Simplification Orderings

The proper subterm ordering $\triangleright$ is defined by $s \triangleright t$ if and only if $s/p = t$ for some position $p \neq \varepsilon$ of $s$. 
Simplification Orderings

A rewrite ordering $\succ$ over $T_\Sigma(X)$ is called simplification ordering, if it has the subterm property:
$s \succ t$ implies $s \succ t$ for all $s, t \in T_\Sigma(X)$.

Example:

Let $R_{\text{emb}}$ be the rewrite system

$$R_{\text{emb}} = \{ f(x_1, \ldots, x_n) \rightarrow x_i \mid f/n \in \Omega, \ n \geq 1, \ 1 \leq i \leq n \}.$$ 

Define $\triangleright_{\text{emb}} = \rightarrow_{R_{\text{emb}}}^+$ and $\triangleright_{\text{emb}} = \rightarrow_{R_{\text{emb}}}^*$

(“homeomorphic embedding relation”).

$\triangleright_{\text{emb}}$ is a simplification ordering.
Simplification Orderings

Lemma 3.31:
If $\not\succ$ is a simplification ordering, then $s \succ_{\text{emb}} t$ implies $s \not\succ t$ and $s \succeq_{\text{emb}} t$ implies $s \succeq t$.

Proof:
Since $\not\succ$ is transitive and $\succeq$ is transitive and reflexive, it suffices to show that $s \rightarrow_{\text{R}_{\text{emb}}} t$ implies $s \not\succ t$.
By definition, $s \rightarrow_{\text{R}_{\text{emb}}} t$ if and only if $s = s[l\sigma]$ and $t = s[r\sigma]$ for some rule $l \rightarrow r \in \text{R}_{\text{emb}}$.
Obviously, $l \succ r$ for all rules in $\text{R}_{\text{emb}}$, hence $l \not\succ r$.
Since $\not\succ$ is a rewrite relation, $s = s[l\sigma] \not\succ s[r\sigma] = t$. 
Simplification Orderings

Goal:

Show that every simplification ordering is well-founded (and therefore a reduction ordering).

Note: This works only for finite signatures!

To fix this for infinite signatures, the definition of simplification orderings and the definition of embedding have to be modified.
Kruskal’s Theorem

A (usually not strict) partial ordering $\succeq$ on a set $A$ is called well-partial-ordering (wpo), if for every infinite sequence $a_1, a_2, a_3, \ldots$ there are indices $i < j$ such that $a_i \preceq a_j$.

Terminology:

An infinite sequence $a_1, a_2, a_3, \ldots$ is called good, if there exist $i < j$ such that $a_i \preceq a_j$; otherwise it is called bad.

Therefore: $\succeq$ is a wpo iff every infinite sequence is good.
**Kruskal’s Theorem**

Lemma 3.32:
If $\succeq$ is a wpo, then every infinite sequence $a_1, a_2, a_3, \ldots$ has an infinite ascending subsequence $a_{i_1} \preceq a_{i_2} \preceq a_{i_3} \preceq \ldots$, where $i_1 < i_2 < i_3 < \ldots$.

Proof:
Let $a_1, a_2, a_3, \ldots$ be an infinite sequence. We call an index $m \geq 1$ terminal, if there is no $n > m$ such that $a_m \preceq a_n$. There are only finitely many terminal indices $m_1, m_2, m_3, \ldots$; otherwise the sequence $a_{m_1}, a_{m_2}, a_{m_3}, \ldots$ would be bad.
Choose $p > 1$ such that all $m \geq p$ are not terminal; define $i_1 = p$; define recursively $i_{j+1}$ such that $i_{j+1} > i_j$ and $a_{i_{j+1}} \succeq a_{i_j}$. 
Kruskal’s Theorem

Lemma 3.33:
If \( \succeq_1, \ldots, \succeq_n \) are wpo’s on \( A_1, \ldots, A_n \), then \( \succeq \) defined by
\[
(a_1, \ldots, a_n) \succeq (a_1', \ldots, a_n') \iff a_i \succeq_i a_i' \text{ for all } i
\]
is a wpo on \( A_1 \times \cdots \times A_n \).

Proof:
The case \( n = 1 \) is trivial.
Otherwise let \((a_1^{(1)}, \ldots, a_n^{(1)}), (a_1^{(2)}, \ldots, a_n^{(2)}), \ldots\) be an infinite sequence. By the previous lemma, there are infinitely many indices \( i_1 < i_2 < i_3 < \ldots \) such that \( a_{n}^{(i_1)} \preceq a_{n}^{(i_2)} \preceq a_{n}^{(i_3)} \preceq \ldots \).
By induction on \( n \), there are \( k < l \) such that \( a_{1}^{(i_k)} \preceq a_{1}^{(i_l)} \land \cdots \land a_{n-1}^{(i_k)} \preceq a_{n-1}^{(i_l)} \). Therefore \((a_1^{(i_k)}, \ldots, a_n^{(i_k)}) \preceq (a_1^{(i_l)}, \ldots, a_n^{(i_l)})\).
Kruskal’s Theorem

Theorem 3.34 (“Kruskal’s Theorem”):
Let \( \Sigma \) be a finite signature, let \( X \) be a finite set of variables. Then \( \succeq_{\text{emb}} \) is a wpo on \( T_\Sigma(X) \).

Proof:
Baader and Nipkow, page 114/115.
Theorem 3.35 (Dershowitz): If $\Sigma$ is a finite signature, then every simplification ordering $\succ$ on $T_{\Sigma}(X)$ is well-founded (and therefore a reduction ordering).

Proof:
Suppose that $t_1 \succ t_2 \succ t_3 \succ \ldots$ is an infinite decreasing chain.

First assume that there is an $x \in \operatorname{var}(t_{i+1}) \setminus \operatorname{var}(t_i)$. Let $\sigma = [t_i/x]$, then $t_{i+1}\sigma \supseteq x\sigma = t_i$ and therefore $t_i = t_i\sigma \succ t_{i+1}\sigma \succeq t_i$, contradicting reflexivity.

Consequently, $\operatorname{var}(t_i) \supseteq \operatorname{var}(t_{i+1})$ and $t_i \in T_{\Sigma}(V)$ for all $i$, where $V$ is the finite set $\operatorname{var}(t_1)$. By Kruskal’s Theorem, there are $i < j$ with $t_i \leq_{\text{emb}} t_j$. Hence $t_i \leq t_j$, contradicting $t_i \succ t_j$. 


Simplification Orderings

There are reduction orderings that are not simplification orderings and terminating TRSs that are not contained in any simplification ordering.

Example:

Let \( R = \{ f(f(x)) \rightarrow f(g(f(x))) \} \).

\( R \) terminates and \( \rightarrow^+ \) is therefore a reduction ordering.

Assume that \( \rightarrow_R \) were contained in a simplification ordering \( \succ \).

Then \( f(f(x)) \rightarrow_R f(g(f(x))) \) implies \( f(f(x)) \succ f(g(f(x))) \),
and \( f(g(f(x))) \trianglerighteq_{\text{emb}} f(f(x)) \) implies \( f(g(f(x))) \succeq f(f(x)) \),
hence \( f(f(x)) \succ f(f(x)) \).
Recursive Path Orderings

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering ("precedence") on $\Omega$.

The lexicographic path ordering $\succ_{lpo}$ on $T_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{lpo} t$ iff

1. $t \in \text{var}(s)$ and $t \neq s$, or
2. $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and
   (a) $s_i \succeq_{lpo} t$ for some $i$, or
   (b) $f \succ g$ and $s \succ_{lpo} t_j$ for all $j$, or
   (c) $f = g$, $s \succ_{lpo} t_j$ for all $j$, and
      $(s_1, \ldots, s_m) \prec_{lpo}(t_1, \ldots, t_n)$. 

Recursive Path Orderings

Lemma 3.36:
$s \succ_{lpo} t$ implies $\text{var}(s) \supseteq \text{var}(t)$.

Proof:
By induction on $|s| + |t|$ and case analysis.
Recursive Path Orderings

Theorem 3.37:
\( \succ_{lpo} \) is a simplification ordering on \( T_\Sigma(X) \).

Proof:
Show transitivity, subterm property, stability under substitutions, compatibility with \( \Sigma \)-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis.
Details: Baader and Nipkow, page 119/120.
Recursive Path Orderings

Theorem 3.38:
If the precedence \( \succ \) is total, then the lexicographic path ordering \( \succ_{\text{lpo}} \) is total on ground terms, i.e., for all \( s, t \in T_{\Sigma}(\emptyset) \):
\[
s \succ_{\text{lpo}} t \lor t \succ_{\text{lpo}} s \lor s = t.
\]

Proof:
By induction on \(|s| + |t|\) and case analysis.
Recursive Path Orderings

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering ("precedence") on $\Omega$. The lexicographic path ordering $\succ_{lpo}$ on $T_{\Sigma}(X)$ induced by $\succ$ is defined by: $s \succ_{lpo} t$ iff

(1) $t \in \text{var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and

   (a) $s_i \succeq_{lpo} t$ for some $i$, or

   (b) $f \succ g$ and $s \succ_{lpo} t_j$ for all $j$, or

   (c) $f = g$, $s \succ_{lpo} t_j$ for all $j$, and

       $(s_1, \ldots, s_m) (\succ_{lpo})_{\text{lex}} (t_1, \ldots, t_n)$. 
Recursive Path Orderings

There are several possibilities to compare subterms in (2)(c):

- compare list of subterms lexicographically left-to-right
  ("lexicographic path ordering (lpo)“, Kamin and Lévy)
- compare list of subterms lexicographically right-to-left
  (or according to some permutation \( \pi \))
- compare multiset of subterms using the multiset extension
  ("multiset path ordering (mpo)“, Dershowitz)

To each function symbol \( f/n \) associate a status \( \in \{\text{mul}\} \cup \{\text{lex}_\pi \mid \pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}\} \)
and compare according to that status
("recursive path ordering (rpo) with status")
The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering ("precedence") on $\Omega$, let $w : \Omega \cup X \rightarrow \mathbb{R}_0^+$ be a weight function, such that the following admissibility conditions are satisfied:

\[ w(x) = w_0 \in \mathbb{R}^+ \text{ for all variables } x \in X; \]
\[ w(c) \geq w_0 \text{ for all constants } c/0 \in \Omega. \]

If $w(f) = 0$ for some $f/1 \in \Omega$, then $f \succeq g$ for all $g \in \Omega$.

$w$ can be extended to terms as follows:

\[ w(t) = \sum_{x \in \text{var}(t)} w(x) \cdot \#(x, t) + \sum_{f \in \Omega} w(f) \cdot \#(f, t). \]
The Knuth-Bendix Ordering

The Knuth-Bendix ordering $\succ_{\text{kbo}}$ on $T_{\Sigma}(X)$ induced by $\succ$ and $w$ is defined by: $s \succ_{\text{kbo}} t$ iff

(1) $\#(x, s) \geq \#(x, t)$ for all variables $x$ and $w(s) > w(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables $x$, $w(s) = w(t)$, and
   (a) $t = x$, $s = f^n(x)$ for some $n \geq 1$, or
   (b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or
   (c) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and
       $(s_1, \ldots, s_m) (\succ_{\text{kbo}})_{\text{lex}} (t_1, \ldots, t_m)$. 


The Knuth-Bendix Ordering

Theorem 3.39:
The Knuth-Bendix ordering induced by \( \succ \) and \( w \) is a simplification ordering on \( T_\Sigma(X) \).

Proof:
Baader and Nipkow, pages 125–129.
3.6 Knuth-Bendix Completion

Completion:

Goal: Given a set \( E \) of equations, transform \( E \) into an equivalent convergent set \( R \) of rewrite rules.

How to ensure termination?

Fix a reduction ordering \( \succ \) and construct \( R \) in such a way that \( \rightarrow_R \subseteq \succ \) (i.e., \( l \succ r \) for every \( l \rightarrow r \in R \)).

How to ensure confluence?

Check that all critical pairs are joinable.
Knuth-Bendix Completion: Inference Rules

The completion procedure is presented as a set of inference rules working on a set of equations $E$ and a set of rules $R$:

$E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$

At the beginning, $E = E_0$ is the input set and $R = R_0$ is empty. At the end, $E$ should be empty; then $R$ is the result.

For each step $E, R \vdash E', R'$, the equational theories of $E \cup R$ and $E' \cup R'$ agree: $\approx_{E \cup R} = \approx_{E' \cup R'}$. 
Knuth-Bendix Completion: Inference Rules

Notations:

The formula \( s \simeq t \) denotes either \( s \simeq t \) or \( t \simeq s \).

\( \text{CP}(R) \) denotes the set of all critical pairs between rules in \( R \).
Knuth-Bendix Completion: Inference Rules

Orient:

\[
\frac{E \cup \{s \equiv t\}, \ R}{E, \ R \cup \{s \rightarrow t\}} \quad \text{if } s \succ t
\]

Note: There are equations \( s \equiv t \) that cannot be oriented, i.e., neither \( s \succ t \) nor \( t \succ s \).
Knuth-Bendix Completion: Inference Rules

Trivial equations cannot be oriented – but we don’t need them anyway:

\[ E \cup \{s \approx s\}, \quad R \]
\[ \frac{E, \quad R}{E, \quad R} \]

Delete:
Knuth-Bendix Completion: Inference Rules

Critical pairs between rules in \( R \) are turned into additional equations:

\[
\frac{E, \ R}{E \cup \{s \approx t\}, \ R} \quad \text{if } \langle s, t \rangle \in \text{CP}(R).
\]

**Deduce:**

Note: If \( \langle s, t \rangle \in \text{CP}(R) \) then \( s \leftarrow_R u \rightarrow_R t \) and hence \( R \models s \approx t \).
Knuth-Bendix Completion: Inference Rules

The following inference rules are not absolutely necessary, but very useful (e.g., to get rid of joinable critical pairs and to deal with equations that cannot be oriented):

**Simplify-Eq:**

\[
\begin{align*}
E \cup \{s \approx t\}, & \quad R \\
\Rightarrow & \quad E \cup \{u \approx t\}, \quad R \\
\text{if } s \rightarrow_R u.
\end{align*}
\]
Knuth-Bendix Completion: Inference Rules

Simplification of the right-hand side of a rule is unproblematic.

**R-Simplify-Rule:**

\[
\begin{align*}
E, & \quad R \cup \{s \rightarrow t\} \\
E, & \quad R \cup \{s \rightarrow u\}
\end{align*}
\]

if \( t \rightarrow_R u \).

Simplification of the left-hand side may influence orientability and orientation. Therefore, it yields an *equation*:

**L-Simplify-Rule:**

\[
\begin{align*}
E, & \quad R \cup \{s \rightarrow t\} \\
E \cup \{u \equiv t\}, & \quad R
\end{align*}
\]

if \( s \rightarrow_R u \) using a rule \( l \rightarrow r \in R \) such that \( s \sqsupset l \) (see next slide).
Knuth-Bendix Completion: Inference Rules

For technical reasons, the lhs of $s \rightarrow t$ may only be simplified using a rule $l \rightarrow r$, if $l \rightarrow r$ cannot be simplified using $s \rightarrow t$, that is, if $s \sqsupseteq l$, where the encompassment quasi-ordering $\sqsupseteq$ is defined by

$$s \sqsupseteq l \text{ if } s/p = l\sigma \text{ for some } p \text{ and } \sigma$$

and $\sqsubset = \sqsupset \setminus \sqsupseteq$ is the strict part of $\sqsupseteq$.

Lemma 3.40:
$\sqsubset$ is a well-founded strict partial ordering.
Knuth-Bendix Completion: Inference Rules

Lemma 3.41:
If $E, R \vdash E', R'$, then $\approx_{EUR} = \approx_{E'UR'}$.

Lemma 3.42:
If $E, R \vdash E', R'$ and $\rightarrow_R \subseteq \succ$, then $\rightarrow_{R'} \subseteq \succ$. 
Knuth-Bendix Completion: Correctness Proof

If we run the completion procedure on a set $E$ of equations, different things can happen:

(1) We reach a state where no more inference rules are applicable and $E$ is not empty.
   ⇒ Failure (try again with another ordering?)

(2) We reach a state where $E$ is empty and all critical pairs between the rules in the current $R$ have been checked.

(3) The procedure runs forever.

In order to treat these cases simultaneously, we need some definitions.
Knuth-Bendix Completion: Correctness Proof

A (finite or infinite sequence) \( E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots \) with \( R_0 = \emptyset \) is called a run of the completion procedure with input \( E_0 \) and \( \succcurlyeq \).

For a run, \( E_\infty = \bigcup_{i \geq 0} E_i \) and \( R_\infty = \bigcup_{i \geq 0} R_i \).

The sets of persistent equations or rules of the run are \( E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j \) and \( R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j \).

Note: If the run is finite and ends with \( E_n, R_n \), then \( E_* = E_n \) and \( R_* = R_n \).
Knuth-Bendix Completion: Correctness Proof

A run is called fair, if \( CP(R_*) \subseteq E_\infty \)
(i.e., if every critical pair between persisting rules is computed at some step of the derivation).

Goal:

Show: If a run is fair and \( E_* \) is empty,
then \( R_* \) is convergent and equivalent to \( E_0 \).

In particular: If a run is fair and \( E_* \) is empty,
then \( \approx E_0 = \approx E_\infty \cup R_\infty = \leftrightarrow E_\infty \cup R_\infty = \downarrow R_* \).
General assumptions from now on:

\[ E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots \text{ is a fair run.} \]

\[ R_0 \text{ and } E_* \text{ are empty.} \]
Knuth-Bendix Completion: Correctness Proof

A proof of $s \simeq t$ in $E_\infty \cup R_\infty$ is a finite sequence $(s_0, \ldots, s_n)$ such that $s = s_0$, $t = s_n$, and for all $i \in \{1, \ldots, n\}$:

1. $s_{i-1} \leftrightarrow_{E_\infty} s_i$, or
2. $s_{i-1} \rightarrow_{R_\infty} s_i$, or
3. $s_{i-1} \leftarrow_{R_\infty} s_i$.

The pairs $(s_{i-1}, s_i)$ are called proof steps.

A proof is called a rewrite proof in $R_*$, if there is a $k \in \{0, \ldots, n\}$ such that $s_{i-1} \rightarrow_{R_*} s_i$ for $1 \leq i \leq k$ and $s_{i-1} \leftarrow_{R_*} s_i$ for $k + 1 \leq i \leq n$.
Knuth-Bendix Completion: Correctness Proof

Idea (Bachmair, Dershowitz, Hsiang):

Define a well-founded ordering on proofs, such that for every proof that is not a rewrite proof in $R_*$ there is an equivalent smaller proof.

Consequence: For every proof there is an equivalent rewrite proof in $R_*$. 
Knuth-Bendix Completion: Correctness Proof

We associate a cost \( c(s_{i-1}, s_i) \) with every proof step as follows:

1. If \( s_{i-1} \leftrightarrow_{E_\infty} s_i \), then \( c(s_{i-1}, s_i) = (\{s_{i-1}, s_i\}, -, -) \), where the first component is a multiset of terms and \(-\) denotes an arbitrary (irrelevant) term.

2. If \( s_{i-1} \rightarrow_{R_\infty} s_i \) using \( l \rightarrow r \), then \( c(s_{i-1}, s_i) = (\{s_{i-1}\}, l, s_i) \).

3. If \( s_{i-1} \leftarrow_{R_\infty} s_i \) using \( l \rightarrow r \), then \( c(s_{i-1}, s_i) = (\{s_i\}, l, s_{i-1}) \).

Proof steps are compared using the lexicographic combination of the multiset extension of reduction ordering \( \succ \), the encompassment ordering \( \sqsupseteq \), and the reduction ordering \( \succ \).
Knuth-Bendix Completion: Correctness Proof

The cost $c(P)$ of a proof $P$ is the multiset of the costs of its proof steps.

The proof ordering $\succ_C$ compares the costs of proofs using the multiset extension of the proof step ordering.

Lemma 3.43:
$\succ_C$ is a well-founded ordering.
Lemma 3.44:
Let $P$ be a proof in $E_\infty \cup R_\infty$. If $P$ is not a rewrite proof in $R_*$, then there exists an equivalent proof $P'$ in $E_\infty \cup R_\infty$ such that $P \succ C P'$.

Proof:
If $P$ is not a rewrite proof in $R_*$, then it contains

(a) a proof step that is in $E_\infty$, or
(b) a proof step that is in $R_\infty \setminus R_*$, or
(c) a subproof $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$ (peak).

We show that in all three cases the proof step or subproof can be replaced by a smaller subproof:
Knuth-Bendix Completion: Correctness Proof

Case (a): A proof step using an equation $s \approx t$ is in $E_\infty$. This equation must be deleted during the run.

If $s \approx t$ is deleted using Orient:

$$\ldots s_{i-1} \leftrightarrow_{E_\infty} s_i \ldots \implies \ldots s_{i-1} \rightarrow_{R_\infty} s_i \ldots$$

If $s \approx t$ is deleted using Delete:

$$\ldots s_{i-1} \leftrightarrow_{E_\infty} s_{i-1} \ldots \implies \ldots s_{i-1} \ldots$$

If $s \approx t$ is deleted using Simplify-Eq:

$$\ldots s_{i-1} \leftrightarrow_{E_\infty} s_i \ldots \implies \ldots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \ldots$$
Knuth-Bendix Completion: Correctness Proof

Case (b): A proof step using a rule $s \rightarrow t$ is in $R_\infty \setminus R_*$. This rule must be deleted during the run.

If $s \rightarrow t$ is deleted using $R$-Simplify-Rule:

\[
\ldots s_{i-1} \rightarrow_{R_\infty} s_i \ldots \quad \Longrightarrow \quad \ldots s_{i-1} \rightarrow_{R_\infty} s' \leftarrow_{R_\infty} s_i \ldots
\]

If $s \rightarrow t$ is deleted using $L$-Simplify-Rule:

\[
\ldots s_{i-1} \rightarrow_{R_\infty} s_i \ldots \quad \Longrightarrow \quad \ldots s_{i-1} \rightarrow_{R_\infty} s' \leftarrow_{E_\infty} s_i \ldots
\]
Knuth-Bendix Completion: Correctness Proof

Case (c): A subproof has the form \( s_{i-1} \xleftarrow{R_*} s_i \xrightarrow{R_*} s_{i+1} \).

If there is no overlap or a non-critical overlap:

\[
\ldots s_{i-1} \xleftarrow{R_*} s_i \xrightarrow{R_*} s_{i+1} \ldots \implies \ldots s_{i-1} \xrightarrow{R_*} s' \xleftarrow{R_*} s_{i+1} \ldots
\]

If there is a critical pair that has been added using \textit{Deduce}:

\[
\ldots s_{i-1} \xleftarrow{R_*} s_i \xrightarrow{R_*} s_{i+1} \ldots \implies \ldots s_{i-1} \xleftarrow{E_\infty} s_i \ldots
\]

In all cases, checking that the replacement subproof is smaller than the replaced subproof is routine.
Knuth-Bendix Completion: Correctness Proof

Theorem 3.45:
Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \ldots$ be a fair run and let $R_0$ and $E_\ast$ be empty. Then

(1) every proof in $E_\infty \cup R_\infty$ is equivalent to a rewrite proof in $R_\ast$,
(2) $R_\ast$ is equivalent to $E_0$, and
(3) $R_\ast$ is convergent.
Knuth-Bendix Completion: Correctness Proof

Proof:
(1) By well-founded induction on $\succ_C$ using the previous lemma.

(2) Clearly $\approx_{E_\infty \cup R_\infty} = \approx_{E_0}$. Since $R_* \subseteq R_\infty$, we get $\approx_{R_*} \subseteq \approx_{E_\infty \cup R_\infty}$.
On the other hand, by (1), $\approx_{E_\infty \cup R_\infty} \subseteq \approx_{R_*}$.

(3) Since $\rightarrow_{R_*} \subseteq \succ$, $R_*$ is terminating.
By (1), $R_*$ is confluent.
Knuth-Bendix Completion: Outlook

Classical completion:

Fails, if an equation can neither be oriented nor deleted.

Unfailing Completion:

Use an ordering $\succ$ that is total on ground terms.

If an equation cannot be oriented, use it in both directions for rewriting (except if that would yield a larger term).

In other words, consider the relation $\leftrightarrow_E \cap \not\prec$.

Special case of superposition (see next chapter).