Two's Company, Three's a Crowd: Stable Family and Threesome Roommates Problems

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Abstract. We investigate Knuth's eleventh open question on stable matchings. In the stable family problem, sets of women, men, and dogs are given, all of whom state their preferences among the other two groups. The goal is to organize them into family units, so that no three of them have incentive to desert their assigned family members to join in a new family. A similar problem, called the threesome roommates problem, assumes that a group of persons, each with their preferences among the combinations of two others, are to be partitioned into triples. Similarly, the goal is to make sure that no three persons want to break up with their assigned roommates.
Ng and Hirschberg were the first to investigate these two problems. In their formulation, each participant provides a strictly-ordered list of all combinations. They proved that under this scheme, both problems are NP-complete. Their paper reviewers pointed out that their reduction exploits *inconsistent* preference lists and they wonder whether these two problems remain NP-complete if preferences are required to be consistent. We answer in the affirmative.
In order to give these two problems a broader outlook, we also consider the possibility that participants can express indifference, on the condition that the preference consistency has to be maintained. As an example, we propose a scheme in which all participants submit two (or just one in the roommates case) lists ranking the other two groups separately. The order of the combinations is decided by the sum of their ordinal numbers. Combinations are tied when the sums are equal. By introducing indifference, a hierarchy of stabilities can be defined. We prove that all stability definitions lead to NP-completeness for existence of a stable matching.

1 Problem Definition

Knuth proposed twelve open questions on the stable matching problem [9]. The eleventh question asks whether the well-studied stable marriage problem [3] can be generalized to the case of three parties, women, men, and dogs. In this paper, we call this problem the *stable family* problem and refer generically to all participants in this problem as "players." Roughly speaking, given sets of women, men, and dogs, all of whom state their preferences among the other two groups, the goal is to organize them into family units so that there is no *blocking triple*: three players each preferring one another to their assigned family members. A problem in a similar vein, which we call the *threesome roommates* problem, assumes that $3n$ students are to be assigned to the dormitory bedrooms in some college. They state their preferences of the combinations of two other persons. The goal is to partition them into sets of size 3. Such a partition (matching) is said to be stable if no three persons each prefer the others to their assigned roommates.

As Knuth does not specify any precise definition of "preference" and "blocking triples," one can conceive a number of ways to define the two problems. One possible formulation is that each player submits a strictly-ordered preference list, ranking all possible combinations that she/he/it can get in a matching. We call such a scheme strictly-ordered-complete-list (**SOCL**) scheme. In this setting, Ng and Hirschberg [10] proved that both problems are NP-complete.

At the end of their paper, Ng and Hirschberg mentioned that their reviewers pointed out their reduction allows preference to be *inconsistent*. For example, man m might rank (w_1, d_1) higher than (w_2, d_1), but he also ranks (w_2, d_2) higher than (w_1, d_2). In other words, he does not consistently prefer woman w_1 over woman w_2 (nor the other way around). Independently, Subramanian [11] gave an alternative NP-completeness proof for stable family, but his reduction also uses inconsistent lists.

The reviewers of Ng and Hirschberg wondered whether these two problems remain NP-complete if inconsistency is disallowed. To answer this open question and to motivate some variants problems we will define, we introduce the notion of *preference posets* and *simple lists*. In stable family, assuming that each player has two simple lists in which two different types of players are ranked separately, a preference poset is a product poset of the two simple lists. In such a poset, the combination (w_1, d_1) precedes another combination (w_2, d_2) only if w_1 ranks at least as high as w_2 and d_1 at least as high as d_2 in the simple lists. If neither combination precedes the other, they are incomparable. Similarly, in threesome roommates, the preference poset is the product poset of the one simple list with itself. By this notion, the question raised by the reviewers of Ng and Hirschberg can be rephrased as follows. Under the **SOCL** scheme, if every player has to submit a preference list which is a *linear extension* of her/his/its preference poset, are the stable family and the threesome roommates still NP-complete? We answer in the affirmative.

In an attempt to give these two problems a broader outlook, we then allow players to express indifference by giving full preference lists containing ties. In particular, to capture the spirit of maintaining consistency in the preferences, we stipulate that the full list must be a *relaxed linear extension* of a preference poset: strict precedence order in the poset has to be observed in the relaxed linear extension; only incomparable elements in the poset can be tied.

We propose the following scheme to make the above concept concrete. Suppose that a player submits two simple lists (or just one in the roommates case). We create a full list, ranking the combinations based on the sums of their ordinal numbers. For example, for man m, the combination of his rank-2 woman and rank-5 dog is as good as that of his rank-4 woman and rank-3 dog; while both of them are inferior to the combination of his top-ranked woman and his top-ranked dog. We call such a scheme precedence-by-ordinal-number (**PON**) scheme. The **PON** scheme produces full preference lists which are relaxed linear extensions of preference posets. Also, one can envisage an even more flexible scheme. For example, instead of giving "ranks," the players can provide "ratings" of other players. The order of the combinations can be decided by the sum of the ratings; two combinations are tied only when the sums of their ratings are equivalent.

Setting theoretical concerns aside for a moment, the above schemes are probably more practicable when n is large, because a player only has to provide lists of $\Theta(n)$ length, while under the **SOCL** scheme, they have to give strictly ordered lists of size $\Theta(n^2)$.

By allowing indifference, we can define 4 different types of blocking triples and, based on them, build up a hierarchy of stabilities. (This hierarchy is similar to that constructed by Irving in the context of 2-party stable matchings [7].)

- Weak Stable Matching: a blocking triple is one in which all three players of the blocking triple strictly prefer the other two members in the triple over their assigned family members (roommates).
- Strong Stable Matching: a blocking triple is one in which at least two players of the blocking triple strictly prefer the other two players in the triple to their assigned family members (roommates), while the remaining player can be indifferent or also strictly prefer the other two players in the triple.
- Super Stable Matching: a blocking triple is one in which at least one player of the blocking triple strictly prefers the other two players in the triple to her/his/its assigned family members (roommates), while the remaining players can be indifferent or also strictly prefer the other two players in the triple.
- Ultra Stable Matching: a blocking triple is one in which all three players in the triple are at least indifferent to the others.

Note that if ties are not allowed in the full preference lists, i.e., the **SOCL** scheme, then blocking triples can only be of degree 3. Thus there can be only one type of stability. For presentational reason, in this case, we refer to the stability under the **SOCL** scheme as the weak stability.

Our Results and Paper Roadmap We will prove in the paper that, if full preference lists are (relaxed) linear extensions of preference posets, the problem of deciding whether weak/strong/super/ultra stable matchings exist is NP-complete in both the stable family problem and the threesome roommates problem. Our reduction techniques are inspired by Ng and Hirschberg's, although the consistency requirement in the preferences makes our construction more involved. In presenting our result, instead of directly answering the open question posed by Ng and Hirschberg's reviewers by studying weak-stability, we make a detour to first study strong/super/ultra stability. Introducing them first helps us to explain our intuition behind the more complex reduction for the former problem.

As is well-known, the stable marriage and the stable roommates problems can be solved in $O(n^2)$ time, by the Gale-Shapley algorithm [3] and by the Irving algorithm [6], respectively. Unfortunately, our results, along with Ng and Hirschberg and Subramanian's, indicate that attempts to efficiently solve the stable matching problem in generalized cases of three (or more) parties are unlikely to be fruitful. This is not surprising, as in theoretical computer science, the fine line between **P** and **NP** is often drawn between the numbers two and three.

We organize the paper as follows. In Section 2, we present necessary notation; Section 3 proves the NP-completeness of strong/super/ultra stable matchings in the stable family problem under the **PON** scheme; Section 4 presents a reduction to transform a stable family problem to a threesome roommate problem, thus establishing the NP-completeness of strong/super/ultra stable matchings in the latter; Section 5 considers the **SOCL** scheme and proves the NP-completeness of (weak) stable matchings, thereby answering the open question posed by the anonymous reviewers of Ng and Hirschberg. Section 6 concludes and discusses related issues. Due to space constraint, we omit some proofs. See [5] for full details.

2 Preliminaries

We use $\mathcal{M}$, $\mathcal{W}$, $\mathcal{D}$ to indicate the sets of men, women, and dogs in stable family; the students in threesome roommates are denoted as $\mathcal{R}$. In stable family, $L_g(p)$ denotes the simple list of player p on the players of type $g \in \{\mathcal{M}, \mathcal{W}, \mathcal{D}\}$. For example $L_{\mathcal{W}}(m)$ is the simple list of man m among women $\mathcal{W}$. In threesome roommates, we simply write $L(m)$, where $m \in \mathcal{R}$, dropping the subscript.

In general, we use the notation $\succ$ to denote the precedence order (in either posets or in linear lists). For example, supposing that p_i ranks higher than p_j in the list l, we write $p_i \succ_l p_j$. In a poset Q, two elements q_i, q_j either one precedes the other, which we write $q_i \succ_Q q_j$ or $q_j \succ_Q q_i$, or they are incomparable, which is expressed as $q_i ||_Q q_j$. The notation $\succ$ is also used to express explicitly the order of players in simple lists. For example, we write $L(p) = q \succ r \succ \cdots$ to show that player p prefers player q to player r. Note also that the notation $\cdots$ denotes the remaining players in arbitrary order. We use the notation $r_p(q)$ to indicate the rank of q on player p's simple list.

We say a blocking triple is of degree i, if i players strictly prefer the triple while the remaining $3-i$ players are indifferent. Unless stated otherwise, in the article, when we say some triple "blocks," it is always a blocking triple of degree 3.

A preference poset constructed from lists l_1 and l_2 is written as $l_1 \times l_2$. To be precise, given lists l_1 and l_2 and the poset $l_1 \times l_2$, supposing that $\{p_i, p_j\}, \{p_{i'}, p_{j'}\} \in l_1 \times l_2$, then $\{p_i, p_j\} \succ_{l_1 \times l_2} \{p_{i'}, p_{j'}\}$ only if (1) $p_i \succ_{l_1} p_{i'}, p_j = p_{j'}$, or (2) $p_j \succ_{l_2} p_{j'}, p_i = p_{i'}$, or (3) $p_i \succ_{l_1} p_{i'}, p_j \succ_{l_2} p_{j'}$. The notation $\pi(X)$ means an arbitrary permutation of elements in the set X. $E_\pi(l_1 \times l_2)$ is an arbitrary linear extension of the preference poset $l_1 \times l_2$.

3 Reducing Three-dimensional Matching to Stable Family

In this section, we focus on the NP-completeness of strong stable matching under the **PON** scheme. Similar results hold for super stable and ultra stable matchings by a straightforward argument and will be discussed at the end of this section.

Our reduction is from the three-dimensional matching problem, one of the 21 NP-complete problems in Karp's seminal paper [8]. The problem instance is given in the form $\Upsilon = (\mathcal{M}, \mathcal{W}, \mathcal{D}, \mathcal{T})$, where $\mathcal{T} \subseteq \mathcal{M} \times \mathcal{W} \times \mathcal{D}$. The goal is to decide whether a perfect matching $\mathcal{M} \subseteq \mathcal{T}$ exists. This problem remains NP-complete even if every player in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ appears exactly 2 or 3 times in the triples of $\mathcal{T}$ [4].

We first explain the intuition behind our reduction. Supposing that man m_i appears in three triples $(m_i, w_{ia}, d_{ia}), (m_i, w_{ib}, d_{ib}), (m_i, w_{ic}, d_{ic})$ in $\mathcal{T}$, we create three *dopplegangers*, m_{i1}, m_{i2}, m_{i3} in the derived stable family problem instance Υ'. We also create four garbage collectors, $w^g_{i1}, d^g_{i1}, w^g_{i2}, d^g_{i2}$. Each doppleganger m_{ij} puts a woman-dog pair, with whom man m_i shares a triple, and the garbage collectors on top of his two simple lists. The goal of our design is that in a stable matching, exactly one doppleganger will be matched to a woman-dog pair with whom m_i shares a triple in $\mathcal{T}$, while the other two dopplegangers will be matched to garbage collectors. In the case that there are only two triples in $\mathcal{T}$ containing man m_i, we artificially make a copy of one of the triples, making the total number of triples three, and treat him as described above.

Now, we will refer to the set of dopplegangers as $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, the set of garbage collectors as $\mathcal{W}_1^g, \mathcal{W}_2^g, \mathcal{D}_1^g, \mathcal{D}_2^g$ and the original set of real women and real dogs as $\mathcal{W}, \mathcal{D}$. Collectively, we refer to them as $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{W} \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g \cup \mathcal{D}$.

To realize our plan, we introduce two gadgets. The first is three sets of "dummy players": $m_1^{\#}, w_1^{\#}, d_1^{\#}, m_2^{\#}, w_2^{\#}, d_2^{\#}, m_3^{\#}, w_3^{\#}, d_3^{\#}$. Their preferences are such that they must be matched to one another in a stable matching. To be precise, for $j \in \{1, 2, 3\}$,

– $L_{\mathcal{W}}(m_j^{\#}) = w_j^{\#} \succ \cdots, L_{\mathcal{D}}(m_j^{\#}) = d_j^{\#} \succ \cdots$
– $L_{\mathcal{M}}(w_j^{\#}) = m_j^{\#} \succ \cdots, L_{\mathcal{D}}(w_j^{\#}) = d_j^{\#} \succ \cdots$
– $L_{\mathcal{M}}(d_j^{\#}) = m_j^{\#} \succ \cdots, L_{\mathcal{W}}(d_j^{\#}) = w_j^{\#} \succ \cdots$

These nine dummy players are used to "pad" the preference lists of other players. Their purpose will be clear shortly.

Another gadget we need is a set of "guard players" for each doppleganger in $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. They will make sure that in a stable matching, a doppleganger m_{ij} will only get a woman-dog pair with whom m_i shares a triple in $\mathcal{T}$ or those garbage collectors. As an example, consider the doppleganger m_{i1}. He has six associated guard players, $m_{i1}^{\flat 1}, w_{i1}^{\flat 1}, d_{i1}^{\flat 1}, m_{i1}^{\flat 2}, w_{i1}^{\flat 2}, d_{i1}^{\flat 2}$ and their preferences are summarized below:

– $L_{\mathcal{W}}(m_{i1}) = w_{i2}^g \succ w_{i1}^g \succ w_{ia} \succ w_{i1}^{\flat 1} \succ w_{i1}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots,$
  $L_{\mathcal{D}}(m_{i1}) = d_{i2}^g \succ d_{i1}^g \succ d_{ia} \succ d_{i1}^{\flat 2} \succ d_{i1}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
– $L_{\mathcal{W}}(m_{i1}^{\flat 1}) = w_{i1}^{\flat 1} \succ \cdots, L_{\mathcal{D}}(m_{i1}^{\flat 1}) = d_{i1}^{\flat 1} \succ \cdots$
  $L_{\mathcal{W}}(m_{i1}^{\flat 2}) = w_{i1}^{\flat 2} \succ \cdots, L_{\mathcal{D}}(m_{i1}^{\flat 2}) = d_{i1}^{\flat 2} \succ \cdots$
– $L_{\mathcal{M}}(w_{i1}^{\flat 1}) = m_{i1} \succ m_{i1}^{\flat 1} \succ \cdots, L_{\mathcal{D}}(w_{i1}^{\flat 1}) = d_{i1}^{\flat 1} \succ d_1^{\#} \succ \cdots$
  $L_{\mathcal{M}}(w_{i1}^{\flat 2}) = m_{i1} \succ m_{i1}^{\flat 2} \succ \cdots, L_{\mathcal{D}}(w_{i1}^{\flat 2}) = d_{i1}^{\flat 2} \succ d_1^{\#} \succ \cdots$
– $L_{\mathcal{M}}(d_{i1}^{\flat 1}) = m_{i1} \succ m_{i1}^{\flat 1} \succ \cdots, L_{\mathcal{W}}(d_{i1}^{\flat 1}) = w_{i1}^{\flat 1} \succ w_1^{\#} \succ \cdots$
  $L_{\mathcal{M}}(d_{i1}^{\flat 2}) = m_{i1} \succ m_{i1}^{\flat 2} \succ \cdots, L_{\mathcal{W}}(d_{i1}^{\flat 2}) = w_{i1}^{\flat 2} \succ w_1^{\#} \succ \cdots$

The following case analysis proves that, in a stable matching M', m_{i1} will get only players from the set $\{w_{i1}^g, w_{i2}^g, w_{ia}, d_{i1}^g, d_{i2}^g, d_{ia}\}$.

– Suppose that m_{i1} gets two players ranking below $w_{i1}^{\flat 1}$ and $d_{i1}^{\flat 1}$ respectively. It can be observed that for both $w_{i1}^{\flat 1}, d_{i1}^{\flat 1}$, the best man is m_{i1}. Therefore, they would prefer m_{i1} and so does he them, inducing a blocking triple to M', a contradiction.
– Suppose that m_{i1} gets a woman $w \in \{w_{ia}, w_{i1}^g, w_{i2}^g\}$ and a dog d ranking below $d_{i1}^{\flat 1}$. In this case, we can be sure that d cannot be $d_1^{\#}$ or $d_2^{\#}$ or $d_3^{\#}$, since their preferences guarantee that they will only be matched to other dummy players. So, $r_{m_{i1}}(w) + r_{m_{i1}}(d) \geq 10$, while $r_{m_{i1}}(w_{i1}^{\flat 1}) + r_{m_{i1}}(d_{i1}^{\flat 1}) = 9$, causing $(m_{i1}, w_{i1}^{\flat 1}, d_{i1}^{\flat 1})$

to become a blocking triple. This example explains why we need to pad the simple lists of m_{i1} with dummy players.

The case that m_{i1} gets a dog $d \in \{d_{ia}, d^g_{i1}, d^g_{i2}\}$ and a woman w ranking lower than $w^{\flat 2}_{i1}$ follows analogous arguments; $(m_{i1}, w^g_{i2}, d^g_{i2})$ will become a blocking triple.

- Suppose that m_{i1} gets only one of the players from the set $\{w^{\flat 1}_{i1}, w^{\flat 2}_{i1}, d^{\flat 1}_{i1}, d^{\flat 2}_{i1}\}$. Without loss of generality, we assume that $(m_{i1}, w^{\flat 1}_{i1}, d^{\phi}), d^{\phi} \neq d^{\flat 1}_{i1}$, is part of the matching. For woman $w^{\flat 1}_{i1}$, dog d^{ϕ} cannot be the dummy player $d^{\#}_1$. Therefore, $r_{w^{\flat 1}_{i1}}(m_{i1}) + r_{w^{\flat 1}_{i1}}(d^{\phi}) \geq 4 > 3 = r_{w^{\flat 1}_{i1}}(m^{\flat 1}_{i1}) + r_{w^{\flat 1}_{i1}}(d^{\flat 1}_{i1})$. Similarly for $d^{\flat 1}_{i1}$, $r_{d^{\flat 1}_{i1}}(m^{\flat 1}_{i1}) + r_{d^{\flat 1}_{i1}}(w^{\flat 1}_{i1}) = 3$, which is better than whatever combination it can get. Therefore, we have that $(m^{\flat 1}_{i1}, w^{\flat 1}_{i1}, d^{\flat 1}_{i1})$ constitutes a blocking triple to M'. This example shows why we need to pad the preference of $w^{\flat 1}_{i1}, d^{\flat 1}_{i1}$ (and also $w^{\flat 2}_{i1}, d^{\flat 2}_{i1}$) with dummy players.
- Suppose that m_{i1} gets $w^{\flat 1}_{i1}$ and $d^{\flat 1}_{i1}$. Note that $w^{\flat 1}_{i1} \succ w^{\flat 2}_{i1}$ and $d^{\flat 2}_{i1} \succ d^{\flat 1}_{i1}$. Therefore, m_{i1} is *indifferent* to the combinations of $w^{\flat 2}_{i1}$ and $d^{\flat 2}_{i1}$, since $r_{m_{i1}}(w^{\flat 1}_{i1}) + r_{m_{i1}}(d^{\flat 1}_{i1}) = 9 = r_{m_{i1}}(w^{\flat 2}_{i1}) + r_{m_{i1}}(d^{\flat 2}_{i1})$. Additionally, $w^{\flat 2}_{i1}, d^{\flat 2}_{i1}$ strictly prefer m_{i1}. Hence $(m_{i1}, w^{\flat 2}_{i1}, d^{\flat 2}_{i1})$ constitutes a blocking triple of degree 2 to M'. This explains why we need two sets of guard players to guarantee that the doppleganger will "behave" in a stable matching.

  The case that m_{i1} gets $w^{\flat 2}_{i1}$ and $d^{\flat 2}_{i1}$ follows analogous arguments.

The other two dopplegangers m_{i2}, m_{i3} also have six associated guard players for each; they, along with their associated guard players, have similar preferences to guarantee that m_{i2} and m_{i3} will only get garbage collectors or the woman-dog pairs with whom m_i shares triples. The only difference in the lists is that m_{i2} and m_{i3} replace w_{ia}, d_{ia} with w_{ib}, d_{ib}, and with w_{ic}, d_{ic}, respectively, in their simple lists. For a summary of the simple lists of members in the set X, see Table 1. It should be noted that w^g_{i1}, d^g_{i1} (and also w^g_{i2}, d^g_{i2}) rank the three dopplegangers in reverse order. This trick guarantees that the dopplegangers will not form blocking triples with the garbage collectors, defeating our purpose. For example, suppose (m_{i1}, w_{ia}, d_{ia}) is part of the matching, we want to avoid $(m_{i1}, w^g_{i1}, d^g_{i1})$ to becoming a blocking triple. It can be easily verified that if w^g_{i1} and d^g_{i1} are matched to m_{i2} or m_{i3}, such a blocking triple will not be formed.

Finally, garbage collectors also use dummy players to pad their simple lists, to avoid the awkward situation that some doppleganger is matched to a real woman and a garbage collector dog (or a real dog and a garbage collector woman). How this arrangement works will be clear in the proof below.

Lemma 1. *Suppose a stable matching M' exists in the derived stable family problem instance Υ'. The following facts hold in M':*

- *Fact A: The three sets of dummy players are matched to one another.*
- *Fact B: For each doppleganger $m_{ij} \in \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$, the ranks of his family members in M' are at least as high as 3 in his simple lists.*
- *Fact C: The six associated guard players of each doppleganger $m_{ij} \in \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ are matched to one another.*

Table 1. The simple lists of all players in the set $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{W} \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g \cup \mathcal{D}$. We assume that there exist three triples $(m_i, w_{ia}, d_{ia}), (m_i, w_{ib}, d_{ib}), (m_i, w_{ic}, d_{ic})$ in $\mathcal{T}$.

Player	Simple Lists
$m_{i1} \in \mathcal{M}_1$	$L_{\mathcal{W}}(m_{i1}) = w_{i2}^g \succ w_{i1}^g \succ w_{ia} \succ w_{i1}^{\flat 1} \succ w_{i1}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$ $L_{\mathcal{D}}(m_{i1}) = d_{i2}^g \succ d_{i1}^g \succ d_{ia} \succ d_{i1}^{\flat 2} \succ d_{i1}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$m_{i2} \in \mathcal{M}_1$	$L_{\mathcal{W}}(m_{i2}) = w_{i2}^g \succ w_{i1}^g \succ w_{ib} \succ w_{i2}^{\flat 1} \succ w_{i2}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$ $L_{\mathcal{D}}(m_{i2}) = d_{i2}^g \succ d_{i1}^g \succ d_{ib} \succ d_{i2}^{\flat 2} \succ d_{i2}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$m_{i3} \in \mathcal{M}_1$	$L_{\mathcal{W}}(m_{i3}) = w_{i2}^g \succ w_{i1}^g \succ w_{ic} \succ w_{i3}^{\flat 1} \succ w_{i3}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$ $L_{\mathcal{D}}(m_{i3}) = d_{i2}^g \succ d_{i1}^g \succ d_{ic} \succ d_{i3}^{\flat 2} \succ d_{i3}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$w_{i1}^g \in \mathcal{W}_1^g$	$L_{\mathcal{M}}(w_{i1}^g) = m_{i1} \succ m_{i2} \succ m_{i3} \succ \cdots$ $L_{\mathcal{D}}(w_{i1}^g) = d_{i1}^g \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$d_{i1}^g \in \mathcal{D}_1^g$	$L_{\mathcal{M}}(d_{i1}^g) = m_{i3} \succ m_{i2} \succ m_{i1} \succ \cdots$ $L_{\mathcal{W}}(d_{i1}^g) = w_{i1}^g \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$
$w_{i2}^g \in \mathcal{W}_2^g$	$L_{\mathcal{M}}(w_{i2}^g) = m_{i1} \succ m_{i2} \succ m_{i3} \succ \cdots$ $L_{\mathcal{D}}(w_{i2}^g) = d_{i2}^g \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$d_{i2}^g \in \mathcal{D}_2^g$	$L_{\mathcal{M}}(d_{i2}^g) = m_{i3} \succ m_{i2} \succ m_{i1} \succ \cdots$ $L_{\mathcal{W}}(d_{i2}^g) = w_{i2}^g \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$
$w \in \mathcal{W}$	$L_{\mathcal{M}}(w) = \cdots$ $L_{\mathcal{D}}(w) = \cdots$
$d \in \mathcal{D}$	$L_{\mathcal{M}}(d) = \cdots$ $L_{\mathcal{W}}(d) = \cdots$

Proof. Fact A follows directly from construction. Fact B is true as we have argued in the case analysis before. Fact C is true because if the guard players are not matched to one another, they will block M', unless $w_{ij}^{\flat 1}, d_{ij}^{\flat 1}$ or $w_{ij}^{\flat 2}, d_{ij}^{\flat 2}$ are matched to m_{ij} in M', but this is impossible because of Fact B. □

Lemma 2. *Suppose a stable matching M' exists in the derived stable family problem instance Υ'. Consider the garbage collectors $w_{i1}^g, d_{i1}^g, w_{i2}^g, d_{i2}^g$ created for man $m_i \in \mathcal{M}$. We must have that w_{i1}^g, d_{i1}^g belong to the same triple t_1 and that w_{i2}^g, d_{i2}^g belong to the same triple t_2 in M'. Moreover, in t_1 and t_2, the man player must be one of the dopplegangers m_{i1}, m_{i2} and m_{i3}.*

Proof. We will prove this lemma by progressively establishing the following facts.

***Fact D**: w_{i2}^g and d_{i2}^g must belong to the same triple t_2 in M'.*

Proof: For a contradiction, suppose that w_{i2}^g and d_{i2}^g are in different triples in M'. We claim that $(m_{i1}, w_{i2}^g, d_{i2}^g)$ forms a blocking triple. It is obvious that m_{i1} and w_{i2}^g prefer such a triple. Now let the man and woman partners of d_{i2}^g be m^ϕ and $w^\phi \neq w_{i2}^g$; then by Fact A in Lemma 1, $r_{d_{i2}^g}(w) \geq 5$. We have that $r_{d_{i2}^g}(m_{i1}) + r_{d_{i2}^g}(w_{i2}^g) = 4 < 6 \leq r_{d_{i2}^g}(m^\phi) + r_{d_{i2}^g}(w^\phi)$. So d_{i2}^g will also prefer m_{i1} and w_{i2}^g, forming a blocking triple with them to M'. This proof also shows why we need to pad the preferences of the garbage collectors.

***Fact E**: w_{i1}^g and d_{i1}^g must belong to the same triple t_1 in M'.*

Proof: For a contradiction, suppose that $(m^{\phi 1}, w_{i1}^g, d^{\phi 1})$ and $(m^{\phi 2}, w^{\phi 2}, d_{i1}^g)$ are triples in M'. There exists at least one doppleganger in $\{m_{i1}, m_{i2}, m_{i3}\}$ preferring the combination of w_{i1}^g and d_{i1}^g (since at most one doppleganger can be matched to w_{i2}^g and d_{i2}^g). Let such a doppleganger be m_{ij}. Then by Fact A in Lemma 1, $r_{w_{i1}^g}(m_{ij}) + r_{w_{i1}^g}(d_{i1}^g) \leq 4 < 6 \leq r_{w_{i1}^g}(m^{\phi 1}) + r_{w_{i1}^g}(d^{\phi 2})$; and similarly, $r_{d_{i1}^g}(m_{ij}) + r_{d_{i1}^g}(w_{i1}^g) \leq 4 < 6 \leq r_{d_{i1}^g}(m^{\phi 2}) + r_{d_{i1}^g}(w^{\phi 2})$, implying that $(m_{ij}, w_{i1}^g, d_{i1}^g)$ blocks M'.

***Fact F**: w_{i2}^g and d_{i2}^g must be matched to one of the dopplegangers of m_i in M', and so are w_{i1}^g and d_{i1}^g.*

Proof: If w_{i2}^g and d_{i2}^g are not matched to a doppleganger of m_i, then any doppleganger m_{ij} will prefer the combination of them over his family members, causing $(m_{ij}, w_{i2}^g, d_{i2}^g)$ to block M'. A similar argument applies to the case of w_{i1}^g and d_{i1}^g, giving the lemma. □

By the previous two lemmas, we have established the correctness of the reduction on one side.

Lemma 3. *(Sufficiency) If there exists a stable matching M' in the derived stable family problem instance Υ', there exists a perfect matching M in the original three-dimensional matching instance Υ.*

To show the necessity, we need to prove one more lemma.

Lemma 4. *In a matching M' in the derived stable family problem instance Υ', suppose that dummy players are matched to one another. Suppose further that the garbage collectors of m_i are matched to two of the dopplegangers of m_i, while the remaining doppleganger m_{ij} is matched to a real woman and a real dog with whom m_i shares a triple in $\mathcal{T}$ in the original three-dimensional matching instance Υ. Then there is no blocking triple in which the dopplegangers m_{i1}, m_{i2}, and m_{i3} are involved.*

Proof. We assume that $(m_{i1}, w_{i2}^g, d_{i2}^g), (m_{i2}, w_{i1}^g, d_{i1}^g), (m_{i3}, w_{ic}, d_{ic}) \in M'$. Other cases follow analogous arguments. We claim that there does not exist a blocking triple of the form $(m_{ij}, w_{i1}^g, d^{\phi 1})$, $(m_{ij}, w_{i2}^g, d^{\phi 2})$, $(m_{ij}, w^{\phi 3}, d_{i1}^g)$, and $(m_{ij}, w^{\phi 4}, d_{i2}^g)$ where $d^{\phi 1} \neq d_{i1}^g$, $d^{\phi 2} \neq d_{i2}^g$, $w^{\phi 3} \neq w_{i1}^g$, and $w^{\phi 4} \neq w_{i2}^g$. We only argue the first case. Since $d^{\phi 1} \notin \{d_1^{\#}, d_2^{\#}, d_3^{\#}\}$, we have $r_{w_{i1}^g}(d^{\phi 1}) \geq 5 > 3 = r_{w_{i1}^g}(d_{i1}^g) + r_{w_{i1}^g}(m_{i2})$. Therefore, w_{i1}^g has no incentive to join the combination of m_{ij} and $d^{\phi 1}$.

Now we only need to consider the three remaining potential blocking triples: $(m_{i2}, w_{i2}^g, d_{i2}^g), (m_{i3}, w_{i2}^g, d_{i2}^g)$, $(m_{i3}, w_{i1}^g, d_{i1}^g)$. It can be easily verified that they do not block M' because the orders of the three dopplegangers in the simple lists of w_{i1}^g and d_{i1}^g (and also w_{i2}^g and d_{i2}^g) are reversed. □

Lemma 5. *(Necessity) Suppose that there is a perfect matching M in the original three-dimensional matching instance Υ. There also exists a stable matching M' in the derived stable family problem instance Υ'.*

Proof. We build a stable matching M' in Υ' as follows. Let the dummy players $\{m^{\#}_j, w^{\#}_j, d^{\#}_j\}, 1 \leq j \leq 3$, be matched to one another. Given any doppleganger m_{ij}, let his guard players $\{m^{\flat 1}_{ij}, w^{\flat 1}_{ij}, d^{\flat 1}_{ij}\}, \{m^{\flat 2}_{ij}, w^{\flat 2}_{ij}, d^{\flat 2}_{ij}\}$ be matched to one another as well. Furthermore, suppose that $(m_i, w_{ix}, d_{ix}) \in M$. Let the doppleganger who lists w_{ix} and d_{ix} above his guard players be matched to w_{ix} and d_{ix}, while the other two dopplegangers be matched to the garbage collectors. By this construction, it can be seen that none of the guard players and dummy players will be part of a blocking triple. This, combined with Lemma 4, completes the proof. □

Suppose that in the given three-dimensional matching instance Υ, $|\mathcal{M}| = |\mathcal{W}| = |\mathcal{D}| = n$. Then in the derived instance Υ', we use in all $3n$ dopplegangers, $18n$ guard players, $4n$ garbage collectors, $2n$ real women and real dogs, and 9 dummy players. Their preferences (in the form of simple lists) can be generated in $O(n^2)$ time. Therefore, this is a polynomial-time reduction. Also, given any matching, we definitely can check its stability in $O(n^3)$ time. Combining the two facts with Lemma 3 and Lemma 5, we can conclude:

Theorem 1. *It is NP-complete to decide whether strong stable matchings exist under the* ***PON*** *scheme. Therefore, the question of deciding existence of strong stable matching is also NP-complete when the full preference lists are consistent, i.e., when they are relaxed linear extensions of preference posets.*

Super Stability and Ultra Stability It can be observed that throughout the proof, all arguments involving blocking triples use those of degree 3. The only exception is the occasion that we argue that a doppleganger cannot be matched to his guard players in a stable matching. To recall, supposing that $(m_{ij}, w^{\flat 1}_{ij}, d^{\flat 1}_{ij})$ is part of a matching, then $(m_{ij}, w^{\flat 2}_{ij}, d^{\flat 2}_{ij})$ is a blocking triple of degree 2. (Or if the latter is part of the matching, the former is a blocking triple of degree 2). Therefore, our reduction only uses blocking triples of degree 2 or 3; both are still blocking triples with regard to super stability and ultra stability. Moreover, when we argue the strong-stability of matchings in the reduction, we never allow blocking triples of degree 0 or degree 1 to exist. Therefore, essentially, our reduction has also established the NP-completeness of super stable matchings and ultra stable matchings.

4 Threesome Roommates with Relaxed Linear Extensions of Preference Posets

In this section, we exhibit a reduction of stable family to threesome roommates, thereby establishing the NP-completeness of strong/super/ultra stable matchings in the latter problem. Instead of the **PON** scheme, we use the more general scheme in which any relaxed linear extension of preference posets is allowed. We choose to use this scheme because the involved reduction technique has a different flavor. Nonetheless, we do have another reduction for the **PON** scheme. See [5] for details.

Let an instance of stable family problem be $\Upsilon = (\mathcal{M}, \mathcal{W}, \mathcal{D}, \Psi)$, where Ψ represents the preferences of the players in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$. We create an instance of threesome roommates $\Upsilon' = (\mathcal{R}', \Psi')$ by copying all players in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ into $\mathcal{R}'$. Regarding the preferences in Ψ', we first build up the simple lists of all players.

– Suppose $m \in \mathcal{M}, L(m) = L_{\mathcal{W}}(m) \succ L_{\mathcal{D}}(m) \succ \pi(\mathcal{M} - \{m\})$.
– Suppose $w \in \mathcal{W}, L(w) = L_{\mathcal{D}}(w) \succ L_{\mathcal{M}}(w) \succ \pi(\mathcal{W} - \{w\})$.
– Suppose $d \in \mathcal{D}, L(d) = L_{\mathcal{M}}(d) \succ L_{\mathcal{W}}(d) \succ \pi(\mathcal{D} - \{d\})$.

In words, a man lists all women and then all dogs, based respectively on their original order in his simple lists in Ψ. He then attaches other fellow men in arbitrary order to the end of his list. Women and dogs have analogous arrangements in their simple lists.

Having constructed the simple lists, we still need to build consistent relaxed linear extensions. We will exploit the following lemma, whose proof can be found in the full version [5].

Lemma 6. *Let l be a strictly-ordered list. Suppose that l is decomposed into nonempty contiguous sublists $(l_1, l_2, \cdots, l_k)$ such that (1) $\bigcup_{i=1}^{k} l_i = l$, (2) if $e \succ_{l_i} f$, then $e \succ_l f$, and (3) if $e \in l_i, f \in l_j, i < j$, then $e \succ_l f$. Then there exists a linear extension of $l \times l$ such that all combinations drawn from $\{l_i, l_j\}$ precede all pairs drawn from $\{l_{i'}, l_{j'}\}$, provided that $i \leq j, i' \leq j'$ and one of the following conditions holds (1) $i < i'$, (2) $i = i', j < j'$.*

By Lemma 6, we can construct the linear extensions as follows:

– Consider $m \in \mathcal{M}$ and assume that $W = L_{\mathcal{W}}(m), D = L_{\mathcal{D}}(m), N = \pi(\mathcal{M} - \{m\})$. His relaxed linear extension is: $E_\pi(W \times W) \succ X \succ E_\pi(W \times N) \succ E_\pi(D \times D) \succ E_\pi(D \times N) \succ E_\pi(N \times N)$, where X is the original relaxed linear extension of man m's preference poset given in Ψ.
– Consider $w \in \mathcal{W}$ and assume that $D = L_{\mathcal{D}}(w), N = L_{\mathcal{M}}(w), W = \pi(\mathcal{W} - \{w\})$. Her relaxed linear extension is: $E_\pi(D \times D) \succ Y \succ E_\pi(D \times W) \succ E_\pi(N \times N) \succ E_\pi(N \times W) \succ E_\pi(W \times W)$, where Y is the original relaxed linear extension of woman w's preference poset given in Ψ.
– Consider $d \in \mathcal{D}$ and assume that $N = L_{\mathcal{M}}(d), W = L_{\mathcal{W}}(d), D = \pi(\mathcal{D} - \{d\})$. Its relaxed linear extension is: $E_\pi(N \times N) \succ Z \succ E_\pi(N \times D) \succ E_\pi(W \times W) \succ E_\pi(W \times D) \succ E_\pi(D \times D)$, where Z is the original relaxed linear extension of dog d's preference poset given in Ψ.

To prove that the reduction from Υ to Υ' is valid, we will rely heavily on the following technical lemma.

Lemma 7. *In the derived instance Υ', if a stable matching M' exists, every triple in M' must contain a man, a woman, and a dog. Moreover, suppose that in a matching M'' in Υ' in which each player gets two other types of players as roommates, then a blocking triple cannot contain two (or three) players of the same type.*

Proof. For the first part, we argue case by case.

1. If $\{m, w_i, w_j\} \in M'$, there exists another man m' who can get neither a woman-woman combination nor a woman-dog combination. By construction, m' would prefer any woman-dog combination to his assigned roommates in M'. Similarly, there exists a dog d' who gets another fellow dog in M'. Such a dog would prefer a man-woman combination to its assigned roommates in M'. Finally, woman w_i and w_j would prefer a dog-man combination. Therefore, both $\{m', w_i, d'\}$ and $\{m', w_j, d'\}$ block M', a contradiction.
2. If $\{m, m_i, m_j\} \in M'$, then there exists a woman w who gets a fellow woman in M' and a dog d who gets a fellow dog in M'. Thus, woman w would prefer a dog-man combination and dog d would prefer a man-woman combination. Therefore, $\{m, w, d\}, \{m_i, w, d\}, \{m_j, w, d\}$ block M', a contradiction.
3. All other cases can be argued similarly.

For the second part, suppose that matching M'' has the stated property. Given any man m, by our construction, if there is a blocking triple containing m and in which there are two players of the same type, the only possibility of a blocking triple is $\{m, w_i, w_j\}$. However, neither w_i nor w_j would prefer such a triple, because in our construction, for a woman, a dog-man combination is better than a man-woman combination. The other potential blocking triples not involving men follow analogous arguments, thus giving us the lemma. □

It is straightforward to use Lemma 7 to prove our reduction is a valid one.

Theorem 2. *Deciding whether strong/super/ultra stable matchings exist in the threesome roommates problem is NP-complete when full preference lists are consistent, i.e., when they are relaxed linear extension of preference posets.*

5 Weak Stability under the SOCL Scheme

Due to space constraint, we can only state our results and leave the details to the full version [5].

Theorem 3. *It is NP-complete to decide whether weak stable matchings exist under the **SOCL** scheme, for both the stable family and the threesome roommates problems. Hence, it is also NP-complete to decide whether a weak stable matching exists when consistent preferences are allowed to contain ties: i.e. the full preferences are relaxed linear extensions of preference posets.*

6 Conclusion and Related Problems

In this paper, we answer the open question of whether the stable family and the threesome roommates problems are NP-complete if all players have to provide consistent preference lists. We introduce a scheme in which players can express indifference on the precondition that their preferences have to be consistent. Under this scheme, a variety of stabilities are defined and we prove that all lead to NP-complete problems.

Since we have proved that the general cases of stable family and threesome roommates are NP-complete, a natural question to ask is whether there are special cases that allow polynomial time solutions. Actually, examples of the two problems that can be solved efficiently do exist.

Consider the following scheme. Every player submits two simple lists. A man evaluates combinations first by the woman he gets, then by the dog; a woman first by the man she gets, then by the dog; a dog first by the man it gets, then by the woman. (Note the asymmetry). It is not hard to see that we can apply the Gale-Shapley algorithm twice to get a weak stable matching: letting the men propose to women and then propose to dogs. Women and dogs make the decision of acceptance or rejection based on their simple lists of men [2]. Merging the two matchings will give a stable matching in the stable family problem.

However, even a little twist can make the above scheme hard to solve. Suppose a man decides first based on the woman he gets and then the dog; a woman first based on the dog she gets and then on the man; a dog decides first based on the man it gets then on the woman. The Gale-Shapley algorithm no longer works [1].

Interestingly, the above scheme is reminiscent of another open problem allegedly originated by Knuth. Suppose that a man has only a simple list for women; a woman has only a simple list for dogs; a dog has only a simple list for men. This problem is called *circular stable matching*. Its complexity is still unknown.

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