2.10 Refutational Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \Rightarrow N \vdash_{Res} \bot,$
  or equivalently: If $N \not\vdash_{Res} \bot,$ then $N$ has a model.

- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).

- Now order the clauses in $N$ according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.

- The limit interpretation can be shown to be a model of $N.$
Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is \textit{total} and \textit{well-founded}. (There exist many such orderings, e.g., the length-based ordering on atoms when these are viewed as words over a suitable alphabet.)

2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

\[
[\neg]A \succ_L [\neg]B, \text{ if } A \succ B
\]
\[
\neg A \succ_L A
\]

3. Extend $\succ_L$ to an ordering $\succ_C$ on ground clauses:

$\succ_C = (\succ_L)_{\text{mul}}$, the multi-set extension of $\succ_L$.

\textit{Notation:} $\succ$ also for $\succ_L$ and $\succ_C$. 
Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$. Then:

\[
A_0 \lor A_1 \\
A_1 \lor A_2 \\
\neg A_1 \lor A_2 \\
\neg A_1 \lor A_4 \lor A_3 \\
\neg A_1 \lor \neg A_4 \lor A_3 \\
\neg A_5 \lor A_5
\]
Properties of the Clause Ordering

Proposition 2.20:

1. The orderings on literals and clauses are total and well-founded.

2. Let $C$ and $D$ be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in $C$.
   
   (i) If $A \succ B$ then $C \succ D$.
   
   (ii) If $A = B$, $A$ occurs negatively in $C$ but only positively in $D$, then $C \succ D$. 
Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:

- All $D$ where $\max(D) = B$
- All $C$ where $\max(C) = A$
Closure of Clause Sets under $\text{Res}$

$\text{Res}(N) = \{ C \mid C \text{ is concl. of a rule in } \text{Res} \text{ w/ premises in } N \}$

$\text{Res}^0(N) = N$

$\text{Res}^{n+1}(N) = \text{Res}(\text{Res}^n(N)) \cup \text{Res}^n(N)$, for $n \geq 0$

$\text{Res}^*(N) = \bigcup_{n \geq 0} \text{Res}^n(N)$

$N$ is called saturated (wrt. resolution), if $\text{Res}(N) \subseteq N$.

Proposition 2.21:

(i) $\text{Res}^*(N)$ is saturated.

(ii) $\text{Res}$ is refutationally complete, iff for each set $N$ of ground clauses:

\[ N \models \bot \iff \bot \in \text{Res}^*(N) \]
Construction of Interpretations

Given: set $N$ of ground clauses, atom ordering $\succ$. 
Wanted: Herbrand interpretation $I$ such that

- “many” clauses from $N$ are valid in $I$;
- $I \models N$, if $N$ is saturated and $\bot \notin N$.

Construction according to $\succ$, starting with the minimal clause.
## Example

Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

<table>
<thead>
<tr>
<th></th>
<th>clauses $C$</th>
<th>$I_C$</th>
<th>$\Delta_C$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\neg A_0$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>true in $I_C$</td>
</tr>
<tr>
<td>2</td>
<td>$A_0 \lor A_1$</td>
<td>$\emptyset$</td>
<td>${A_1}$</td>
<td>$A_1$ maximal</td>
</tr>
<tr>
<td>3</td>
<td>$A_1 \lor A_2$</td>
<td>${A_1}$</td>
<td>$\emptyset$</td>
<td>true in $I_C$</td>
</tr>
<tr>
<td>4</td>
<td>$\neg A_1 \lor A_2$</td>
<td>${A_1}$</td>
<td>${A_2}$</td>
<td>$A_2$ maximal</td>
</tr>
<tr>
<td>5</td>
<td>$\neg A_1 \lor A_4 \lor A_3 \lor A_0$</td>
<td>${A_1, A_2}$</td>
<td>${A_4}$</td>
<td>$A_4$ maximal</td>
</tr>
<tr>
<td>6</td>
<td>$\neg A_1 \lor \neg A_4 \lor A_3$</td>
<td>${A_1, A_2, A_4}$</td>
<td>$\emptyset$</td>
<td>$A_3$ not maximal; \textit{min. counter-ex.}</td>
</tr>
<tr>
<td>7</td>
<td>$\neg A_1 \lor A_5$</td>
<td>${A_1, A_2, A_4}$</td>
<td>${A_5}$</td>
<td></td>
</tr>
</tbody>
</table>

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set

$\Rightarrow$ there exists a \textit{counterexample}. 


Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.

- When considering $C$, one already has a partial interpretation $I_C$ (initially $I_C = \emptyset$) available.

- If $C$ is true in the partial interpretation $I_C$, nothing is done. ($\Delta_C = \emptyset$).

- If $C$ is false, one would like to change $I_C$ such that $C$ becomes true.
Main Ideas of the Construction

- Changes should, however, be *monotone*. One never deletes anything from $I_C$ and the truth value of clauses smaller than $C$ should be maintained the way it was in $I_C$.

- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, $C$ is false in $I_C$, if $A$ occurs positively in $C$ (*adding $A$ will make $C$ become true*) and if this occurrence in $C$ is strictly maximal in the ordering on literals (*changing the truth value of $A$ has no effect on smaller clauses*).
Resolution Reduces Counterexamples

\[ \neg A_1 \lor A_4 \lor A_3 \lor A_0 \quad \neg A_1 \lor \neg A_4 \lor A_3 \]
\[ \neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0 \]

Construction of \( I \) for the extended clause set:

<table>
<thead>
<tr>
<th>clauses ( C )</th>
<th>( I_C )</th>
<th>( \Delta_C )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \neg A_0 )</td>
<td>( \emptyset )</td>
<td>( \emptyset )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( A_0 \lor A_1 )</td>
<td>( \emptyset )</td>
<td>{ ( A_1 ) }</td>
<td>{ ( A_1 ) }</td>
</tr>
<tr>
<td>( A_1 \lor A_2 )</td>
<td>{ ( A_1 ) }</td>
<td>( \emptyset )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( \neg A_1 \lor A_2 )</td>
<td>{ ( A_1 ) }</td>
<td>{ ( A_2 ) }</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0 )</td>
<td>{ ( A_1, A_2 ) }</td>
<td>( \emptyset )</td>
<td>A(_3) occurs twice minimal counter-ex.</td>
</tr>
<tr>
<td>( \neg A_1 \lor A_4 \lor A_3 \lor A_0 )</td>
<td>{ ( A_1, A_2 ) }</td>
<td>{ ( A_4 ) }</td>
<td>counterexample</td>
</tr>
<tr>
<td>( \neg A_1 \lor \neg A_4 \lor A_3 )</td>
<td>{ ( A_1, A_2, A_4 ) }</td>
<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor A_5 )</td>
<td>{ ( A_1, A_2, A_4 ) }</td>
<td>{ ( A_5 ) }</td>
<td></td>
</tr>
</tbody>
</table>

The same \( I \), but smaller counterexample, hence some progress was made.
Factorization Reduces Counterexamples

\[
\begin{align*}
\neg A_1 &\lor \neg A_1 \lor A_3 \lor A_3 \lor A_0 \\
\neg A_1 &\lor \neg A_1 \lor A_3 \lor A_0
\end{align*}
\]

Construction of \( I \) for the extended clause set:

<table>
<thead>
<tr>
<th>clauses ( C )</th>
<th>( I_C )</th>
<th>( \Delta_C )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \neg A_0 )</td>
<td>( \emptyset )</td>
<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>( A_0 \lor A_1 )</td>
<td>( \emptyset )</td>
<td>( { A_1 } )</td>
<td></td>
</tr>
<tr>
<td>( A_1 \lor A_2 )</td>
<td>( { A_1 } )</td>
<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor A_2 )</td>
<td>( { A_1 } )</td>
<td>( { A_2 } )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor \neg A_1 \lor A_3 \lor A_0 )</td>
<td>( { A_1, A_2 } )</td>
<td>( { A_3 } )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0 )</td>
<td>( { A_1, A_2, A_3 } )</td>
<td>( \emptyset )</td>
<td>true in ( I_C )</td>
</tr>
<tr>
<td>( \neg A_1 \lor A_4 \lor A_3 \lor A_0 )</td>
<td>( { A_1, A_2, A_3 } )</td>
<td>( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>( \neg A_1 \lor \neg A_4 \lor A_3 )</td>
<td>( { A_1, A_2, A_3 } )</td>
<td>( \emptyset )</td>
<td>true in ( I_C )</td>
</tr>
<tr>
<td>( \neg A_3 \lor A_5 )</td>
<td>( { A_1, A_2, A_3 } )</td>
<td>( { A_5 } )</td>
<td></td>
</tr>
</tbody>
</table>

The resulting \( I = \{ A_1, A_2, A_3, A_5 \} \) is a model of the clause set.
Construction of Candidate Models Formally

Let $N, \succ$ be given. We define sets $I_C$ and $\Delta_C$ for all ground clauses $C$ over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N, C = C' \lor A, A \succ C', I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that $C$ produces $A$, if $\Delta_C = \{A\}$.

The candidate model for $N$ (wrt. $\succ$) is given as $I_N^\succ := \bigcup_C \Delta_C$.

We also simply write $I_N$, or $I$, for $I_N^\succ$ if $\succ$ is either irrelevant or known from the context.
Let $A \succ B$; producing a new atom does not affect smaller clauses.
Proposition 2.22:

(i) $C = \neg A \lor C' \Rightarrow$ no $D \succeq C$ produces $A$.

(ii) $C$ productive $\Rightarrow I_C \cup \Delta_C \models C$.

(iii) Let $D' \succ D \succeq C$. Then

$$I_D \cup \Delta_D \models C \Rightarrow I_{D'} \cup \Delta_{D'} \models C \text{ and } I_N \models C.$$ 

If, in addition, $C \in N$ or $\max(D) \succ \max(C)$:

$$I_D \cup \Delta_D \not\models C \Rightarrow I_{D'} \cup \Delta_{D'} \not\models C \text{ and } I_N \not\models C.$$
(iv) Let $D' \triangleright D \triangleright C$. Then

$$I_D \models C \Rightarrow I_{D'} \models C \text{ and } I_N \models C.$$ 

If, in addition, $C \in N$ or $\max(D) \triangleright \max(C)$:

$$I_D \not\models C \Rightarrow I_{D'} \not\models C \text{ and } I_N \not\models C.$$ 

(v) $D = C \lor A$ produces $A \Rightarrow I_N \not\models C$. 

Some Properties of the Construction
Theorem 2.23 (Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let $N$ be saturated wrt. $Res$, and suppose that $\bot \not\in N$. Then $I_{N}^{\succ} \models N$.

Corollary 2.24:
Let $N$ be saturated wrt. $Res$. Then $N \models \bot \iff \bot \in N$. 
Model Existence Theorem

Proof of Theorem 2.23:
Suppose \( \bot \notin N \), but \( I_N^\prec \not\models N \). Let \( C \in N \) minimal (in \( \succ \) ) such that \( I_N^\prec \not\models C \). Since \( C \) is false in \( I_N \), \( C \) is not productive. As \( C \neq \bot \) there exists a maximal atom \( A \) in \( C \).

Case 1: \( C = \neg A \lor C' \) (i.e., the maximal atom occurs negatively)
\[ \Rightarrow I_N \models A \text{ and } I_N \not\models C' \]
\[ \Rightarrow \text{some } D = D' \lor A \in N \text{ produces } A. \text{ As } \frac{D' \lor A}{D' \lor C', \neg A \lor C'}, \text{ we infer that } D' \lor C' \in N, \text{ and } C \succ D' \lor C' \text{ and } I_N \not\models D' \lor C' \]
\[ \Rightarrow \text{contradicts minimality of } C. \]

Case 2: \( C = C' \lor A \lor A \). Then \( \frac{C' \lor A \lor A}{C' \lor A} \) yields a smaller counterexample \( C' \lor A \in N. \Rightarrow \text{contradicts minimality of } C. \)
Compactness of Propositional Logic

Theorem 2.25 (Compactness):
Let $N$ be a set of propositional formulas. Then $N$ is unsatisfiable, if and only if some finite subset $M \subseteq N$ is unsatisfiable.

Proof:

“$\leftarrow$”: trivial.

“$\rightarrow$”: Let $N$ be unsatisfiable.

$\Rightarrow \ Res^*(N)$ unsatisfiable

$\Rightarrow \bot \in Res^*(N)$ by refutational completeness of resolution

$\Rightarrow \exists n \geq 0 : \bot \in Res^n(N)$

$\Rightarrow \bot$ has a finite resolution proof $P$;
choose $M$ as the set of assumptions in $P$. 