Exercise 11.1: (2 P)
Use the polynomial interpretation $\mathcal{A}$ with $U_{\mathcal{A}} = \mathbb{N}\setminus\{0,1,2\}$ and $P_f := X^2 + XY$ to show that the term rewriting system:

\[
\begin{align*}
    f(f(x,y), z) & \rightarrow f(x, f(y, z)) \\
    f(y, f(x, z)) & \rightarrow f(x, x)
\end{align*}
\]

terminates.

Exercise 11.2: (4 P)
Find polynomial orderings that prove the termination of the following term rewriting systems:

(1) \[(x + y) \cdot z \rightarrow (x \cdot z) + (y \cdot z);\]

(2) \[
\begin{align*}
    x + 0 & \rightarrow x \\
    x + s(y) & \rightarrow s(x + y) \\
    d(0) & \rightarrow 0 \\
    d(s(x)) & \rightarrow s(d(x));
\end{align*}
\]

(3) \[
\begin{align*}
    (x + y) + z & \rightarrow x + (y + z) \\
    h(x) + h(y) & \rightarrow h(x + y);
\end{align*}
\]

(4) \[
\begin{align*}
    (\neg x) \sqcup y & \rightarrow x \sqcup y \\
    x \sqcup (y \sqcup z) & \rightarrow y \sqcup (x \sqcup z) \\
    (\neg x) \sqcup (y \sqcup z) & \rightarrow y \sqcup (x \sqcup z).
\end{align*}
\]

Exercise 11.3: (2 P)
Has every polynomial ordering $>$ (cf. the definitions in Section 3.5, slides 24–27) the subterm property? Justify your answer. How could one modify the requirements 1 and/or 2 on slides 25, 26 to ensure that the corresponding polynomial orderings have the subterm property?
Exercise 11.4: (2 P)
Let $\mathcal{A}$ be a monotone polynomial interpretation. For $P, Q \in \mathbb{N}[X_1, \ldots, X_n]$ we write $P \geq_{\mathcal{A}} Q$ iff $P(a_1, \ldots, a_n) \geq Q(a_1, \ldots, a_n)$ holds for all $a_1, \ldots, a_n \in U_{\mathcal{A}}$. (Note that $\geq_{\mathcal{A}}$ is not the reflexive closure of $>_{\mathcal{A}}$.)

1. Show that $s \preceq_{\text{emb}} t$ implies $P_s \geq_{\mathcal{A}} P_t$.

2. Show that this implies that a polynomial ordering cannot be used to prove termination of a term rewriting system $R$ if there are two terms $s, t$ such that $s \xrightarrow{R} t$ and $s \preceq_{\text{emb}} t$.

Put your solution into the mail box at the door of room 607 in the MPI building (46.1) before Friday, July 9, 11:00. Don’t forget to write your name and the name of your tutorial group (A, B, C) on your solution.

Note: Joint solutions, prepared by up to three persons together, are allowed. Joint solutions should be submitted only once, and all the authors should be indicated.