Tutorials for “Automated Reasoning”
Exercise sheet 9

Exercise 9.1: (2 P)
Compute a closed tableau for
\[
\{ \left( \forall x \exists y \left( p(x, x) \rightarrow \neg p(x, y) \right) \right) \land \left( \exists x \forall y p(x, y) \right) \}\]

Exercise 9.2: (2 P)
Prove that the following set of formulae is unsatisfiable by using first-order semantic tableaux:
\[\{p(a), \ \forall x(p(x) \rightarrow p(f(x))), \ \neg p(f(f(a)))\}\]

Exercise 9.3: (2 P)
Prove the following subcase of Theorem 2.52: If \( T \) is a satisfiable tableau, then the tableau \( T' \) that results by applying the \( \delta \)-expansion rule to \( T \) is also satisfiable.

Exercise 9.4: (4 P)
Let \( F \) be a closed first-order formula with equality over a signature \( \Sigma = (\Omega, \Pi) \). Let \( \sim \notin \Omega \) be a new binary relation symbol (written as an infix operator). Let the set \( Eq(\Sigma) \) contain the formulas
\[
\forall x (x \sim x)
\forall x, y (x \sim y \rightarrow y \sim x)
\forall x, y, z (x \sim y \land y \sim z \rightarrow x \sim z)
\]
and for every \( f/n \in \Omega \) the formula
\[
\forall x_1, \ldots, x_n, y_1, \ldots, y_n (x_1 \sim y_1 \land \cdots \land x_n \sim y_n \rightarrow f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n))
\]
and for every \( p/n \in \Pi \) the formula
\[
\forall x_1, \ldots, x_n, y_1, \ldots, y_n (x_1 \sim y_1 \land \cdots \land x_n \sim y_n \land p(x_1, \ldots, x_n) \rightarrow p(y_1, \ldots, y_n)).
\]
Let $\tilde{F}$ be the formula that one obtains from $F$ if every occurrence of the equality symbol $\approx$ is replaced by the relation symbol $\sim$.

(a) Let $A$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Show that the interpretation $\sim_A$ of $\sim$ in $A$ is a congruence relation. (It is enough if you prove one of the properties of congruence relations, say symmetry; the other properties are proved analogously.)

(b) Let $A$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Use the congruence relation $\sim_A$ to construct a model of $F$ and prove that it is a model.

(c) Prove that a formula $F$ is satisfiable if and only if $Eq(\Sigma) \cup \{\tilde{F}\}$ is satisfiable.

Put your solution into the mail box at the door of room 607 in the MPI building (46.1) before Friday, June 25, 11:00. Don’t forget to write your name and the name of your tutorial group (A, B, C) on your solution.

**Note:** Joint solutions, prepared by up to three persons together, are allowed. Joint solutions should be submitted only once, and all the authors should be indicated.