Abstract: An acyclic edge coloring of a graph is a proper edge coloring such that there are no bichromatic cycles. The acyclic chromatic index of a graph is the minimum number \( k \) such that there is an acyclic edge coloring using \( k \) colors and is denoted by \( a'(G) \). It was conjectured by Alon, Sudakov and Zaks (and earlier by Fiamcik) that \( a'(G) \leq \Delta + 2 \), where \( \Delta = \Delta(G) \) denotes the maximum degree of the graph. Alon et al. also raised the question whether the complete graphs of even order are the only regular graphs which require \( \Delta + 2 \) colors to be acyclically edge colored. In this article, using a simple counting argument we observe not only that this is not true, but in fact all \( d \)-regular graphs with \( 2n \) vertices and \( d > n \), requires at least \( d + 2 \) colors. We also show that \( a'(K_{n,n}) \geq n + 2 \), when \( n \) is odd using a more non-trivial argument. (Here \( K_{n,n} \) denotes the complete bipartite graph with \( n \) vertices on each side.) This lower bound for \( K_{n,n} \) can be shown to be tight for some families of complete bipartite graphs and for small values
of \( n \). We also infer that for every \( d, n \) such that \( d \geq 5 \), \( n \geq 2d + 3 \) and \( dn \) even, there exist \( d \)-regular graphs which require at least \( d + 2 \)-colors to be acyclically edge colored. © 2009 Wiley Periodicals, Inc. J Graph Theory 63: 226–230, 2010

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All graphs considered in this article are finite and simple. A proper edge coloring of \( G = (V, E) \) is a map \( c: E \to C \) (where \( C \) is the set of available colors) with \( c(e) \neq c(f) \) for any adjacent edges \( e, f \). The minimum number of colors needed to properly color the edges of \( G \) is called the chromatic index of \( G \) and is denoted by \( \chi'(G) \). A proper edge coloring \( c \) is called acyclic if there are no bichromatic cycles in the graph. In other words, an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in \( G \). The acyclic edge chromatic number (also called acyclic chromatic index), denoted by \( a'(G) \), is the minimum number of colors required to acyclically edge color \( G \). The concept of acyclic coloring of a graph was introduced by Grünbaum [6]. Let \( \Delta = \Delta(G) \) denote the maximum degree of a vertex in graph \( G \). By Vizing’s theorem, we have \( \Delta \leq \chi'(G) \leq \Delta + 1 \) (see [4] for proof). Since any acyclic edge coloring is also proper, we have \( a'(G) \geq \chi'(G) \geq \Delta \).

It has been conjectured by Alon et al. [2] that \( a'(G) \leq \Delta + 2 \) for any \( G \). We were informed by Alon that the same conjecture was raised earlier by Fiamcik [5]. Using probabilistic arguments Alon et al. [1] proved that \( a'(G) \leq 60\Delta \). The best known result up to now for arbitrary graph is by Molloy and Reed [7] who showed that \( a'(G) \leq 16\Delta \).

The complete graph on \( n \) vertices is denoted by \( K_n \) and the complete bipartite graph with \( n \) vertices on each side is denoted by \( K_{n,n} \). We denote the sides of the bi-partition by \( A \) and \( B \). Thus \( V(K_{n,n}) = A \cup B \).

Our Result. Alon et al. [2] suggested a possibility that complete graphs of even order are the only regular graphs which require \( \Delta + 2 \) colors to be acyclically edge colored. Nešetřil and Wormald [8] supported the statement by showing that the acyclic edge chromatic number of a random \( d \)-regular graph is asymptotically almost surely equal to \( d + 1 \) (when \( d \geq 2 \)). In this article, we show that this is not true in general. More specifically we prove the following Theorems:

**Theorem 1.** Let \( G \) be a \( d \)-regular graph with \( 2n \) vertices and \( d > n \), then \( a'(G) \geq \Delta + 2 = \Delta(G) + 2 \).

**Theorem 2.** For any \( d \) and \( n \) such that \( dn \) is even and \( d \geq 5 \), \( n \geq 2d + 3 \), there exists a connected \( d \)-regular graphs that requires \( d + 2 \) colors to be acyclically edge colored.

**Theorem 3.** \( a'(K_{n,n}) \geq n + 2 = \Delta + 2 \), when \( n \) is odd.

Remarks.

1. It is interesting to compare the statement of Theorem 1 to the result of [8], namely that almost all \( d \)-regular graphs for a fixed \( d \), require only \( d + 1 \) colors to be
acyclically edge colored. From the introduction of [8], it appears that the authors expect their result for random \(d\)-regular graphs would extend to all \(d\)-regular graphs except for \(K_n\), \(n\) even. From Theorems 1 and 2 it is clear that this is not true: There exists a large number of \(d\)-regular graphs which require \(d+2\) colors to be acyclically edge colored, even when \(d\) is fixed.

2. The complete bipartite graph, \(K_{n+2,n+2}\) is said to have a perfect 1-factorization if the edges of \(K_{n+2,n+2}\) can be decomposed into \(n+2\) disjoint perfect matchings such that the union of any two perfect matchings forms a hamiltonian cycle. It is obvious from Lemma 1 that \(K_{n+2,n+2}\) does not have perfect 1-factorization when \(n\) is even. When \(n\) is odd, some families have been proved to have perfect 1-factorization (see [3] for further details). It is easy to see that if \(K_{n+2,n+2}\) has a perfect 1-factorization then \(K_{n+2,n+1}\) and therefore \(K_{n+1,n+1}\) has a acyclic edge coloring using \(n+2\) colors. Therefore the statement of Theorem 3 cannot be extended to the case when \(n\) is even in general.

3. Clearly if \(K_{n+2,n+2}\) has a perfect 1-factorization, then \(a'(K_{n,n})=n+2\). It is known that (see [3]), if \(n+2\in \{p,2p-1,p^2\}\), where \(p\) is an odd prime or when \(n+2<50\) and odd, then \(K_{n+2,n+2}\) has a perfect 1-factorization. Thus the lower bound in Theorem 3 is tight for the above-mentioned values of \(n+2\).

**Proof of Theorem 1.** Observe that two different color classes cannot have \(n\) edges each, since that will lead to a bichromatic cycle. Therefore at most one color class can have \(n\) edges while all other color classes can have at most \(n-1\) edges. Thus the number of edges in the union of \(\Delta(G)+1=d+1\) color classes is at most \(n+d(n-1)<dn\), when \(d>n\). (Note that \(dn\) is the total number of edges in \(G\).) Thus \(G\) needs at least one more color. Thus \(a'(G)\geq d+2=\Delta(G)+2\). ■

**Remark.** It is clear from the proof that if \(n+d(n-1)+x<brdn\) then even after removing \(x\) edges from the given graph, the resulting graph still would require \(d+2\) colors to be acyclically edge colored.

**Proof of Theorem 2.** If \(d\) is odd, let \(G'=K_{d+1}\). Else if \(d\) is even let \(G'\) be the complement of a perfect matching on \(d+2\) vertices. Let \(H\) be any \(d\)-regular graph on \(N=n-n'\) vertices. Now remove an edge \((a,a')\) from \(G'\) and an edge \((b,b')\) from \(H\). Now connect \(a\) to \(b\) and \(a'\) to \(b'\) to create a \(d\)-regular graph \(G\). Clearly \(G\) requires \(d+2\) colors to be acyclically edge colored since otherwise it would mean that \(G'=\{(a,a')\}\) is \(d+1\) colorable, a contradiction in view of the Remark following Theorem 1, for \(d\geq 5\). ■

Complete bipartite graphs offer an interesting case since they have \(d=n\). Observe that the above counting argument fails. We deal with this case in the next section.

**COMPLETE BIPARTITE GRAPHS**

**Lemma 1.** If \(n\) is even, then \(K_{n,n}\) does not contain three disjoint perfect matchings \(M_1, M_2, M_3\) such that \(M_i\cup M_j\) forms a hamiltonian cycle for \(i,j\in \{1,2,3\}\) and \(i\neq j\).
Proof. Observe that a perfect matching of $K_{n,n}$ corresponds to a permutation of \{1,2,\ldots,n\}. Let the perfect matching $M_i$ corresponds to permutation $\pi_i$. Without loss of generality, we can assume that $\pi_1$ is the identity permutation by renumbering the vertices of one side of $K_{n,n}$.

Suppose $K_{n,n}$ contains three perfect matchings $M_1, M_2, M_3$ such that $M_i \cup M_j$ forms a hamiltonian cycle for $i,j \in \{1,2,3\}$ and $i \neq j$.

Now we study the permutation $\pi_i^{-1} \pi_j$. Since $M_i \cup M_j$ induces a hamiltonian cycle in $K_{n,n}$, it is easy to see that the smallest $t \geq 1$ such that $(\pi_i^{-1} \pi_j)^t(1) = 1$ equals $n$.

It follows that, in the cycle structure of $\pi_i^{-1} \pi_j$, there exists exactly one cycle and this cycle is of length $n$. The sign of a permutation is defined as: \(\text{sign}(\pi) = (-1)^k\), where $k$ is the number of even cycles in the cycle structure of the permutation $\pi$. Recalling that $n$ is even, we have the following claim:

\textbf{Claim 1.} \(\text{sign}(\pi_i^{-1} \pi_j) = -1\) for $i,j \in \{1,2,3\}$ and $i \neq j$.

Now with respect to $\pi_i^{-1} \pi_j$, taking $\pi_i = \pi_1$ (the identity permutation) and $\pi_j = \pi_2$ (or $\pi_3$), we infer that $\text{sign}(\pi_2) = -1$ and $\text{sign}(\pi_3) = -1$. Now $\text{sign}(\pi_2^{-1} \pi_3) = \text{sign}(\pi_2^{-1}) \text{sign}(\pi_3) = (-1)(-1) = 1$, a contradiction in view of Claim 1. \(\blacksquare\)

\textbf{Proof of Theorem 3.} Since $K_{n,n}$ is a regular graph, $\Delta(K_{n,n}) \geq \Delta + 1 = n + 1$.

Suppose $n+1$ colors are sufficient. This can be achieved only in the following way:

One color class contains $n$ edges and the remaining color classes contain $n - 1$ edges each. Let $\alpha$ be the color class that has $n$ edges. Thus color $\alpha$ is present at every vertex on each side $A$ and $B$. Any other color is missing at exactly one vertex on each side.

\textbf{Observation 1.} Let $\theta \neq \alpha$ be a color class. The subgraph induced by color classes $\theta$ and $\alpha$ contains $2n - 1$ edges and since there are no bichromatic cycles, the subgraph induced is a hamiltonian path. We call this an $(\alpha, \theta)$ hamiltonian path.

\textbf{Observation 2.} Let $\theta_1$ and $\theta_2$ be color classes with $n - 1$ edges each. The subgraph induced by color classes $\theta_1$ and $\theta_2$ contains $2n - 2$ edges. Since there are no bichromatic cycles, the subgraph induced consists of exactly two paths.

Note that there is a unique color missing at each vertex on each side of $K_{n,n}$. Let $m(u)$ be the color missing at vertex $u$. For $a_1 \in A$ and $b_1 \in B$, let $m(a_1) = m(b_1) = \beta$. Let the color of the edge $(a_1, b_1) = \gamma$. Clearly $\gamma \neq \alpha$ since otherwise there cannot be an $(\alpha, \beta)$ hamiltonian path, a contradiction to \textbf{Observation 1}. For $a_2 \in A$ and $b_2 \in B$, let $m(a_2) = m(b_2) = \gamma$. It is clear that $a_1 \neq a_2$ and $b_1 \neq b_2$. Consider the subgraph induced by the colors $\beta$ and $\gamma$. In view of \textbf{Observation 2} it consists of exactly two paths. One of them is the single edge $(a_1, b_1)$. The other path has length $2n - 3$ and has $a_2$ and $b_2$ as end points.

Now we construct a $K_{n+1,n+1}$ from the above $K_{n,n}$ by adding a new vertex, $a_{n+1}$, to side $A$ and a new vertex, $b_{n+1}$, to side $B$. Now for $u \in B$ color each edge $(a_{n+1}, u)$ by the color $m(u)$ and for $v \in A$ color each edge $(b_{n+1}, v)$ by the color $m(v)$. Assign the color $\alpha$ to the edge $(a_{n+1}, b_{n+1})$. Clearly, the coloring thus obtained is a proper coloring.

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Now we know that there existed an \((x, \beta)\) Hamiltonian path in \(K_{n,n}\) with \(a_1\) and \(b_1\) as end points. Recalling that \(m(a_1) = m(b_1) = \beta\), we have \(\text{color}(a_{n+1}, b_1) = \beta\). It is easy to see that in \(K_{n+1,n+1}\), this path along with the edges 
\((a_1, b_{n+1}), (b_{n+1}, a_{n+1})\) and \((a_{n+1}, b_1)\) forms an \((x, \beta)\) Hamiltonian cycle. In a similar way, for \((x, \gamma)\) Hamiltonian path that existed in \(K_{n,n}\), we can see that in \(K_{n+1,n+1}\), we have a corresponding \((x, \gamma)\) Hamiltonian cycle.

Recall that there was a \((\beta, \gamma)\) bichromatic path starting from \(a_2\) and ending at \(b_2\) in \(K_{n,n}\). In the \(K_{n+1,n+1}\) we created, we have \(c(a_2, a_{n+1}) = \gamma, c(a_1, b_{n+1}) = \beta, c(a_{n+1}, b_1) = \beta\) and \(c(a_{n+1}, b_2) = \gamma\). Thus the above \((\beta, \gamma)\) bichromatic path in \(K_{n,n}\) along with the edges \((a_2, b_{n+1}), (b_{n+1}, a_1), (a_1, b_1), (b_1, a_{n+1}), (a_{n+1}, b_2)\) in that order forms a \((\beta, \gamma)\) bichromatic Hamiltonian cycle. Thus, we have 3 perfect matchings induced by the color classes \(x, \beta\) and \(\gamma\) whose pairwise union gives rise to Hamiltonian cycles in \(K_{n+1,n+1}\), a contradiction to Lemma 1 since \(n+1\) is even.

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