Upper and Lower Bounds for Arithmetic Circuits

Sébastien Tavenas

supervised by Pascal Koiran and Natacha Portier

July 9, 2014

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- A problem: multiply 12345 by 378267
- Algorithm with 40 operations.
  $\Rightarrow$ Upper bound on the minimal cost.
- Proof: it is not possible with 10 operations.
  $\Rightarrow$ Lower bound on the minimal cost.
- Easier model of computation: Arithmetic circuits.

Outline

1. Arithmetic circuits
2. Depth Reduction
3. Real τ-conjecture
4. Sevostyanov's problem

Outline

Arithmetic circuits

Polynomials

$f(x, y) = 4x^2 - 2xy$

Representation by circuits (gates: $+$, $\times$, variables and real constants):

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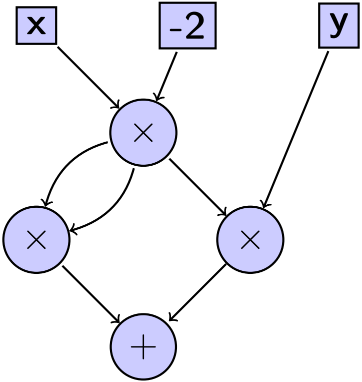
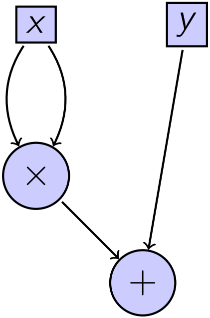
Representation by circuits (gates: $+$, $\times$, variables and real constants):



Complexity of the circuits - Examples



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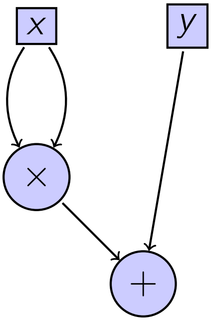

- Computed polynomial: $P(x) = x^2 + y$.
- Size: $s = 4$.
- Depth: $l = 2$.

Complexity of the circuits - Examples

- Example:
  $P_n(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$

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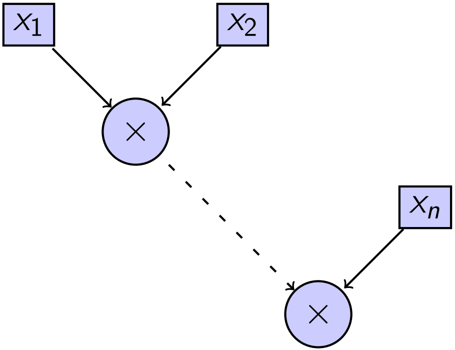

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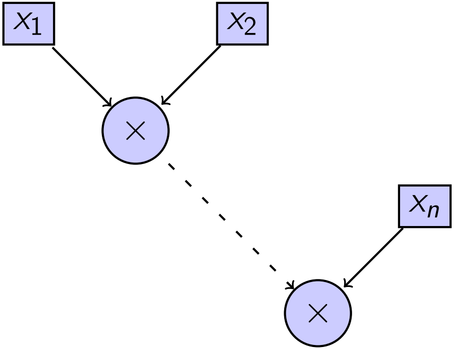

- Size: $s = 2n - 1$.
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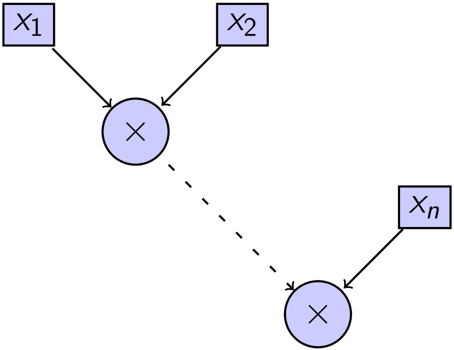

- Size: $s = 2n - 1$.
- Depth: $l = n - 1$.
- Sequence: homogeneous polynomials
  - $x^3 - 2xy^2 \rightarrow$ hom.
  - $x^2y^2 + y^4 - 3 \rightarrow$ not hom.
- Homogeneous circuit: all gates compute homogeneous polynomials.

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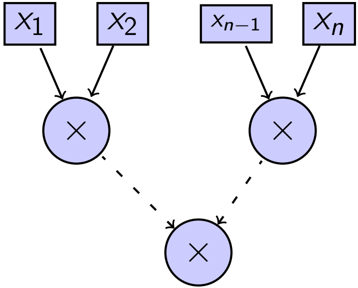

- Size: $s = 2n - 1$.
- Depth: $l = \log n$.

Valiant's conjecture

Class VP

(f_n) : there exists c s.t. for every $n \geq 2$

- f_n is computed by circuits of size $\leq n^c$
- degree is bounded by n^c.

$$\mathrm{Det}_n((x_{i,j})_{i,j \leq n}) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} x_{i,\sigma(i)}$$

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Valiant's conjecture (1979)

$\mathrm{VP} \neq \mathrm{VNP}$.

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Reduction of the depth

Theorem (Valiant,Skyum,Berkowitz,Rackoff (1983))

f: homogeneous, of degree d, circuits of size s.
Then f computed by homogeneous circuits of size $(sd)^{O(1)}$ and of depth $O(\log s \log d)$.

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f: homogeneous, degree d, circuits of size s.
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such that

- *Fan-in of $\times$-gates bounded by 5,*
- *If α is a $\times$-gate of degree d and of children $\alpha_1, \ldots, \alpha_r$ ($r \leq 5$), then $\forall i, d^\circ(\alpha_i) \leq \frac{d}{2}$.*

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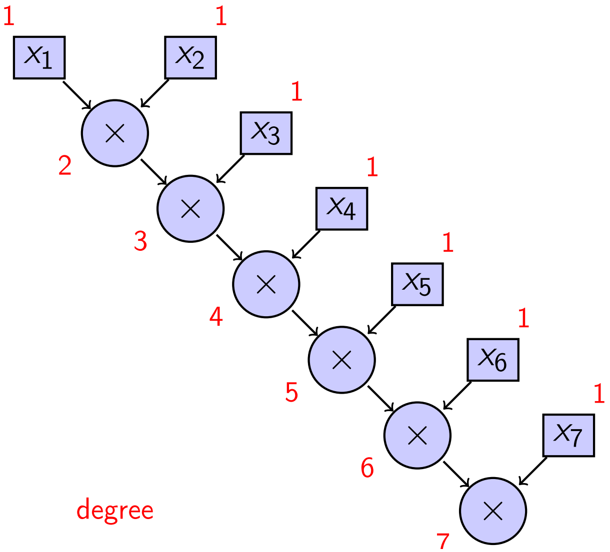

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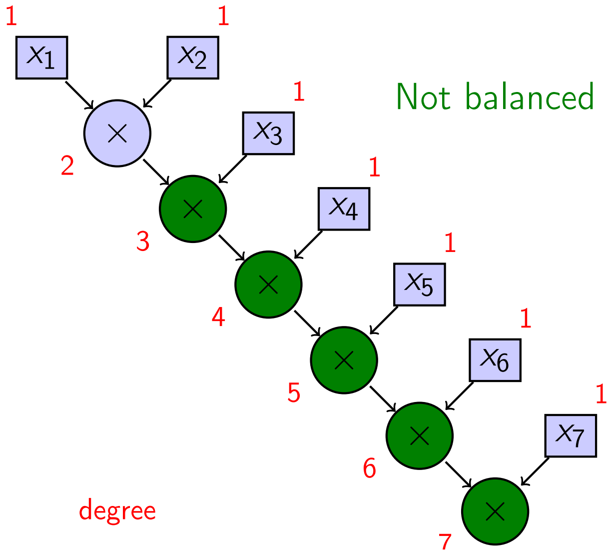

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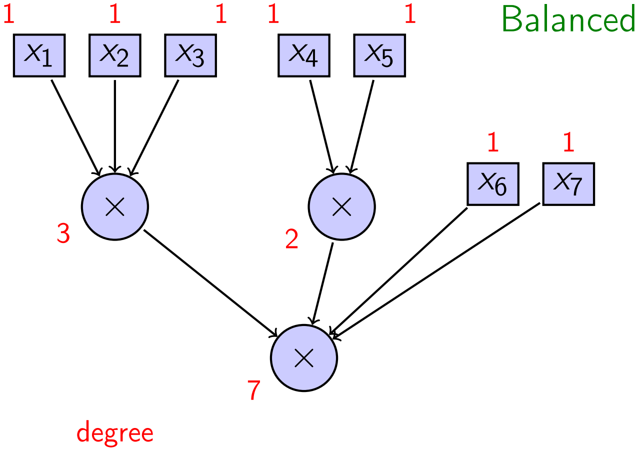

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In the following, $s, d = n^{O(1)}$.
Is it possible to reduce the depth?

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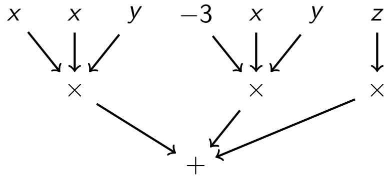

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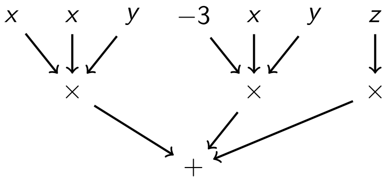

- Depth 3 $\rightarrow$ a subexponential reduction to depth 3 using the reduction to depth 4.

Reduction to depth 4

Theorem (Koiran (2012))

If f has polynomial-size (homogeneous) circuits,
then f has depth 4 (homogeneous) circuits of size $2^{O(\sqrt{d}\log^2 n)}$.

Idea:

$$\underset{s,d}{\mathcal{C}} \longrightarrow \underset{2^{O(\log d \log s)}}{\text{Iterated matrix product}} \longrightarrow \underset{2^{O(\sqrt{d}\log d \log s)}}{\text{Depth 4 circuits}}$$

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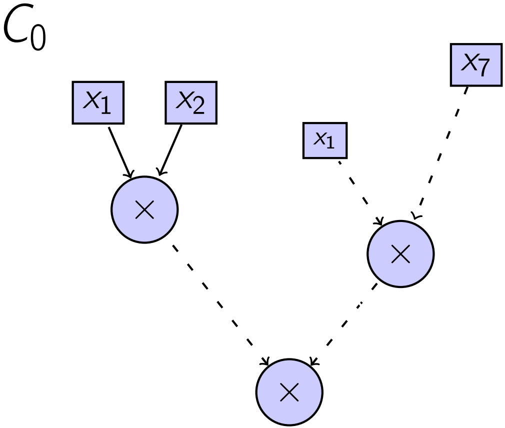

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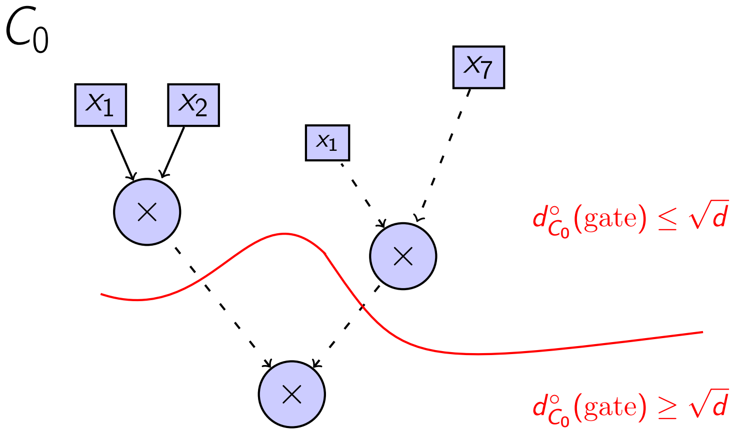

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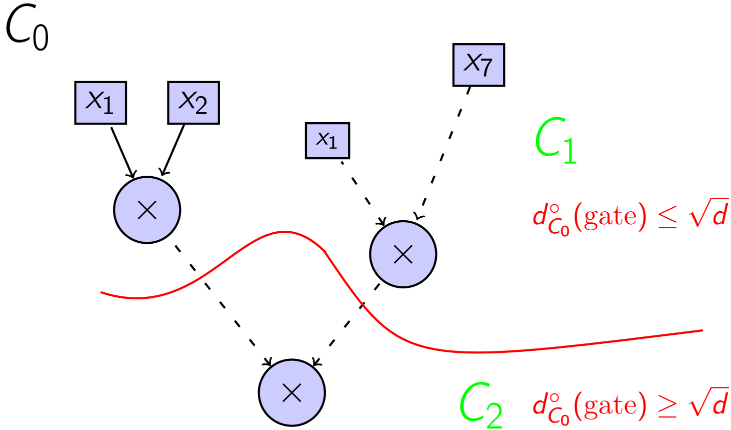
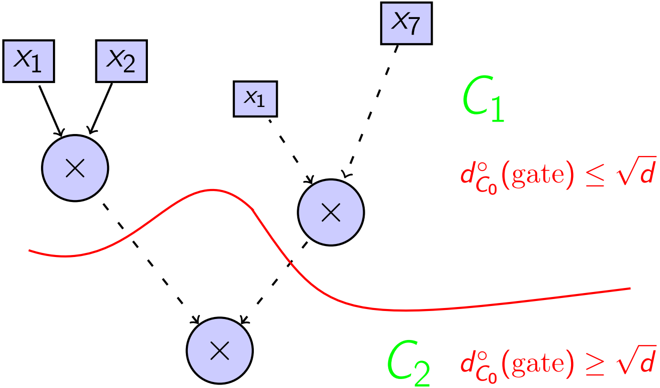


Idea of the proof 2/5



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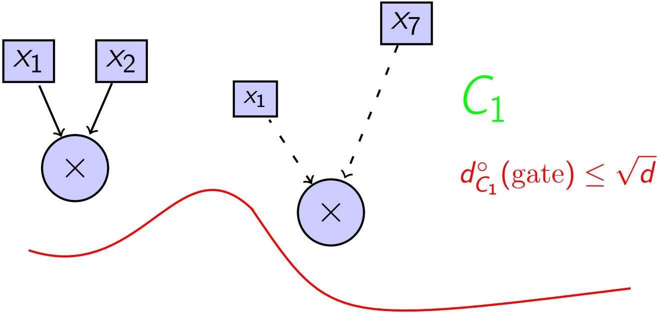

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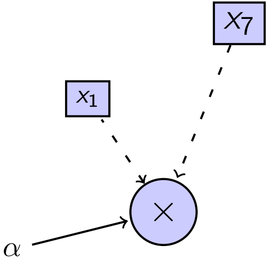

$d^{\circ}_{C_1}(\text{gate}) \leq \sqrt{d}$

α computed by a depth-2 circuit of size $n^{O(\sqrt{d})}$

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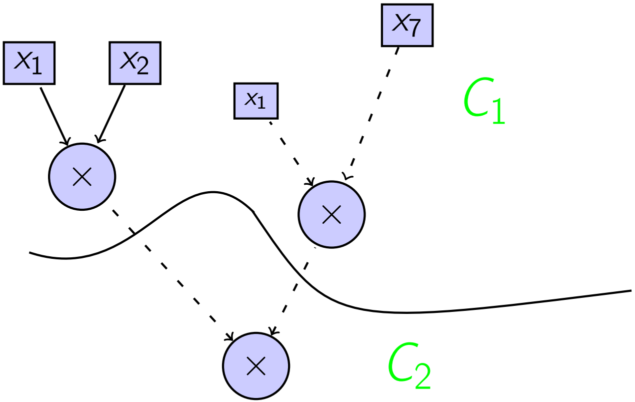

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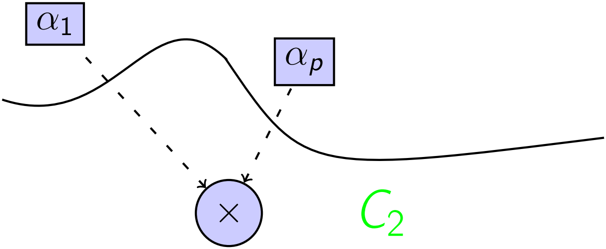

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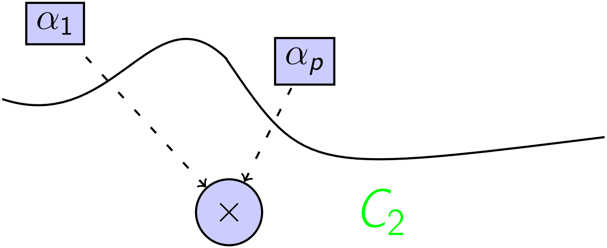

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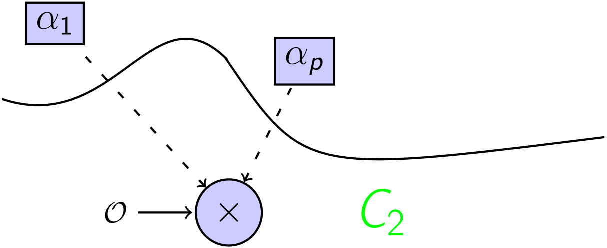

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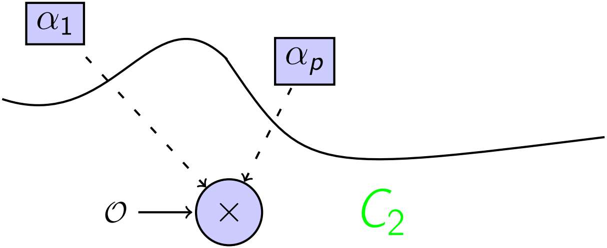

Let us suppose that $d^{\circ}_{C_2}(\mathcal{O}) = O(\sqrt{d})$

depth-2 circuit of size $n^{O(\sqrt{d})}$ for C_2 $\rightarrow$ QED

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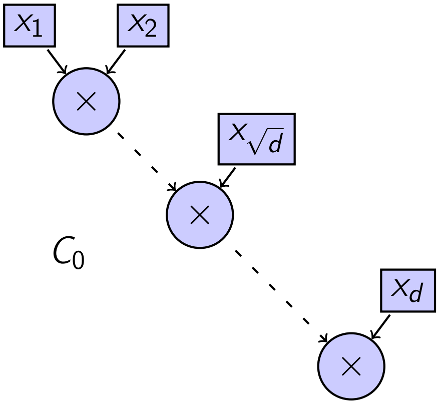

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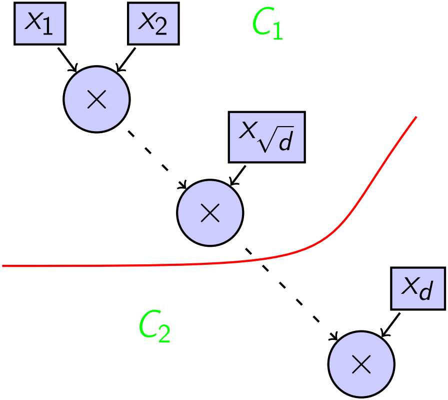

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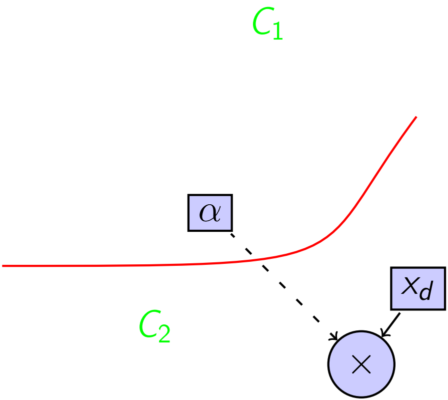

C_2 computes $\alpha \cdot x_{\sqrt{d}+1} \cdot \ldots \cdot x_d$

of degree $d - \sqrt{d} + 1 >> \sqrt{d}$

Idea of the proof 5/5: ... and if C_0 is balanced.

Lemma

If C_0 is balanced, then the degree of $C_2 \leq O(\sqrt{d})$.

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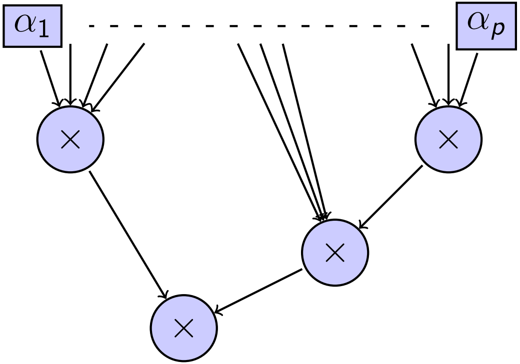
A monomial in C_2



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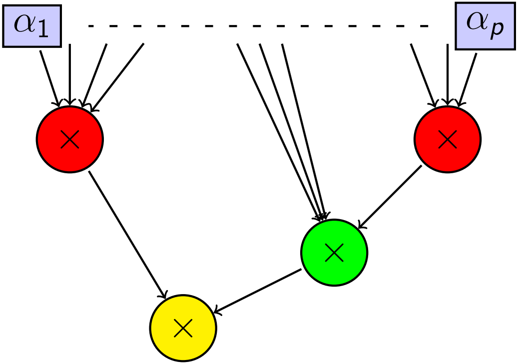

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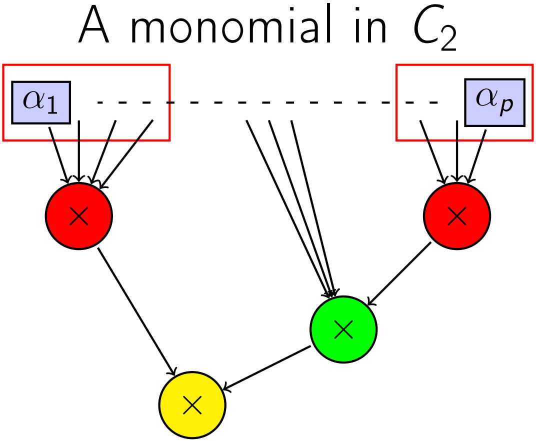

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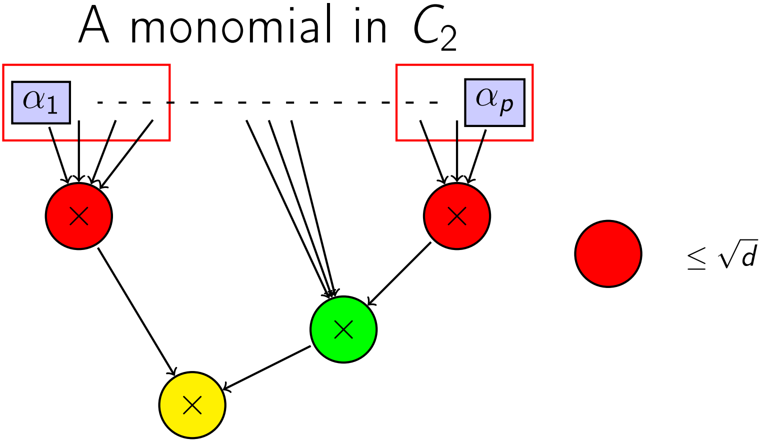

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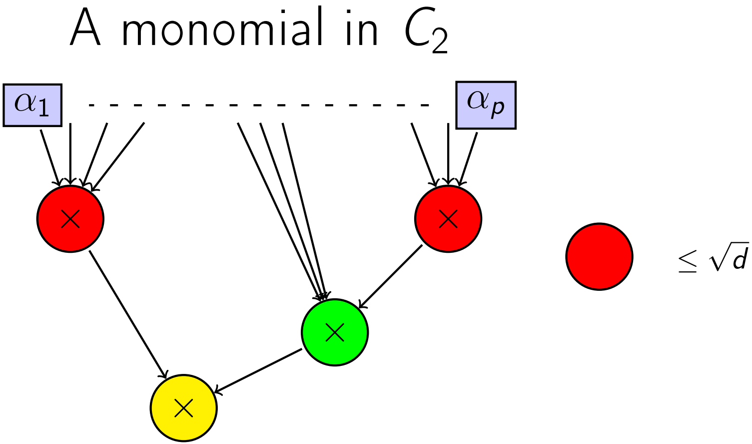

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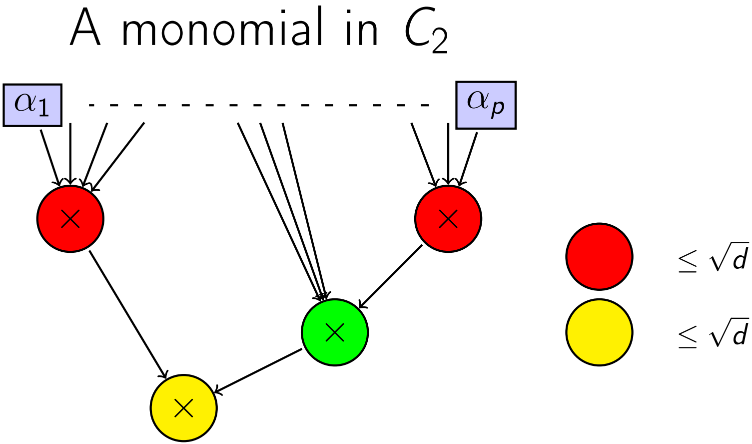

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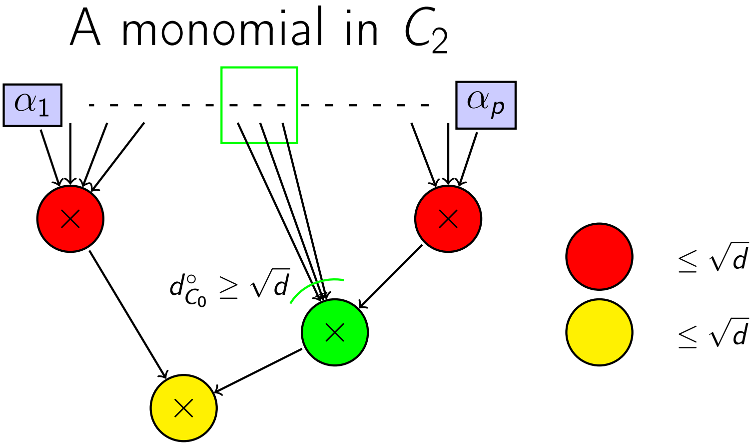

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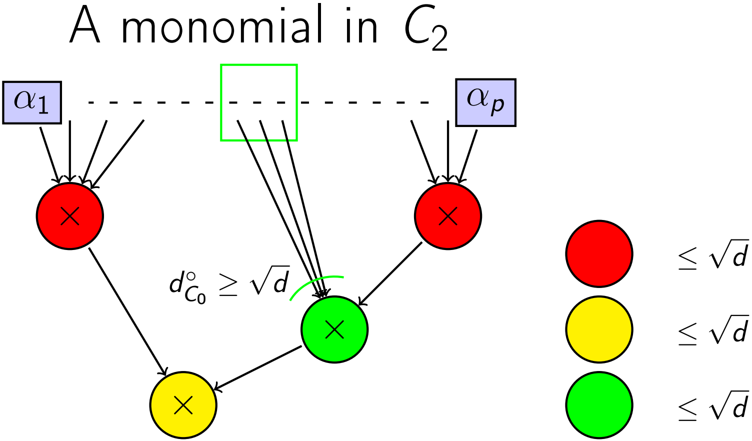

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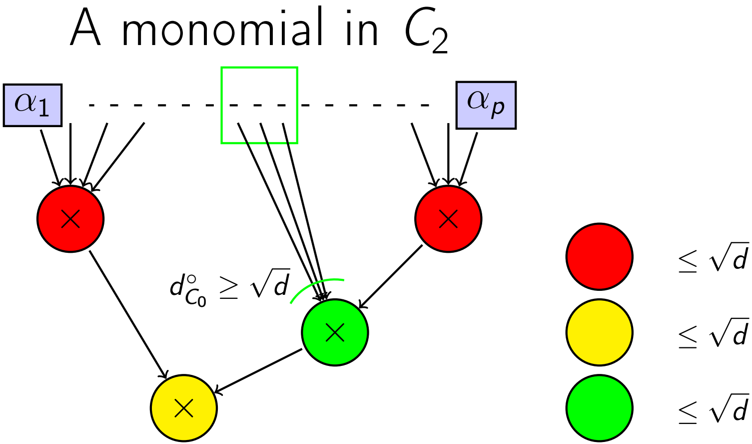


So $d^\circ(C_2) \leq 15\sqrt{d}$ $\Rightarrow$ theorem is proved.

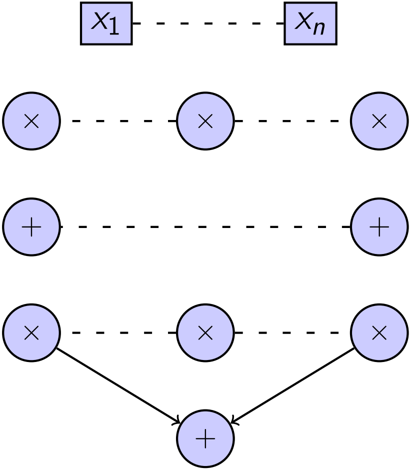
Case of the Permanent

Corollary

If Perm_n *has no depth-4 homogeneous circuits of size* $2^{O(\sqrt{n}\log n)}$, *then* $\mathsf{VP} \neq \mathsf{VNP}$.

Lower bounds for Perm_n in depth 4

Homogeneous circuits:



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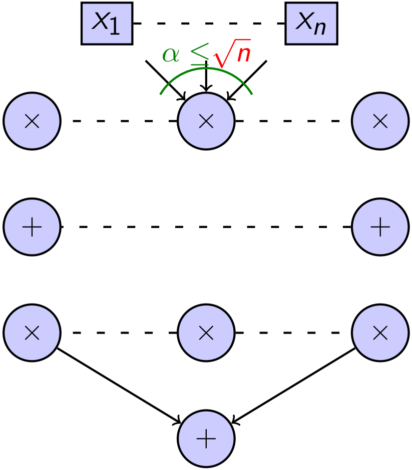

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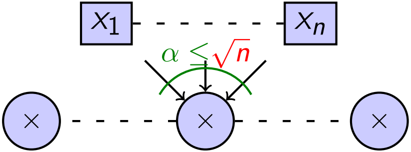
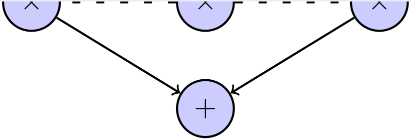


Theorem (Gupta,Kamath,Kayal,Saptharishi (2013))

If C is an homogeneous depth-4 circuit with $\alpha \leq O(\sqrt{n})$ which computes Perm_n*,*
then, $size(C) \geq 2^{\Omega(\sqrt{n})}$.



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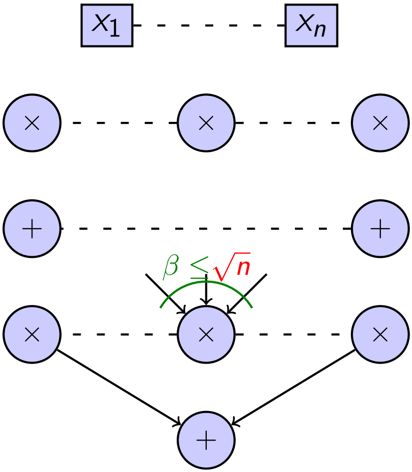

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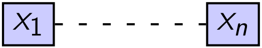
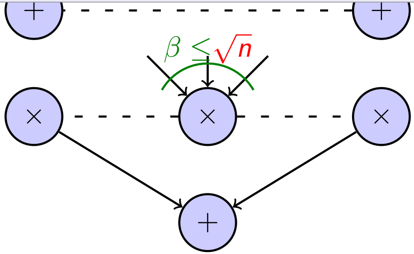


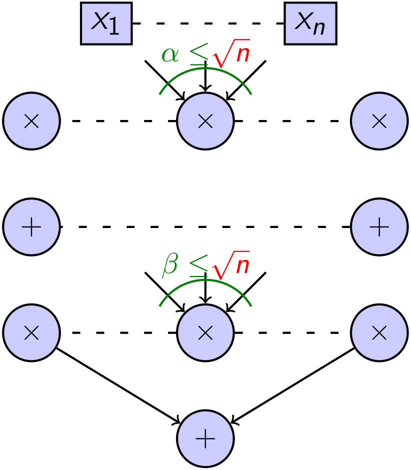
Proposition (Monomials counting)

If C is an homogeneous depth-4 circuit with $\beta \leq O(\sqrt{n})$ which computes Perm_n,
then, $size(C) \geq 2^{\Omega(\sqrt{n}\log n)}$.



Lower bounds for Det_n in depth 4

Homogeneous circuits:



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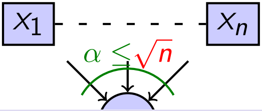
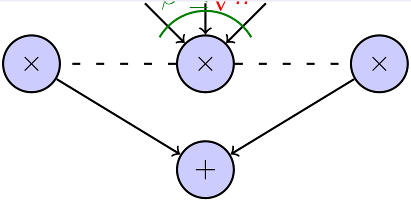


Corollary

If C is a minimal homogeneous depth-4 circuit with α and $\beta \leq O(\sqrt{n})$ computing Det_n,
then, $size(C) = 2^{\Theta(\sqrt{n}\log n)}$.



Generalization to constant depth

Theorem (T.)

If f_n has polynomial-size homogeneous circuits,
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Application (Adaptation of Gupta, Kamath, Kayal and Saptharishi's proof):

Proposition (T.)

If f_n has polynomial-size homogeneous circuits, then f_n has depth-4 non-homogeneous circuits of size $2^{O(\sqrt[3]{d}\log n)}$.

Outline

τ-conjecture

Conjecture (τ-conjecture, Shub-Smale (1995))

There exists an universal constant c such that if $P(X)$ is an **univariate** *polynomial computed by a circuit of size s, then the number of integer roots of P is bounded by s^c.*

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Theorem (Bürgisser (2007), almost...)

If the τ-conjecture is true, then $\mathsf{VP} \neq \mathsf{VNP}$*.*

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Theorem (Bürgisser (2007))

If the τ-conjecture is true, then $\mathsf{VP}^0 \neq \mathsf{VNP}^0$.

- VP^0, VNP^0 classes using only the constant -1.

Bürgisser's proof

Let us assume $\mathsf{VP}^0 = \mathsf{VNP}^0$.

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Conjecture

There exists c such that the univariate polynomial $\sum_{i=1}^{k}\prod_{j=1}^{m} f_{i,j}$ *(with* $f_{i,j}$ *t-sparse) has at most* $(m+k+t)^c$ *integer roots.*

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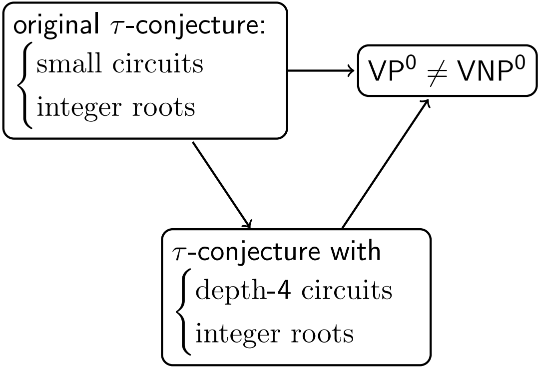

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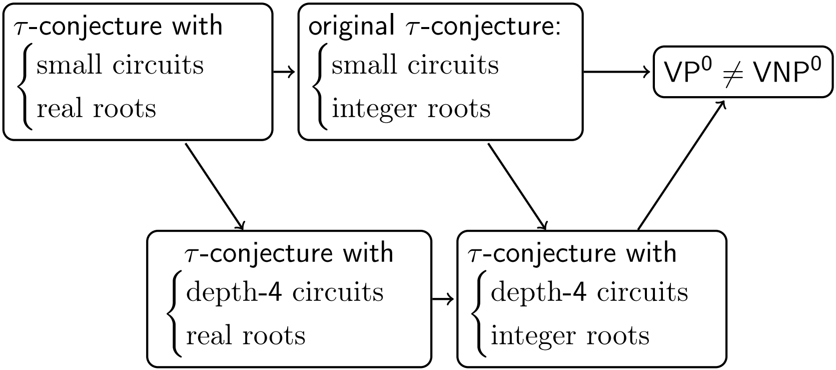

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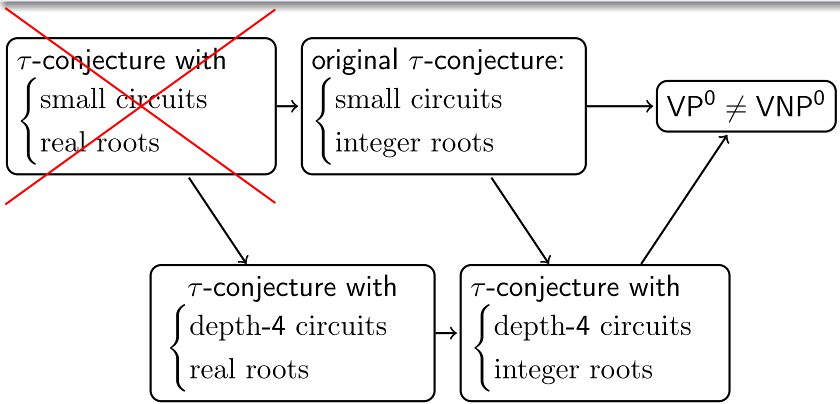

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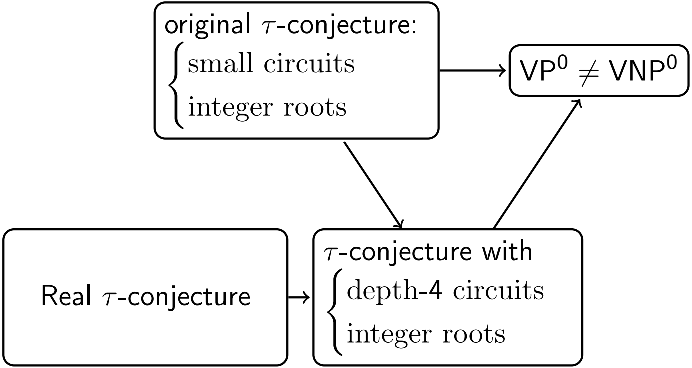

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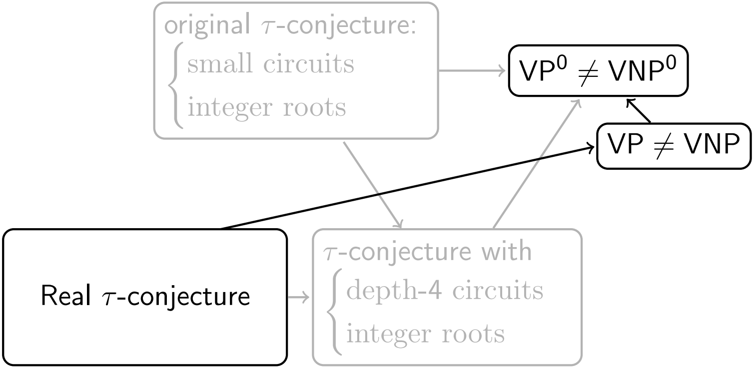

Outline

Descartes' rule

Fundamental theorem of algebra

A complex polynomial P of degree d has exactly d roots counted with multiplicity.

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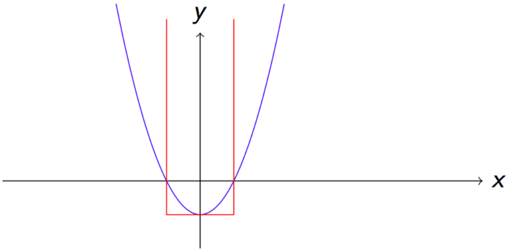
- $f(x) = x^2 - 1$ and $g(x) = x^{200} - 1$?



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Descartes' estimate

If P is a real t-sparse polynomial, then P has at most $t - 1$ positive roots (counted with multiplicity).
So, at most $2t - 1$ distinct roots on $\mathbb{R}$.

In fact Descartes' rule is more precise.

Bézout's Theorem

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The following $n \times n$ system

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \dots \\ f_n(x_1, \dots, x_n) = 0 \end{cases}$$

has at most $d_1 d_2 \dots d_n$ nondegenerate complex solutions.

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- Similar result for sparse polynomials and real solutions?
- Kushnirenko's question (1977): bounded by a function $N(t_1, \ldots, t_n)$? What is the optimal bound?

An initial case

Sevostyanov's problem (1977)

Let f and g be two real bivariate polynomials.
f is of degree d and g is t-sparse.
Is the number of distinct isolated real solutions of the system

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

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According to Kushnirenko, Sevostyanov proved the existence of $N(d,t)$ in 1978.

Fewnomial bounds

Theorem (Khovanskiĭ (1983))

System of n equations and n variables
with ony $n + l + 1$ distinct monomials.
Then, number of positive real solutions bounded by

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In particular,

- Kushnirenko's question: $N(t_1, \ldots, t_n) \leq 2^{\binom{t_1+\ldots+t_n}{2}}(n+1)^{t_1+\ldots+t_n}$

Improvement

Khovanskiĭ's Theorem was improved by Bihan and Sottile.

Theorem (Bihan, Sottile (2007))

System of n equations and n variables
with only $n + l + 1$ distinct monomials.
Then, number of positive real solutions bounded by

$$\frac{e^2 + 3}{4} 2^{\binom{l}{2}} n^l.$$

Intersection of a trinomial curve with a sparse curve

Theorem (Li, Rojas, Wang (2003))

$f(x, y)$ is a trinomial and $g(x, y)$ is t-sparse.
Then the system $f(x, y) = g(x, y) = 0$ has at most $2^t - 2$ real solutions.

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Theorem (Koiran, Portier, T. *MEGA* 2013)

$f(x, y)$ is a trinomial and $g(x, y)$ is t-sparse.
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Sevostyanov's problem

Theorem (Koiran, Portier, T. *Discrete & Computational Geometry*, to appear)

*$f(x, y)$ non-zero polynomial of degree d and $g(x, y)$ t-sparse.
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The constraint $f \neq 0$ is important.

Stronger hypothesis

Theorem

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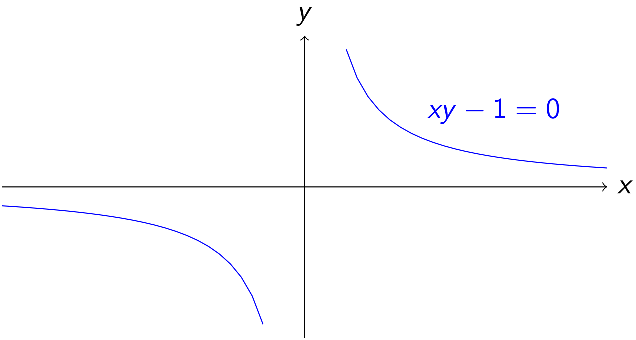

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Lemma

$f(x,y)$ non-zero polynomial of degree d and $g(x,y)$ t-sparse.
Assume that:

- *f irreducible in $\mathbb{C}[X,Y]$*
- *and finite number of solutions.*

Then the system $f(x,y) = g(x,y) = 0$ has at most $O(d^3t + d^2t^3)$ distinct real solutions.

From the lemma to the theorem

Hypothesis

- finite number of solutions
- and f irreducible in $\mathbb{C}[X, Y]$.

From the lemma to the theorem

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1. If the number of solutions is infinite then f divides g.
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2. Irreducibility of f.
   - $f = f_1 f_2 \dots f_k$

$$\bigcup_i \text{solutions of } \begin{cases} f_i = 0 \\ g = 0 \end{cases}$$

Outline of the proof

Lemma

$f(x, y)$ irreducible polynomial of degree d and $g(x, y)$ t-sparse. Then if the system $f(x, y) = g(x, y) = 0$ has a finite number of real solutions, it is at most $O(d^3 t + d^2 t^3)$ solutions.

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1. Rewrite $f(x, y) = 0$ as $y = h(x)$ (h nice).
   To bound the number of roots of $g(x, (h(x))) = \sum^k a_i x^{\alpha_i} h(x)^{\beta_i}$.

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2. To bound the number of roots of a sum.

Cylindrical algebraic decomposition

- Solutions of $f(x, y) = 0$.

Cylindrical algebraic decomposition

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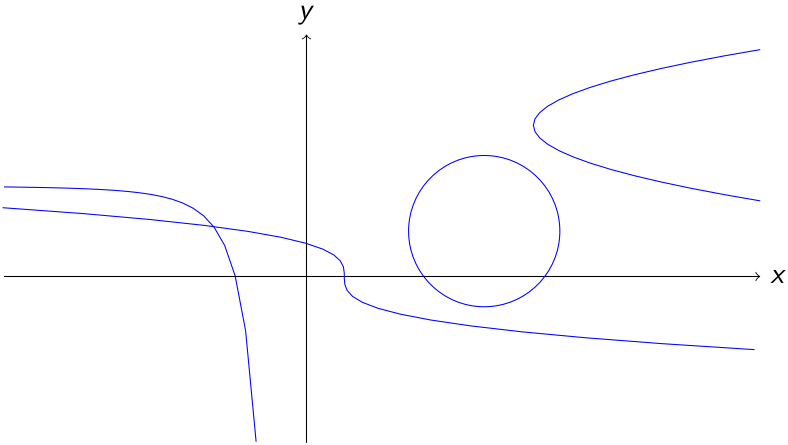

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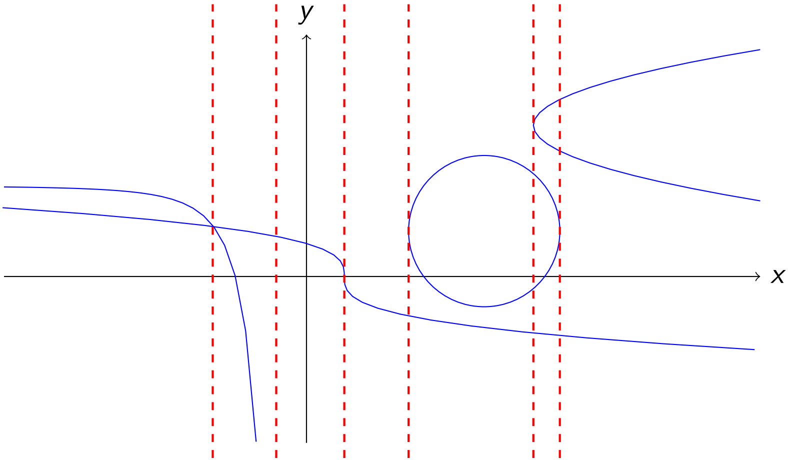

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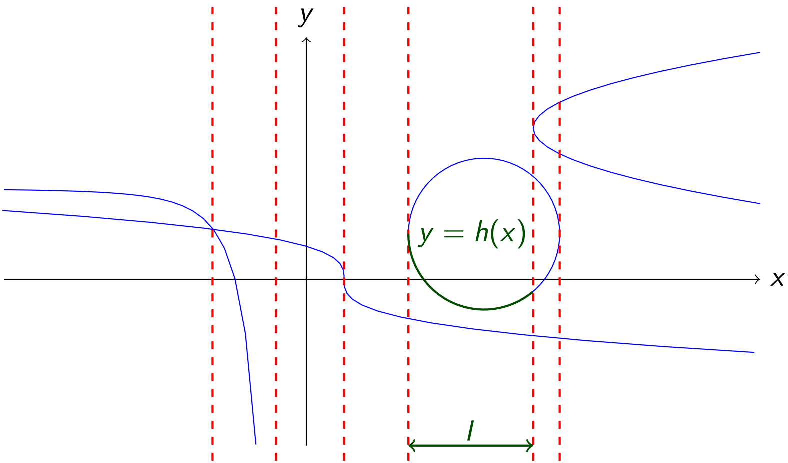

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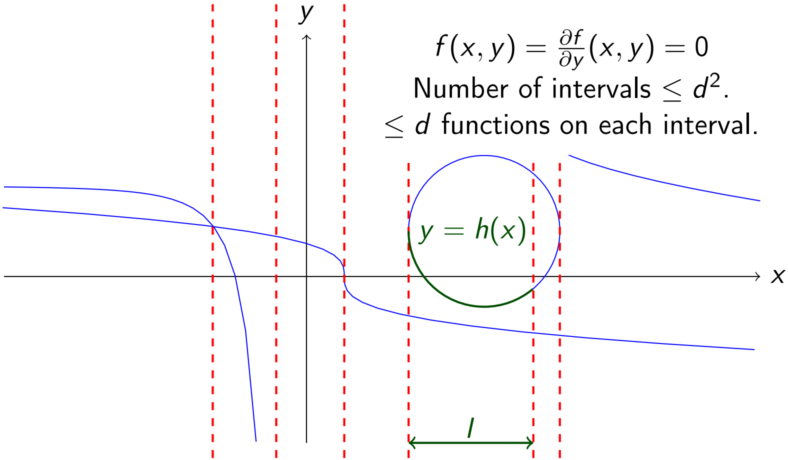

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We will use the Wronskian.

The key of the proof: the Wronskian

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Definition: Let $f_1, \ldots, f_k \in \mathcal{C}^{k-1}(I)$ with $I \subseteq \mathbb{R}$. The *Wronskian* of the family is the determinant of the matrix:

$$W(f_1, \ldots, f_k) = \det \begin{bmatrix} f_1 & f_2 & \ldots & f_k \\ f_1' & f_2' & \ldots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \ldots & f_k^{(k-1)} \end{bmatrix}$$

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- Idea: upper bound the number of roots of a sum by the number of roots of some particular Wronskians.

From sum to Wronskians

Lemma

Let I be a real interval.
If $W(f_1), W(f_1, f_2), \ldots, W(f_1, f_2, \ldots, f_k)$ have no zero on I, then $Z_I(f_1 + \ldots + f_k) \leq k - 1$.

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$$W\left(\left(\frac{f_2}{f_1}\right)', \ldots, \left(\frac{f_q}{f_1}\right)'\right) = \left(\frac{1}{f_1}\right)^p W(f_1, \ldots, f_q).$$

From sum to Wronskians

Theorem (Koiran, Portier, T. *MFCS 2013*)

$$Z(f_1 + \ldots + f_k) \leq k - 1 + 2\sum_{j=1}^{k-2} Z(W(f_1, \ldots, f_j))$$

What remains to be done

Theorem

$f(x, y)$ polynomial of degree d and $g(x, y)$ t-sparse.
Then the system $f(x, y) = g(x, y) = 0$ has at most $O(d^3 t + d^2 t^3)$ real solutions.

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2. To bound the number of roots of
   $W_s = W(a_1 x^{\alpha_1} h^{\beta_1}(x), \dots, a_s x^{\alpha_s} h^{\beta_s}(x))$.

Bounds for W_3

$$\det \begin{bmatrix} x^{\alpha_1} h^{\beta_1} & x^{\alpha_2} h^{\beta_2} & x^{\alpha_3} h^{\beta_3} \\ \left(x^{\alpha_1} h^{\beta_1}\right)' & \left(x^{\alpha_2} h^{\beta_2}\right)' & \left(x^{\alpha_3} h^{\beta_3}\right)' \\ \left(x^{\alpha_1} h^{\beta_1}\right)'' & \left(x^{\alpha_2} h^{\beta_2}\right)'' & \left(x^{\alpha_3} h^{\beta_3}\right)'' \end{bmatrix}$$

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$$\det \begin{bmatrix} x^{\alpha_1} h^{\beta_1} & x^{\alpha_2} h^{\beta_2} & x^{\alpha_3} h^{\beta_3} \\ \left(x^{\alpha_1 - 1} h^{\beta_1 - 1}\right)\left(\alpha_1 h + \beta_1 x h'\right) & \dots & \dots \\ \left(x^{\alpha_1 - 2} h^{\beta_1 - 2}\right)\left(P_2(x, h, h', h'')\right) & \dots & \dots \end{bmatrix}$$

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$$x^{\alpha_1+\alpha_2+\alpha_3-6} h^{\beta_1+\beta_2+\beta_3-6} \det \begin{bmatrix} x^2 h^2 & x^2 h^2 & x^2 h^2 \\ xh(\alpha_1 h + \beta_1 x h') & \dots & \dots \\ P_2(x, h, h', h'') & \dots & \dots \end{bmatrix}$$

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Case of the real τ-conjecture? $\sum^k \prod^m f_{i,j}(x)$?
If the number of distinct $f_{i,j}$ is very small (constant?)?

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Let us consider $f(X) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}}(X)$ with f_j t-sparse polynomial.

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Theorem (Koiran, Portier, T.)

If f is not zero, then it has at most $t^{O(mk^2)}$ real roots.

Perspectives

- Fournier, Limaye, Malod, Srinivasan 2013: If a circuit $\sum^{[s]}\prod^{[O(D)]}\sum\prod^{[\sqrt{d}]}$ computes $\mathsf{IMM}_{n,d}$, then $s \geq 2^{\Omega(\sqrt{d}\log(n/D))}$.
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- Generalisations of Sevostyanov's problem
  - $f_1 = \ldots = f_n = g = 0$ (f_i dense, g sparse)
  - $f = g = 0$ (f and g sparse)

Thank you!

Publications

- Improved Bounds for Reduction to Depth 4 and Depth 3. *MFCS* 813-824, 2013. To appear in a Special Issue of *Information and Computation*.
- With P. Koiran, N. Portier: A Wronskian Approach to the real τ-conjecture. *MEGA*, 2013. To appear in a Special Issue of *Journal of Symbolic Computation*.
- With P. Koiran, N. Portier, S. Thomassé: A τ-conjecture for Newton polygons. To appear in the Special Issue of *Foundations of Computational Mathematics honoring Mike Shub's 70th birthday*.
- With P. Koiran, N. Portier: On the intersection of a sparse curve and a low-degree curve : A polynomial version of the lost theorem. To appear in *Discrete & Computational Geometry*.
- With E. Diot, N. Trotignon: Detecting wheels. *Applicable Analysis and Discrete Mathematics*, 8:111–122, 2014.