# Proving Termination: Monotone Mappings

Let $(A, >_A)$ and $(B, >_B)$ be partial orderings.
A mapping $\varphi : A \to B$ is called monotone,
if $x >_A y$ implies $\varphi(x) >_B \varphi(y)$ for all $x, y \in A$.

Lemma:
If $\varphi : A \to B$ is a monotone mapping from $(A, >_A)$ to $(B, >_B)$ and $(B, >_B)$ is well-founded, then $(A, >_A)$ is well-founded.

## Proving Termination: Lexicographic Product

Let $(A, >_A)$ and $(B, >_B)$ be partial orderings.
The lexicographic product $>_{A \times B}$ on $A \times B$ is defined by

$$(x, y) >_{A \times B} (x', y') \text{ iff } (x >_A x') \vee (x = x' \wedge y >_B y').$$

Lemma:
The lexicographic product of two partial orderings is a partial ordering.

# Proving Termination: Lexicographic Product

Lemma:
The lexicographic product of two well-founded partial orderings is a well-founded partial ordering.

Proof:
Assume that there is an infinite decreasing chain

$$(a_0, b_0) >_{A\times B} (a_1, b_1) >_{A\times B} \dots .$$

This implies $a_0 \geq_A a_1 \geq_A \dots .$
Since $>_A$ is well-founded, this chain can only contain finitely many strict steps $a_i >_A a_{i+1}$.
Hence there is a $k$ such that $a_i = a_{i+1}$ for all $i \geq k$. But then $b_i >_B b_{i+1}$ for all $i \geq k$, contradicting the well-foundedness of $>_B$.

## Proving Termination: Lexicographic Product

Lemma:

The lexicographic product of two strict total orderings
is a strict total ordering.

Proof:

by case analysis.

## Proving Termination: Lexicographic Product

The lexicographic product $>^n_{lex}$ of partial orderings $(A_i, >_i)$ with $1 \leq i \leq n$ can be defined analogously for $n$-tuples with $n > 2$.

The resulting relation is again a partial ordering;
it is well-founded if the orderings $(A_i, >_i)$ are well-founded,
and it is total if the orderings $(A_i, >_i)$ are total.

## Proving Termination: Lexicographic Product

Note: Given an ordering $(A, >_A)$, one can define a lexicographic ordering $>_{Lex}$ on $A^* = \bigcup_{i \geq 0} A^i$ by

$$\begin{aligned} w >_{Lex} w' \;\text{ iff }\; & (w = w'v \;\wedge\; |v| > 0) \\ & \vee\; (w = uxv \;\wedge\; w' = ux'v' \;\wedge\; x >_A x'). \end{aligned}$$

However, this ordering is not well-founded!

# Proving Termination: Lexicographic Product

To get a well-founded ordering on $A^*$, one has to compare the length of tuples first ("length/lexicographic combination"):

$$
\begin{aligned}
w >^*_{llex} w' \quad \text{iff} \quad & (|w| > |w'|) \\
& \vee \; (|w| = |w'| = n \; \wedge \; w >^n_{lex} w'),
\end{aligned}
$$

where $>^n_{lex}$ is the lexicographic ordering on $n$-tuples.

# Proving Termination: Multisets

A multiset $M$ over $A$ is a function $M : A \longrightarrow \mathbb{N}$.

Intuitively, a multiset is a set with (finitely often) repeated elements; $M(x)$ is the number of copies of $x$ in $M$.

We use similar notation as for sets; for instance we write $\{a, c, c\}$ for the multiset $\{a \mapsto 1,\ b \mapsto 0,\ c \mapsto 2\}$.

# Proving Termination: Multisets

A multiset $M$ is called finite, if $\{ x \in A \mid M(x) > 0 \}$ is finite.

$\mathcal{M}(A)$ denotes the set of all finite multisets over $A$.

From now on we will consider only finite multisets.

# Proving Termination: Multisets

Notations:

Element: $x \in M$ iff $M(x) > 0$

Submultiset: $M \subseteq N$ iff for all $x \in A$: $M(x) \leq N(x)$

Union: $(M \cup N)(x) = M(x) + N(x)$

Difference: $(M \setminus N)(x) = M(x) \dot{-} N(x)$

Intersection: $(M \cap N)(x) = \min\{M(x), N(x)\}$

where $m \dot{-} n = m - n$ if $m \geq n$, and $m \dot{-} n = 0$ otherwise.

# Proving Termination: Multisets

Multiset extension:

Let $(A, >)$ be a partial ordering. We define an ordering $>_{mul}$ over $\mathcal{M}(A)$ as follows:

$$
\begin{array}{lll}
M >_{mul} N & \text{iff} & \text{there exist } X, Y \in \mathcal{M}(A) \text{ such that} \\
& & \emptyset \neq X \subseteq M \text{ and} \\
& & N = (M \setminus X) \cup Y \text{ and} \\
& & \forall y \in Y\ \exists x \in X :\ x > y.
\end{array}
$$

# Proving Termination: Multisets

Lemma:

The multiset extension $>_{mul}$ of a partial ordering $>$ is a partial ordering.

Proof:

Baader and Nipkow, page 22/23.

## Proving Termination: Multisets

Lemma ("König's Lemma"):
A finitely branching tree is infinite, if and only if it contains an infinite path.

Proof:
"if": trivial.
"only if": by well-founded induction over the subtree relation.

# Proving Termination: Multisets

Theorem:

The multiset extension of a partial ordering $>$ is well-founded if and only if $>$ is well-founded.

Proof:

Baader and Nipkow, page 23/24.

# Proving Termination: Multisets

Lemma: $M >_{mul} N$ if and only if $M \neq N$ and for every $n \in N \setminus M$ there is an $m \in M \setminus N$ such that $m > n$.

Proof:

Baader and Nipkow, page 24/25.

# Proving Termination: Multisets

Corollary: If the ordering $>$ is total, then its multiset extension $>_{mul}$ is total.

Proof:
Let $>$ be total.
If the multisets $M$ and $N$ are different, then there exists a greatest element $m \in A$ such that $M(m) \neq N(m)$.
W.o.l.o.g, let $M(m) > N(m)$, hence $m \in M \setminus N$.
Then for every $n \in N \setminus M$ we have $m > n$, hence $M >_{mul} N$.

# 3 Rewrite Systems

## Rewrite Relations

Let $E$ be a set of equations.

The rewrite relation $\rightarrow_E \subseteq \mathsf{T}_\Sigma(X) \times \mathsf{T}_\Sigma(X)$ is defined by

$$
\begin{aligned}
s \rightarrow_E t \quad \text{iff} \quad & \text{there exist } (l \approx r) \in E,\ p \in \mathrm{Pos}(s), \\
& \text{and } \sigma : X \to \mathsf{T}_\Sigma(X), \\
& \text{such that } s/p = l\sigma \text{ and } t = s[r\sigma]_p.
\end{aligned}
$$

## Rewrite Relations

An equation $l \approx r$ is also called a rewrite rule, if $l$ is not a variable and $\mathrm{Var}(l) \supseteq \mathrm{Var}(r)$.

Notation: $l \rightarrow r$.

A set of rewrite rules is called a term rewrite system (TRS).

## Rewrite Relations

We say that a set of equations $E$ or a TRS $R$ is terminating, if the rewrite relation $\longrightarrow_E$ or $\longrightarrow_R$ has this property.

(Analogously for other properties of abstract reduction systems).