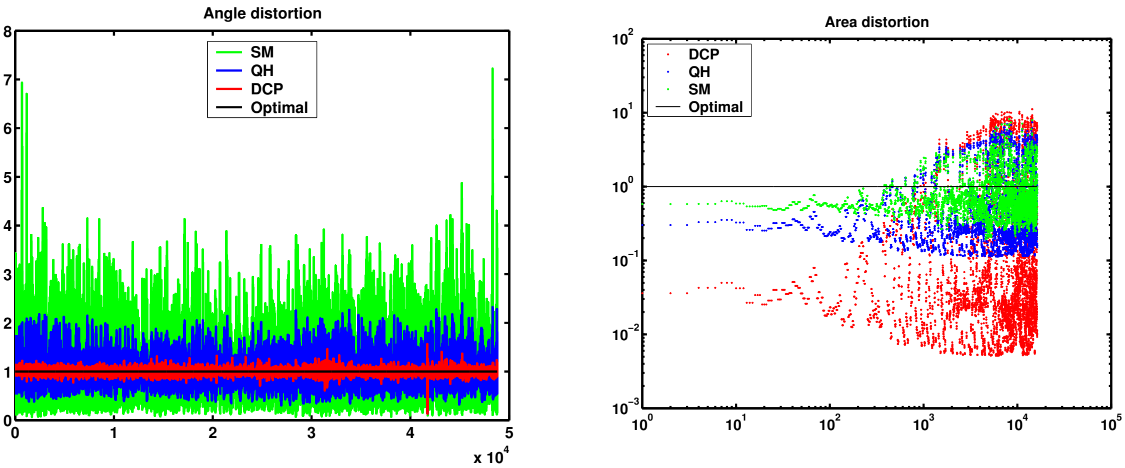
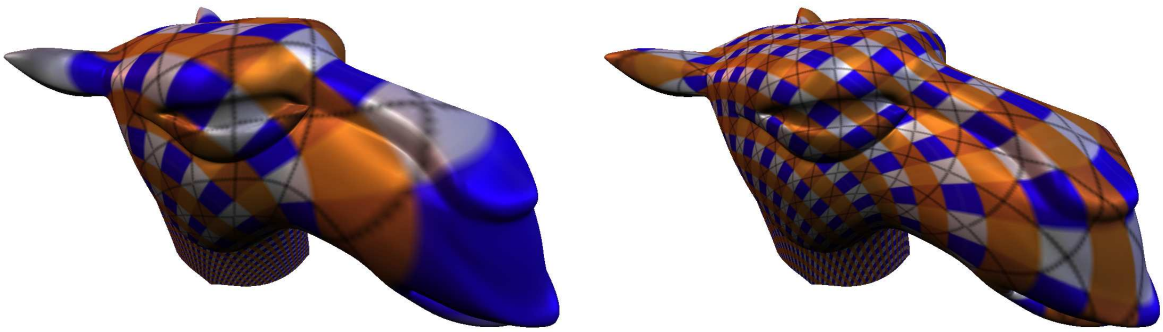
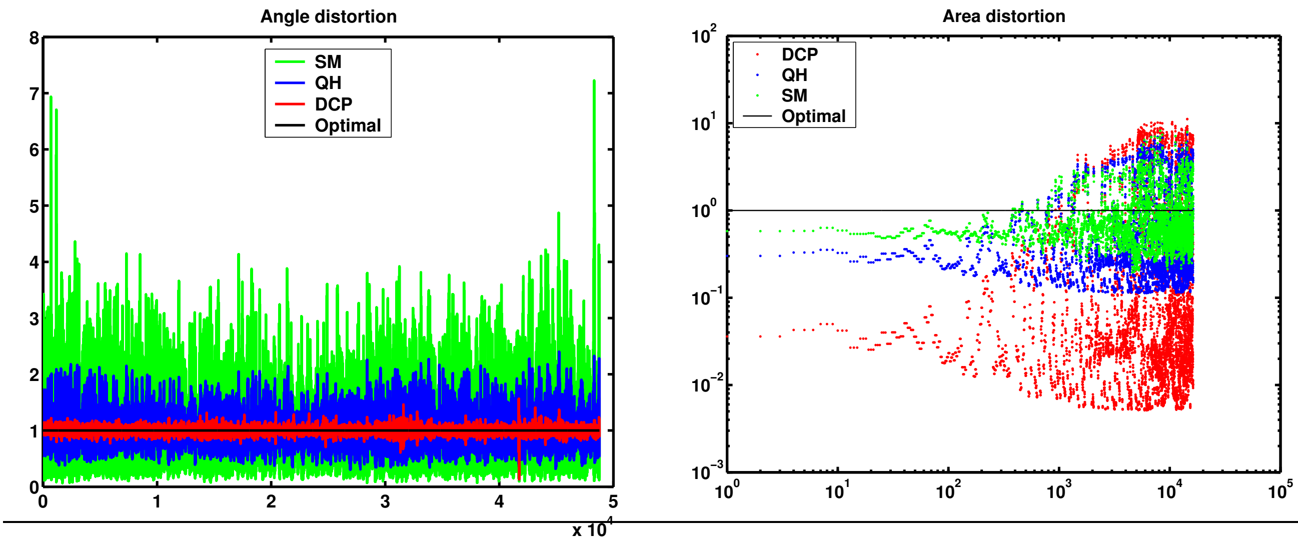


Figure 11. Distortion of angles and triangle areas for the *mannequin head* model. Red: discrete conformal map ([19, 6]). Green: stretch minimization using [28], blue: quasi-harmonic map.




Angle distortion
SM
QH
DCP
Optimal
0 1 2 3 4 5 6 7 8
0 1 2 3 4 5
x 10^4
Area distortion
DCP
QH
SM
Optimal
10^-3 10^-2 10^-1 10^0 10^1 10^2
10^0 10^1 10^2 10^3 10^4 10^5