1 Introducing a special variant of Linear Arrow Debreu Markets

In this lecture we study Linear Arrow Debreu markets under certain assumptions (which may seem to be economically unnatural, but they are algorithmically intuitive). You will show how to remove these assumptions in the exercises. We state the model more formally. We are given a set \( B \) of \( n \) rational agents and a set \( G \) of \( n \) divisible goods (there is one unit of every good). Agent \( i \) owns only good \( i \) completely (This means that agent \( i \) does not own any fraction of any other good and no other agent owns any fraction of good \( i \)). In the lecture notes we may also refer to agent \( i \) as \( b_i \), and good \( i \) as \( g_i \) (It should be clear from the context). Additionally we also have a Utility matrix \( v \), where \( v_{ij} \) denotes the utility agent \( i \) derives from one unit of good \( j \). Additionally for every set \( B' \subset B \) of agents, there exists an agent \( i \) in \( B' \) and a good \( j \), such that \( v_{ij} > 0 \) and agent \( j \) is not in \( B' \). This assumption prevents any proper subset of agents to invest only in the goods they own (and form a local equilibrium among themselves). Notice that in this scenario the budget/ money of buyer \( i \) equals the price of good \( i \). Now, our goal is to compute a market clearing price vector \( p \) and an allocation matrix (you may also refer to this as the money flow matrix) \( x \), such that,

- \( x_{ij} > 0 \) only if \( \frac{v_{ij}}{p_j} \geq \frac{v_{ik}}{p_k} \) for all \( k \in [n] \). (Follows from Linearity of the utility function of the buyers (fractional knapsack)).
- \( \sum_{j \in [n]} x_{ij} = p_i \) for all \( i \in [n] \) (All buyers invest all of their budget).
- \( \sum_{i \in [n]} x_{ij} = p_j \) for all \( j \in [n] \) (All goods are completely sold).

2 The intuitive structure of the algorithm

Walras argued that if a price vector \( p \) is not a market clearing price, then there must exist a good, where demand exceeds supply. Therefore, increase the price of this good until demand equals supply. We first analyze the consequences of such an update. Increasing the price of a good in demand will decrease its demand (since the buyers may either run out of budget as they are forced to invest more or find interest in some other good - the latter case will also increase the demand of few goods that have low demand). This seems to help us move closer to the equilibrium. However increasing the price of the high demand good will also result in an increased budget/ money for the agent that owns the same good (This is exactly where the Arrow Debreu Market differs from the Fisher Market and is therefore more general). This may also increase the demand of other goods that are in high demand. It is therefore not clear whether we are moving closer to the equilibrium or not. However Walras claimed that such a process will eventually converge...
The capacity of the edges \((b_i, g_j)\) is \(\infty\).

Equality Network depends on the Price vector \(p\).

Figure 1: The market clearing price vector in the view of an equality network. \(p\) is a market clearing price vector iff the max-flow in \(N_p\) has magnitude \(\sum_{i \in [n]} p_i\).

(without proof). The Duan Mehlhorn Algorithm intends to perform such multiplicative updates on the prices carefully.

3 Tools for the Algorithm

In the previous section we talked about demands and supply. We now formally define what it means for a good to be in high demand. Towards this goal we introduce two crucial tools.

3.1 Equality Networks \(N_p\)

For a given price vector \(p\), the equality network is a graph with vertex set \(s \cup t \cup B \cup G\) and edges

- \((s, b_i)\) with capacity \(p_i\) for all \(i \in [n]\).
- \((g_i, t)\) with capacity \(p_i\) for all \(i \in [n]\).
- \((b_i, g_j)\) with capacity \(\infty\) iff \(\frac{v_{ij}}{p_k} \geq \frac{v_{ik}}{p_k}\) for all \(k \in [n]\) (Bang per Buck Edge. The goods an agent invests in is also called bang per buck goods for the agent).

**Proposition 1: Market Clearing Prices in the light of Equality networks**

\(p\) is a market clearing price vector iff the max-flow in \(N_p\) has magnitude \(\sum_{i \in [n]} p_i\).

3.2 Balanced Flows

We need the concept of balanced flows to talk about convergence of these multiplicative update rules. Additionally this will also help us to be precise with the notions of demand and supply of goods. To talk about balanced flows we need to introduce the notion of surpluses of agents and goods. For any flow \(f\) in \(N_p\) we define the surplus of an agent \(i\) as \(r_f(b_i) = p_i - \sum_{j \in [n]} f(b_i, g_j)\) where
\( f(b_i, g_j) \) indicates the flow from node \( b_i \) to \( g_j \) in \( N_p \). Analogously we define the surplus of a good \( j \) as 
\[
 r_f(g_j) = p_i - \sum_{j'i \in [n]} f(b_{j'i}, g_i).
\]
We also define the surplus vector \( r_f = (r_f(b_1), r_f(b_2), \ldots, r_f(b_n)) \).

Any flow \( f \) is a balanced flow if it minimizes the L2 norm of the surplus vector. We now discuss few essential properties of balanced flows,

**Lemma 1: Balanced flows and max-flow**

Every balanced flow is a maximum flow.

**Proof.** If not then there exists an augmenting path in \( N_p \) from \( s \) to \( t \) and augmenting any positive flow along any of these paths will result in decreasing the surplus of one of the agents and therefore reducing the L2 norm subsequently.

**Lemma 2: Balanced flows and equi-surplus investors**

A valid flow \( f \) is a balanced flow in \( N_p \), only if \( f(b_i, g_j) > 0 \) and \( f(b_k, g_j) > 0 \), imply \( r_f(b_i) = r_f(b_k) \). Consequently, if \( (b_i, g_j) \in N_p \) and \( (b_k, g_j) \in N_p \), and \( r(b_i) > r(b_k) \), then \( f(b_k, g_j) = 0 \).

**Proof.** If not, assume \( r_f(b_i) = r + \delta \) and \( r_f(b_k) = r - \delta \) for some positive \( r \) and \( \delta \) and (without loss in generality). We modify \( f \) by decreasing \( f(b_i, g_j) \) by \( \varepsilon < \min(f(b_i, g_j), f(b_i, g_k)) \) and increasing \( f(b_i, g_k) \) by the same. It is trivial to check that the new flow satisfies all flow conditions and its L2 norm is
\[
 \sqrt{|r_f|^2 + (r + \delta - \varepsilon)^2 + (r - \delta + \varepsilon)^2 - (r + \delta)^2 - (r - \delta)^2} = \sqrt{|r_f|^2 - 2((\delta)^2 - (\delta - \varepsilon)^2)} < |r_f|_2.
\]
This suggests that \( f \) was not a balanced flow, which is a contradiction.

Intuitively the concept of balanced flow is to be *fair* to all buyers interested in a particular good (Since unlike the L1 norm, the L2 norm depends on the allocation). In general, the L2 norm of a vector reduces if we make the magnitudes of the individual components of the vector closer while keeping the L1 norm constant. We would see later that this would exactly be our strategy in the algorithm.

### 3.2.1 Computing balanced flows in Equality Networks

Balanced flows are also not difficult to compute.

**Lemma 3: Algorithms for Balanced flow**

A balanced flow in an Equality network can be computed with at most \( n \) max-flow iterations.

We just give the high level idea of the proof (Since it is not hard to see that a polynomial time algorithm is possible). The algorithm follows from the fact that we can *in a way* identify agents with high surpluses in a balanced flow easily without effectively computing the balanced flow. Since the high surplus agents and the low surplus agents do not invest on the same goods (by lemma 2)

\( 3 \)
we may claim that the balanced flow is the union of the balanced flows in the two networks (one involving high surplus agents and the one involving low surplus agents) and use divide and conquer.

### 3.3 Goods in Demand in the light of balanced flow

Here we revisit the argument made by Walrus, but a bit more carefully. We say that a good is in demand iff the agents investing on it have high surplus (w.r.t a balanced flow in $N_p$). Intuitively even after giving every agent interested in the good a fair share (defining the allocation with the balanced flow), the investors of the good still have high budgets left to invest. Now we define the goods in demand more formally If $B' \subset B$, then $\Gamma(B')$ defines the neighborhood of $B'$ in $N_p$ or equivalently $\Gamma(B') = \{ g_j | (b_i, g_j) \in N_p \}$.

- Compute the balanced flow $f$.
- Sort the agents in decreasing order of their surpluses and let $\langle b_{\pi(1)}, b_{\pi(2)}, ... b_{\pi(n)} \rangle$ be this order ($\pi$ is a permutation).
- Determine the minimal $l$ such that $r_f(b_{\pi(l)}) > (1 + \frac{1}{n}) \cdot r_f(b_{\pi(l+1)})$.
- Let $S \leftarrow \{ b_{\pi(1)}, b_{\pi(2)}, ... b_{\pi(l)} \}$.
- Goods in demand $\leftarrow \Gamma(S)$.

Let us highlight a subtle but crucial aspect in the above definition of the good in demand. A conventional measure of demand may be the sum of surpluses (excess budgets) of the agents investing upon the good. However in the above definition, it is intuitively the average surplus of the investors after giving each one of them a fair share of the good (allocation based on the balanced flow). For example consider the following scenario - there are 50 agents investing on a good, each with surplus say 1, and there is another good on which only one agent is investing but with surplus 2. Although the sum of surpluses for the former good is far larger, we label the latter good as the good in demand in our setting.

**Proposition 2: How far are the surpluses of the buyers?**

| If $b_i$ and $b_j$ belong to $S$, then $r(b_i) \leq e \cdot r(b_j)$ |

*Proof.* For every $b_i$ in $S$, there exists a $b_k \in S$ such that $\frac{r_f(b_i)}{r_f(b_k)} \leq 1 + \frac{1}{n}$. Thus the ratio between the surpluses of any two buyers in $S$ is at most $(1 + \frac{1}{n})^n \leq e$. $\square$

**Proposition 3: Zero surplus for goods in demand**

| For every good $g_j$ in $\Gamma(S)$, $r_f(g_j) = 0$ |

*Proof.* If not, then we can augment some flow along the path $s \rightarrow b_i \rightarrow g_j \rightarrow t$ for some $b_i \in S$ and reduce the surplus of agent $b_i$ and henceforth the L2 norm of the surplus vector, contradicting that $f$ was a balanced flow. $\square$
4 Strategy of Walrus revisited - Adjusting prices and flows

Now that we have defined the Goods in demand let us keep following strategy of Walrus (increasing the prices of the goods in demand) with an additional constraint.

<table>
<thead>
<tr>
<th>Strategy 1: Goods once sold are always sold</th>
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<tr>
<td>While we increase the prices of the goods in demand, we want to ensure that once the surplus of a good becomes zero, it always stays zero. (A good completely sold, remains completely sold)</td>
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We achieve this by modifying $f$ and $p$ as follows (Also see Figure 2)

$$f'(b_i, g_j) = \begin{cases} x \cdot f(b_i, g_j) & g_j \in \Gamma(S) \\ f(b_i, g_j) & g_j \not\in \Gamma(S) \end{cases}$$ (1)

$$f'(s, b_i) = \begin{cases} x \cdot f(s, b_i) & b_i \in S \\ f(s, b_i) & b_i \not\in S \end{cases}$$ (2)

$$f'(g_j, t) = \begin{cases} x \cdot f(g_j, t) & g_j \in \Gamma(S) \\ f(g_j, t) & g_j \not\in \Gamma(S) \end{cases}$$ (3)

$$p'_i = \begin{cases} x \cdot p_i & g_j \in \Gamma(S) \\ p_i & g_j \not\in \Gamma(S) \end{cases}$$ (4)

At the moment there may be concerns that $f'$ may not be a valid flow in $N_p$. Since we changed the price vector, certain edges of $N_p$ may disappear and certain new edges may appear. Also, for sufficiently large values of $x$, certain flows may exceed the capacities (The flow values in (3)). Therefore, we put some additional constraints on $x$ (will be discussed immediately) to ensure that $f'$ is still a valid flow in $N_{p'}$ and all goods having zero surplus in $N_p$ have zero surplus in $N_{p'}$ as well.

How big should $x$ be? To answer this we need to study the events that may occur while we gradually increase $x$ (The events seem to be having some sort of monotonicity with $x$).

- Increasing the price of a good $g_j$ decreases the bang per buck of the good for its investor ($\frac{v_{ij}}{p_j}$ becomes $\frac{v_{ij}}{p_j'}$). This may lead to the appearance of a new equality edge in the network (when an agent gets interested in another good) and also the disappearance of an existing equality edge (when an interested agent loses interest). Since we are increasing the prices of the goods in $\Gamma(S)$, the only equality edges that may appear are the ones that involve an agent from $S$ and a good from $G \setminus \Gamma(S)$. We also wish not to increase the prices of the goods in demand beyond this. Therefore we define

$$x_{eq} = \min \left\{ \frac{v_{ij}}{p_j} \cdot \frac{p_k}{v_{ik}} \mid b_i \in S, (b_i, g_j) \in N_p, g_k \not\in \Gamma(S) \right\}$$

and our $x \leq x_{eq}$. Here $x_{eq}$ is the minimum $x$ at which an equality edge appears in $N_p$. Additionally, since the prices of all the goods in $\Gamma(S)$ increases by $x$, no equality edge involving an agent from $S$ (since we increase the prices of all the bang per buck goods of this agent) or
a good from $G \setminus \Gamma(S)$ will disappear. Therefore, only equality edges that may disappear from $N_p$ are the ones that involve an agent $b_i$ from $B \setminus S$ and a good $g_j$ from $\Gamma(S)$. However, there exists an agent $b_k \in S$ and $(b_k, g_j) \in N_p$ and $r_f(b_k) > r_f(b_i)$ (since $b_i \notin S$). Therefore by lemma 2, $f(b_i, g_j) = 0$. Thus the only Equality edges that disappear are the ones that carry zero flow and this is harmless, since this does not force us to change the allocation at all.

- Increasing the price of a good also means increasing the budget of the agent that owns the good. Also increasing the incoming flow will also increase the expenditure of some of the agents, thereby reducing their surpluses. Clearly we wish to avoid negative surpluses (as we want to keep $f'$ as a feasible flow in $N_p'$). We may classify the agents into 4 categories based on how their surplus changes (See Figures 2 and 3)

  - **Type-1**: $b_i \in S$ and $g_i \in \Gamma(S)$. The inflow (budget) and outflow (expenditure) (in $N_p$) of the buyer increases by the same multiplicative factor $x$. Its new surplus is $x \cdot r_f(b_i)$.

  - **Type-2**: $b_i \in S$ and $g_i \notin \Gamma(S)$. Only the outflow (expenditure) (in $N_p$) of the buyer increases by a multiplicative factor $x$. Its new surplus is $r_f(b_i) - (x-1) \cdot f(s, b_i)$.

  - **Type-3**: $b_i \notin S$ and $g_i \in \Gamma(S)$. Only the inflow (budget) (in $N_p$) of the buyer increases by a multiplicative factor $x$. Its new surplus is $r_f(b_i) + (x-1) \cdot p_i$.

  - **Type-4**: $b_i \notin S$ and $g_i \notin \Gamma(S)$. Neither the inflow (budget) nor the outflow (expenditure) (in $N_p$) of the buyer changes. Its new surplus is the same $r_f(b_i)$.

Also note that the surpluses of the agents are linear functions in $x$. The surplus of a Type-2 agent moves closer to that of a Type-3 and Type-4. We do not wish to increase $x$ beyond the point where the two surpluses meet. Thus we define

$$x_{23} = \min \left\{ \frac{p_i + p_j - r_f(b_j)}{p_i + p_j - r_f(b_i)} | b_i \in \text{Type-2 and } b_j \in \text{Type-3} \right\}$$

$$x_{24} = \min \left\{ \frac{p_i - r_f(b_j)}{p_i - r_f(b_i)} | b_i \in \text{Type-2 and } b_j \in \text{Type-4} \right\}$$

and our $x \leq \min(x_{23}, x_{24})$. (See Figure 4)

**Strategy 2: Determining $x$ Exactly**

$x \leq \min(x_{eq}, x_{23}, x_{24}, x_{max})$, where $x_{max} = 1 + \frac{1}{Cn^3}$, where $C$ is some constant.

**Proposition 4: Flow Validity**

If $x \leq \min(x_{eq}, x_{23}, x_{24}, x_{max})$, then $f'$ is a valid flow in $N_p'$ ($f'$ and $p'$ are defined in (1)-(4)). $f'$ also happens to be a max-flow in $N_p'$.

**Proof.** Exercise
Proposition 5: L1 norm is non-increasing

\[ \sum_{i \in [n]} r_f(b_i) = \sum_{i \in [n]} r_{f'}(b_i) \]

Proof. Note that the if \( g_j \in \Gamma(S) \), then \( r_f(g_j) = 0 \). Since \( f \) is a flow in \( N_p \), \( \sum_{i \in [n]} r_f(b_i) = \sum_{j \in [n]} r_f(g_j) \). Now we update the flow \( f \) to \( f' \) such that \( \sum_{j \in [n]} r_f(g_j) = \sum_{j \in [n]} r_{f'}(b_j) \) (as we increase the prices and the incoming flow of only zero surplus goods). Therefore the sum of surpluses of the buyers w.r.t \( f' \) also remains constant.

We now revisit the arguments made in section 2 regarding the convergence of these updates to equilibrium. Since an equality edge always appears from an agent \( b_i \) in \( S \) (high surplus agents) to a good \( g_j \) in \( G \setminus \Gamma(S) \), this helps us to reduce the L2 norm of the surplus vector by balancing the surpluses between \( b_i \) and the current investors of \( g_j \) if any (that are low surplus agents). Similarly even if no equality edges appear, the surpluses of the Type-2 and Type-3, Type-4 agents move closer to each other, helping in again reducing the L2 norm of the surplus vector. However there are Type-1 agents who have already have an initial high surplus (since they are in \( S \)) and their surpluses increase further after adjustment. However we will see a way to address this issue as well.
Multiply the prices of the goods in demand (and the budgets of the corresponding agents) by \( x > 1 \).

(a) \( b_1, b_2 \) and \( b_3 \) are the agents in \( S \) and \( g_2, g_3 \) and \( g_4 \) are the goods in demand (\( \Gamma(S) \)). We increase the budgets of \( b_2, b_3 \) and \( b_4 \) and the prices of \( g_2, g_3 \) and \( g_4 \).

Adjust the Flows in order to Strategy 1 and maintain \( f \) also as a valid flow.

(b) Multiply all incoming flows of the goods in demand by \( x \).

Type-1 Agents. Surplus increases

(c) Budget and Expenditure increase by the same factor \( x \)- hence the surplus also increase by a factor of \( x \).

Figure 2: Increasing the prices of the goods in demand and also noting the effect on the surplus vector.
Type-2 Agents.
Surplus decreases

(a) The budget of the agent remains unchanged, but the expenditure increases. Thus surplus decreases.

Type-3 Agents.
Surplus increases

(b) The budget of the agent increases, but the expenditure is unchanged. Thus surplus increases.

Type-4 Agents.
Surplus unchanged

(c) Both the budget and the expenditure is unchanged. Thus surplus is constant.

Figure 3: Increasing the prices of the goods in demand and also noting the effect on the surplus vector
Figure 4: Variation of the surpluses of different buyers as we gradually increase $x$. Type-1 and Type-2 agents have initial high surplus and as we gradually increase $x$ the surplus of Type-2 decreases and that of Type-1 increases. Type-3 and Type-4 have initially lower surplus. Type remains constant and Type-3 increases (more steeply than Type-1).

5 Duan Mehlhorn Algorithm

Now we are finally ready to present the algorithm,

**Algorithm 1** Duan Mehlhorn’s Multiplicative Price Update algorithm

1: Set $p \leftarrow [1]^n$.
2: Set $C \leftarrow 256$.
3: Set $\varepsilon \leftarrow \frac{1}{8n^4U^{3n}}$.
4: while $||r_f||_2 > \varepsilon$, when $f$ is a balanced flow in $N_p$ do
5: Identify $S$ w.r.t $f$ in $N_p$.
6: $x \leftarrow \min(x_{eq}, x_{23}, x_{24}, 1 + \frac{1}{C^2n^3})$.
7: Multiply prices of goods in $\Gamma(S)$ by $x$ and update $f$, $p$ to $f'$ and $p'$ like in (1)-(4) and $N_p$ accordingly.
8: If $x = x_{eq}$ update $f'$ further to $f''$ as shown in Algorithm 2.
9: end while
10: Round $p$ to equilibrium prices.

First we state certain invariants throughout the algorithm.

**Proposition 6: Invariants throughout the algorithm**

- $\sum_{i \in [n]} r_f(b_i) \leq n$.
- The maximum price of any good is less than $(nU)^{n-1}$

*Proof.* Exercise
6 Analysis of the iterations of the Algorithm

We would separately discuss two types of iterations -

- **$x_{\text{max}}$ iterations**: Iterations of DM algorithm when $x = x_{\text{max}}$. Intuitively these are the iterations responsible for increasing the L2 norm of the surplus vector, since there is no substantial balancing in these iterations.

- **Balancing iterations**: All other iterations. These iterations help the prices to converge to the market clearing prices, since they involve sufficient decrease in the L2 norm of the surplus vector.

6.1 Bounding $x_{\text{max}}$ iterations

Notice that the prices of the goods are only increased in every iteration of the algorithm. Thus every $x_{\text{max}}$ iteration will involve a multiplicative increase of $1 + \frac{1}{\sqrt{n^4}}$ for at least one good. Again since the market clearing prices are upper bounded by proposition 6, we may polynomially bound such iterations.

**Lemma 4: Bounding $x_{\text{max}}$ iterations**

The number of $x_{\text{max}}$ iterations is $O(n^5 \cdot \log(nU))$

**Proof.** A single good can belong to $\Gamma(S)$ for at most $\log(1 + \frac{1}{\sqrt{n^4}})(nU)^{n-1} \in O(n^4 \log(nU))$ iterations. Since every $x_{\text{max}}$ iterations involves at least one good in $\Gamma(S)$, the total number of such iterations is in $O(n^5 \cdot \log(nU))$.

As mentioned earlier, $x_{\text{max}}$ iterations may (most likely) involve an increase in the L2 norm of the surplus vector. Since the L1 norm of the surplus vector is non-increasing (by proposition 4), the decrease in the surplus of Type-2 agents must be at least as large as (in this case equal) the sum of increase in the surpluses of Type-1 and Type-3 agents. Since Type-3 agents have lower surpluses than Type-2 agents the sum of squares of the surpluses of the Type-2, Type-3 and Type-4 agents always decreases. The increase in the sum of squares of the surpluses of Type-1 agents is at most $(1 + \frac{1}{\sqrt{n^4}})^2$ and therefore the increase in the L2 norm of the surplus vector after an $x_{\text{max}}$ iteration is at most $1 + \frac{1}{\sqrt{n^4}}$.

**Lemma 5: Bounding the total multiplicative increase in the L2 norm of the surplus vector**

The total multiplicative increase in the L2 norm of the surplus vector in all $x_{\text{max}}$ iterations is $(nU)^{O(n^2)}$

**Proof.** In every $x_{\text{max}}$ iteration, the increase in the L2 norm of the surplus vector is at most $1 + \frac{1}{\sqrt{n^4}}$. By lemma 4, there are $O(n^5 \cdot \log(nU))$ of them. Thus the total multiplicative increase in all such iterations is bounded by $(1 + \frac{1}{\sqrt{n^4}})^{O(n^5 \cdot \log(nU))} \in (nU)^{O(n^2)}$. □
6.2 Bounding the balancing iterations

We will show that the L2 norm of the surpluses decrease in such scenarios.

6.2.1 Balancing iterations where \( x = x_{23} \) or \( x = x_{24} \)

In such scenarios we will show that the balancing of the surpluses of Type-2, Type-3 and Type-4 agents outweighs the increase in the surpluses of the Type-1 agents. Let \( r_{\min} \) denote the minimum surplus of a Type-2 agent and \( r_{\max} \) indicate the maximum surplus of a Type-3 or Type-4 agent prior to the price and flow adjustment in \( N_p \) in this iteration. Note that \( r_{\min} > (1 + \frac{1}{n}) \cdot r_{\max} \Rightarrow r_{\max} < (1 - \frac{1}{n+1}) r_{\min} \). Now, by proposition 2 the maximum surplus of a Type-1 agent prior to price and flow adjustment is \( e \cdot r_{\min} \). The surpluses of the Type-1 agents can increase at most by a factor of \( (1 + \frac{1}{Cn^2}) \) (since \( x < x_{\max} \) in balancing iterations). Thus the total increase in the sum of squares of the surpluses of the buyers is \( n \times ((1 + \frac{1}{Cn^2})^2 - 1) \cdot (e^2 r_{\min}^2) < \frac{3 e^2 r_{\min}^2}{Cn^2} \).

Now we quantify the effect of balancing of surpluses of Type-2, Type-3 and Type-4 buyers.

**Lemma 6: Bounding the multiplicative decrease in the L2 norm of the surplus vector**

The multiplicative decrease in the L2 norm of the surplus vector when \( x = \min(x_{23}, x_{24}) \) is

\[ 1 - \Omega\left(\frac{1}{n^3}\right). \]

**Proof.** Let \( \delta_i \) denote the decrease in surplus of a Type-2 agent \( b_i \) after price and flow adjustment. Similarly let \( \mu_i \) denote the increase in the surplus of a Type-3 or Type-4 (\( \mu_i = 0 \) if agent is Type-4) agent. Note that \( r_f(b_i) - \delta_i \geq r_f(b_j) + \mu_j \), where \( b_i \) is a Type-2 agent and \( b_j \) is a Type-3 or Type-4 agent. Also note that since, \( x = x_{23} \) or \( x = x_{24} \), there exists a Type-2 agent \( b_i \) and a Type-3 or Type-4 agent \( b_j \) such that \( r_f(b_i) - \delta_i = r_f(b_j) + \mu_j \) or equivalently \( \delta_i + \mu_j = r_f(b_i) - r_f(b_j) \geq r_{\min} - r_{\max} \). This also suggests that \( (\sum_{b_i \in T_2} \delta_i + \sum_{b_i \in T_3 \cup T_4} \mu_i) \geq r_{\min} - r_{\max} \), where \( T_2, T_3 \) and \( T_4 \) refer to the set of Type-2, Type-3 and Type-4 agents respectively. Since the sum of surpluses is non-increasing (in this case it is constant), \( \sum_{b_i \in T_2} \delta_i \geq \sum_{b_i \in T_3 \cup T_4} \mu_i \). Therefore we can write the change in the sum of squares of the surpluses of these agents as,
\[= \sum_{b_i \in T_2} ((r_f(b_i) - \delta_i)^2 - r_f(b_i)^2) + \sum_{b_i \in T_3 \cup T_4} ((r_f(b_i) +)^2 - r_f(b_i)^2)\]
\[= \sum_{b_i \in T_2} (-2\delta_i \cdot r_f(b_i) + \delta_i^2) + \sum_{b_i \in T_3 \cup T_4} (2\mu_i \cdot r_f(b_i) + \mu_i^2)\]
\[= -\sum_{b_i \in T_2} \delta_i \cdot r_f(b_i) + \sum_{b_i \in T_3 \cup T_4} \mu_i \cdot r_f(b_i) - \sum_{b_i \in T_2} \delta_i (r_f(b_i) - \delta_i) + \sum_{b_i \in T_3 \cup T_4} \mu_i (r_f(b_i) + \mu_i)\]
\[\leq -r_{\min} \sum_{b_i \in T_2} \delta_i + r_{\max} \sum_{b_i \in T_3 \cup T_4} \mu_i\]
\[\leq -\left(\frac{r_{\min} - r_{\max}}{2}\right) \cdot \left(\sum_{b_i \in T_2} \delta_i + \sum_{b_i \in T_3 \cup T_4} \mu_i\right)\]
\[\leq -\frac{(r_{\min} - r_{\max})^2}{2}\]
\[\leq -\frac{r_{\min}^2}{2(n+1)^2}\]
\[\leq -\frac{4e^2 r_{\min}^2}{C n^2}\]

Therefore the sum of squares of the surpluses of the agents decreases by at least \(\frac{4e^2 r_{\min}^2}{C n^2} - \frac{3e^2 r_{\min}^2}{C n^2} = \frac{1}{C n^2} e^2 r_{\min}^2 \leq \frac{1}{C n^2} ||r_f||_2^2\). Hence the new L2 norm of the surpluses is at most \((1 - \Omega(\frac{1}{n^2})) ||r_f||_2\). \(\Box\)

**6.2.2 Balancing iterations where \(x = x_{eq}\)**

Intuitively this is the best situation as we decrease the demand of good in high demand and increase that of one which is not in high demand. The algorithm involves a balanced flow computation in \(N_p\) after the equality edge appears. We show that there exists a flow \(f''\) such that \(||f''||_2 \leq (1 - \Omega(\frac{1}{n^2})) ||r_f||_2\). We determine the new flow \(f''\) as follows,

**Algorithm 2 Flow Update when new Equality edge appears in \(N_p\)**

1. Let \((b_i, g_j)\) be a new equality edge where \(b_i \in S\) and \(g_j \in \Gamma(S)\).
2. Set \(w \leftarrow \) The largest surplus of an agent in \(B \setminus S\). \(w = 0\) if \(S = B\).
3. Let \(f''\) denote the current flow during augmentation.
4. **Augment** along \((b_i, g_j)\) until \(r_f''(b_i) = w\) or \(r_f''(g_j) = 0\).
5. If \(r_f''(b_i) = w\) then exit.
6. for all agents \(b_k\) that invest in \(g_j\) in \(f''\) in any order do
7. **Augment** along \((b_i, g_j, b_k)\) gradually until \(r_f''(b_i) = \max(r_f''(b_k), w)\) or \(f''(b_k, g_j) = 0\).
8. Set \(w = \max(r_f''(b_k), w)\)
9. If \(r_f''(b_k) = w\) then exit.
10. end for

**Lemma 7: Bounding the multiplicative decrease in the L2 norm of the surplus vector**

The multiplicative decrease in the L2 norm of the surplus vector when \(x = \min(x_{23}, x_{24}, x_{eq})\) is \(1 - \Omega(\frac{1}{n^2})\).

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So basically whenever a new equality edge \((b_i, g_j)\) appears we try to balance the surplus of a Type-2/Type-1 agent \(b_i\) against the surpluses of the current investors of \(g_j\) (if any), until the surplus of \(b_i\) equals \(r_{\text{max}}\) or \(b_i\) is the sole (only) investor in \(g_j\). We will now show that in both case there is a sufficient decrease in the L2 norm of the surplus vector.

**Proof.** When \(x = \min(x_{23}, x_{24})\) then the proof follows from lemma 6. Like earlier, until the new equality edge appears the total increase in the sum of squares of the surpluses of Type-1 agents is at most \(\frac{3e^2r_{\text{min}}^2}{Cn^2}\). In the augmentation (in algorithm 2) that follows after the equality edges appear, there is no increase in surplus of any agents in Type-1 or Type-2. However there could be increase in surpluses of agents in \(B \setminus S\). As mentioned earlier, in \(f''\) either the surplus of an agent not in \(S\) or agent \(b_i\) owns \(g_j\) completely - in the latter case the surplus of \(b_i\) decreases by \(p_j \geq 1\) post augmentation. For both cases we can use re-use the proof of lemma 6 entirely with just small changes,

- Let \(\delta_i\) denote the decrease in surplus of a Type-1 or Type-2 agent \(b_i\). (Note that for a Type-1 agent \(\delta_i\) is only defined if there is a decrease in its surplus post augmentation mentioned in algorithm 2, otherwise it is simply zero)

- Let \(\mu_i\) denote the increase in surplus of Type-3 or Type-4 agents (unlike earlier this time there could be an increase in the surpluses of type-4 agents post augmentation mentioned in algorithm 2).

- \(r_{\text{min}}\) denote the minimum surplus of an agent in \(S\).

- \(r_{\text{max}}\) denote the maximum surplus of an agent in \(B \setminus S\)

Note that all surplus references in the above definitions refer to the surpluses at the beginning of the balancing iteration (not just before the equality edge appears- defined w.r.t \(f\) and not \(f''\)). Notice that in this case \(\sum_{b_i \in S} \delta_i + \sum_{b_i \in B \setminus S} \mu_i \geq \min(r_{\text{min}} - r_{\text{max}}, p_j) \geq \min(r_{\text{min}} - r_{\text{max}}, 1) \frac{r_{\text{min}} - r_{\text{max}}}{n}\). Therefore the result from lemma 6 also holds here.

**Lemma 8: Bounding the balancing iterations**

The number of balancing iterations is at most \(\mathcal{O}(n^5 \cdot \log(nU))\)

**Proof.** We know that the total multiplicative increase in the L2 norm of the surplus vector is at most \((nU)^{\mathcal{O}(n^2)}\) (from lemma 5). Each balancing iteration reduces the L2 norm of the surplus vector by \(1 - \Omega(\frac{1}{Cn^8})\). The initial value of the L2 norm of the surplus vector is at most \(\sqrt{n}\) (when we start the algorithm) and the final value is \(\varepsilon\) (see the DM algorithm). Thus the number of balancing iterations are \(\log((1-\Omega(\frac{\sqrt{nU}}{n^7}))^{-1}) \frac{(\sqrt{nU})^{\mathcal{O}(n^2)}}{\varepsilon} \in \mathcal{O}(n^5 \cdot \log(nU)).\)

**Theorem 1: The total number of iterations of the algorithm**

The total number of iterations of the DM algorithm is \(\mathcal{O}(n^5 \cdot \log(nU))\)

**Proof.** Directly from lemmas 4 and 8
Theorem 2: The total number of iterations of the algorithm [1]

The total number of arithmetic operations of the DM algorithm is $O(n^9 \cdot \log(nU))$.

Proof. Each iteration involves a balanced flow computation which takes $O(n \times n^3)$ arithmetic operations ($n$ maximum flow computations and each maximum flow costs $O(n^3)$ arithmetic operations) and determining $x$, which takes $O(n^2)$ comparisons. There are $O(n^5 \cdot \log(nU))$ iterations (from Theorem 1).

References
