The Superposition Calculus

One problem:

In the completeness proof for the resolution calculus, the following property holds:

If $C = C' \vee A$ with a strictly maximal and positive literal A is false in the current interpretation, then adding A to the current interpretation cannot make any literal of C' true.

This does not hold for superposition:

Let $a > b > c$.
Assume that the current rewrite system (representing the current interpretation) contains the rule $b \to c$.
Now consider the clause $a \approx b \vee a \approx c$.

The Superposition Calculus

We need a further inference rule to deal with clauses of this kind, either the "Merging Paramodulation" rule of Bachmair and Ganzinger or the following "Equality Factoring" rule due to Nieuwenhuis:

Equality Factoring: $$\frac{C' \lor s \approx t' \lor s \approx t}{C' \lor t \not\approx t' \lor s \approx t'}$$

Note: This inference rule subsumes the usual factoring rule.

The Superposition Calculus

How do the non-ground versions of the inference rules for superposition look like?

Main idea as in the resolution calculus:

Replace identity by unifiability.

Apply the mgu to the resulting clause.

In the ordering restrictions, replace $>$ by $\not\leq$.

The Superposition Calculus

However:

As in Knuth-Bendix completion, we do not want to consider overlaps at or below a variable position.

Consequence: there are inferences between ground instances $D\theta$ and $C\theta$ of clauses D and C which are *not* ground instances of inferences between D and C.

Such inferences have to be treated in a special way in the completeness proof.

The Superposition Calculus

Until now, we have seen most of the ideas behind the superposition calculus and its completeness proof.

We will now start again from the beginning giving precise definitions and proofs.

The Superposition Calculus

Inference rules (part 1):

Pos. Superposition:

$$\frac{D' \vee t \approx t' \qquad C' \vee s[u] \approx s'}{(D' \vee C' \vee s[t'] \approx s')\sigma}$$

where $\sigma = \text{mgu}(t, u)$ and u is not a variable.

Neg. Superposition:

$$\frac{D' \vee t \approx t' \qquad C' \vee s[u] \not\approx s'}{(D' \vee C' \vee s[u] \not\approx s')\sigma}$$

where $\sigma = \text{mgu}(t, u)$ and u is not a variable.

The Superposition Calculus

Inference rules (part 2):

Equality Resolution:

$$\frac{C' \vee s \not\approx s'}{C'\sigma}$$

where $\sigma = \mathrm{mgu}(s, s')$.

Equality Factoring:

$$\frac{C' \vee s' \approx t' \vee s \approx t}{(C' \vee t \not\approx t' \vee s \approx t')\sigma}$$

where $\sigma = \mathrm{mgu}(s, s')$.

The Superposition Calculus

Theorem:

All inference rules of the superposition calculus are correct, i. e., for every rule

$$\frac{C_n, \ldots, C_1}{C_0}$$

we have $\{C_1, \ldots, C_n\} \models C_0$.

Proof:

Exercise.

The Superposition Calculus

Orderings:

Let $>$ be a reduction ordering that is total on ground terms.

To a positive literal $s \approx t$, we assign the multiset $\{s, t\}$,
to a negative literal $s \not\approx t$ the multiset $\{s, s, t, t\}$.
The literal ordering $>_L$ compares these multisets using the multiset extension of $>$.

The clause ordering $>_C$ compares clauses by comparing their multisets of literals using the multiset extension of $>_L$.

The Superposition Calculus

Inferences have to be computed only if the following ordering restrictions are satisfied:

- the last literal in each premise is maximal in the respective premise (i. e., there exists no greater one) (strictly maximal for positive literals in superposition inferences, i. e., there exists no greater or equal one),
- in these literals, the lhs is not smaller than the rhs (neither smaller nor equal in superposition inferences), and
- in superposition inferences, after applying the unifier to both premises, the left premise is not greater than the second one.

The Superposition Calculus

A ground inference is called redundant w. r. t. a set of ground clauses N, if its conclusion follows from clauses in N that are smaller than the largest (i. e., rightmost) premise.

An inference is redundant w. r. t. a set of clauses N, if all its ground instances are redundant w. r. t. $\overline{N}$, where $\overline{N}$ is the set of all ground instances of clauses in N.

N is called saturated up to redundancy, if every inference from clauses in N is redundant w. r. t. N.

Superposition: Refutational Completeness

For a set E of ground equations, $\mathrm{T}_\Sigma(\emptyset)/E$ is an E-interpretation (or E-algebra) with universe $\{\, [t] \mid t \in \mathrm{T}_\Sigma(\emptyset) \,\}$.

One can show (similar to the proof of Birkhoff's Theorem) that for every *ground* equation $s \approx t$ we have $\mathrm{T}_\Sigma(\emptyset)/E \models s \approx t$ if and only if $s \leftrightarrow_E^* t$.

In particular, if E is a convergent set of rewrite rules R and $s \approx t$ is a ground equation, then $\mathrm{T}_\Sigma(\emptyset)/R \models s \approx t$ if and only if $s \downarrow_R t$. By abuse of terminology, we say that an equation or clause is valid (or true) in R if and only if it is true in $\mathrm{T}_\Sigma(\emptyset)/R$.

Superposition: Refutational Completeness

Model construction:

Let N be a set of clauses not containing $\bot$.
Using induction on the clause ordering we define sets of rewrite rules E_C and R_C for all $C \in \overline{N}$ as follows:

Assume that E_D has already been defined for all $D \in \overline{N}$ with $D <_C C$. Then $R_C = \bigcup_{D <_C C} E_D$.

Superposition: Refutational Completeness

The set E_C contains the rewrite rule $s \rightarrow t$, if

(a) $C = C' \vee s \approx t$.

(b) $s \approx t$ is strictly maximal in C.

(c) $s > t$.

(d) C is false in R_C.

(e) C' is false in $R_C \cup \{s \rightarrow t\}$.

(f) s is irreducible w. r. t. R_C.

In this case, C is called productive. Otherwise $E_C = \emptyset$.

Finally, $R_\infty = \bigcup_{D \in \bar{N}} E_D$.

Superposition: Refutational Completeness

Lemma:
If $E_D = \{u \rightarrow v\}$ and $E_C = \{s \rightarrow t\}$, then $C >_C D$ if and only if $s > u$.

Superposition: Refutational Completeness

Corollary:

The rewrite systems R_C and R_∞ are convergent.

Proof:

Obviously, $s > t$ for all rules $s \rightarrow t$ in R_C and R_∞.

Furthermore, it is easy to check that there are no critical pairs between any two rules: Assume that there are rules $u \rightarrow v$ in E_D and $s \rightarrow t$ in E_C such that u is a subterm of s. As $>$ is a reduction ordering that is total on ground terms, we get $u < s$ and therefore $D <_C C$ and $E_D \subseteq R_C$. But then s would be reducible by R_C, contradicting condition (f).

Superposition: Refutational Completeness

Lemma:

If $D \leq_C C$ and $E_C = \{s \rightarrow t\}$, then $s > u$ for every term u occurring in a negative literal in D and $s \geq v$ for every term v occurring in a positive literal in D.

Superposition: Refutational Completeness

Corollary:
If $D \in \bar{N}$ is true in R_D, then D is true in R_∞ and R_C for all $C >_C D$.

Proof:
If a positive literal of D is true in R_D, then this is obvious.

Otherwise, some negative literal $s \not\approx t$ of D must be true in R_D, hence $s \not\downarrow_{R_D} t$. As the rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than s and t, they cannot be used in a rewrite proof of $s \downarrow t$, hence $s \not\downarrow_{R_C} t$ and $s \not\downarrow_{R_\infty} t$.

Superposition: Refutational Completeness

Corollary:
If $D = D' \vee u \approx v$ is productive, then D' is false and D is true in R_∞ and R_C for all $C >_C D$.

Proof:
Obviously, D is true in R_∞ and R_C for all $C >_C D$.

Since all negative literals of D' are false in R_D, it is clear that they are false in R_∞ and R_C. For the positive literals $u' \approx v'$ of D', condition (e) ensures that they are false in $R_D \cup \{u \to v\}$. Since $u' \leq u$ and $v' \leq u$ and all rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than u, these rules cannot be used in a rewrite proof of $u' \downarrow v'$, hence $u' \not\downarrow_{R_C} v'$ and $u' \not\downarrow_{R_\infty} v'$.

Superposition: Refutational Completeness

Lemma ("Lifting Lemma"):
Let C be a clause and let θ be a substitution such that $C\theta$ is ground. Then every equality resolution or equality factoring inference from $C\theta$ is a ground instance of an inference from C.

Proof:
Exercise.

Superposition: Refutational Completeness

Lemma ("Lifting Lemma"):
Let $D = D' \vee u \approx v$ and $C = C' \vee [\neg]\, s \approx t$ be two clauses (without common variables) and let θ be a substitution such that $D\theta$ and $C\theta$ are ground.

If there is a superposition inference between $D\theta$ and $C\theta$ where $u\theta$ and some subterm of $s\theta$ are overlapped, and $u\theta$ does not occur in $s\theta$ at or below a variable position of s, then the inference is a ground instance of a superposition inference from D and C.

Proof:
Exercise.

Superposition: Refutational Completeness

Theorem:
Let N be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then we have for every ground clause $C\theta \in \overline{N}$:

(i) $E_{C\theta} = \emptyset$ if and only if $C\theta$ is true in $R_{C\theta}$.

(ii) $C\theta$ is true in R_∞ and R_{C_0} for all $C_0 >_C C\theta$.

Proof:
We use induction on $>_C$ and assume that (i) and (ii) are already satisfied for all clauses in $\overline{N}$ that are smaller than $C\theta$.

Note: the "if" part of (i) is obvious from the model construction and (ii) follows directly from (i) and the two corollaries above.