We are initiating the third part of this course, which will cover some basics of one of the most prominent applications of game theory nowadays — auctions.

1 Combinatorial Auctions and Valuation Functions

In combinatorial auction, an auctioneer sells a number of indivisible items to a number of bidders. By “indivisible”, it means an item is sold wholly to one of the bidders — situations like two bidders each get half are not feasible.

Let $B$ denote the set of bidders, $G$ denote the set of items. Let $m := |B|$, and $n := |G|$. Each bidder $i \in B$ has a valuation function $v_i : 2^G \to \mathbb{R}$ that represents the bidder’s valuation on each subset of $G$. In applications, we often make the following convenient assumptions on $v_i$:

- $v_i(\emptyset) = 0$;
- free disposal: if $S_1 \subset S_2$, then $v_i(S_1) \leq v_i(S_2)$.

While in many applications the valuation functions of bidders are quite simple, one should note that to represent $v_i$ in general, its dimension is as large as $2^n - 1$. The generality permits capturing complicated preferences of some bidders. Next, we provide a few simple and common examples of valuation functions.

**Example 1: Some Valuation Functions**

A valuation function $v$ is additive if there are $n$ non-negative real numbers $u_1, u_2, \ldots, u_n$ such that

$$v(S) = \sum_{j \in S} u_j.$$  

In some applications there is a cap, i.e., there exists $c \in \mathbb{R}$ such that

$$v(S) = \min \left\{ \sum_{j \in S} u_j , c \right\}.$$  

A valuation function $v$ is unit-demand if there are $n$ non-negative real numbers $u_1, u_2, \ldots, u_n$ such that $v(\emptyset) = 0$, and for non-empty $S$,

$$v(S) = \max_{j \in S} u_j.$$  

A natural generalization is the following. There is a quota $\ell \geq 1$, such that

$$v(S) = \max_{S' \subseteq S, |S'| \leq \ell} \sum_{j \in S'} u_j.$$  

A covering valuation function $v$ is described by a collection of set-value tuple $(S_1, u_1), (S_2, u_2), \ldots, (S_{\ell}, u_{\ell})$, where each $S_k$ is a subset of $G$ and $u_k$ is a positive real number, such that

$$v(S) = \sum_{k: S_k \subseteq S} u_k.$$  

This can be used to capture complementary relation between items, e.g., $S_1$ may include cheese, pasta sauce and spaghetti, while $S_2$ may include bread and jam.

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1 In general, some of the items can be identical, so precisely we should say the multiset of items.
2 $2^n$ is the collection of all subsets of $G$, including the empty set.
In a combinatorial auction,

- each bidder \( i \) communicates a bid \( b_i \) to the auctioneer;
- then the auctioneer, based on the bids she receives from all bidders, decides a feasible allocation \( G_1, G_2, \cdots, G_m \) to all bidders.

Obviously, the allocation needs to satisfy that each item is allocated to at most one bidder; in other words, in the allocation, every two sets \( G_{i_1}, G_{i_2} \) are disjoint.

- Also, the auctioneer needs to determine payments of money \( q_1, q_2, \cdots, q_m \) from each bidder to the auctioneer.

Generally, it is possible that there is a payment from the auctioneer to some bidder, although it is uncommon in practice; in such case, the corresponding \( q \)-value is negative.

Note that up to this point, we do not provide clear specification on the bids \( b_i \). This is deliberate, since the specification of the bids may vary with applications. A natural choice is to ask each bidder \( i \) to report her valuation function \( v_i \). However, as said before, the full dimension of \( v_i \) can be up to \( 2^n - 1 \), so bidding the full valuation function might be infeasible when \( n \) is large. When the bidders’ valuation functions are known to be simple, e.g., those given in Example 1, the specification of bids can be the parameters \( u_j, c, \ell \) etc. in Example 1.

**Example 2: First-Price Auction**

When talking about “auction”, the very first example is the single item first-price auction. There is only one item. The bid of each bidder is a non-negative real number. The item is allocated to the bidder with the highest bid; if there is a tie, apply arbitrary tie-breaking rule. The bidder \( i \) who gets the item is required to send a payment of amount \( b_i \) to the auctioneer. Every other bidder gets nothing and pays nothing.

An auctioneer may design a less natural (and even crazy) rule to decide the allocations and payments. For instance, the auctioneer may ask each bidder to report, in addition to the bid above, whether the bidder likes Trump or not. The rule is to allocate the item to the bidder with the second highest bid (denoted by bidder \( i^* \)), and the payment is

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\begin{align*}
\frac{1}{2} (b_{\text{highest}} + b_{\text{lowest}}), & \quad \text{if bidder } i^* \text{ likes Trump;} \\
\frac{1}{4} b_{\text{highest}} + b_{\text{lowest}}, & \quad \text{otherwise},
\end{align*}
\]

where \( b_{\text{highest}} \) is the highest bid value and \( b_{\text{lowest}} \) is the lowest bid value.

Of course, the above auction is unlikely to appear in practice. But as a matter of fact, some complicated auctions have been seen in practice (and the complication might not serve any clear agenda/intention of the auctioneer), e.g., the spectrum auction in Germany.

### 2 Truthful Mechanism

In this section, we consider an auction in which the bid of each bidder is supposed to be her valuation function. We have learnt game theory. Clearly, we may view an auction as a game, where the bidders are strategic players, and bids are strategies. Thus, if a bidder sees reporting an alternative bid (i.e., a bid which is not the same as her true valuation function) will improve her benefit, she will do so. This is viewed by many as an unfavourable phenomenon, and thus a natural question to ask is whether we can design a truthful mechanism which discourages bidders from reporting false valuation function. More precisely, the benefit of a bidder \( i \) is measured by the quantity \( v_i(G_i) - q_i \). A truthful mechanism means a bidder always maximizes this quantity by reporting her true valuation function.
Definition 1: Truthful Mechanism

A mechanism takes the bids of all bidders as input parameters, and outputs allocation to each bidder and the payment to be collected from each bidder. Formally, we write the allocation to bidder $i$ as $G_i(b_1, b_2, \cdots, b_m)$, and the payment collected from bidder $i$ as $q_i(b_1, b_2, \cdots, b_m)$.

A truthful mechanism satisfies the following: for any bidder $i$, let $v_i$ be her true valuation function. Then for any $b'_i$,

$$v_i(G_i(b_1, \cdots, v_i, \cdots, b_m)) - q_i(b_1, \cdots, v_i, \cdots, b_m) \geq v_i(G_i(b_1, \cdots, b'_i, \cdots, b_m)) - q_i(b_1, \cdots, b'_i, \cdots, b_m).$$

Since bidder $i$ concerns her own benefit and she certainly knows her own valuation function, the red part in the above inequality is the same on both sides of the inequality.

While the definition of a truthful mechanism is intuitively clear right now, first learners might doubt if it is too good to exist. We will address this question in the general setting in the next section. Now it is good to show you that in the simplest scenario of one-item auction, truthful mechanism does exist, and its description is very simple.

Example 3: Second-Price Auction

It is clear that the first-price auction is not truthful. For instance, when the highest bid is 100 and the second highest bid is 10, the highest bidder would want to reduce her own bid to a value just a bit larger than 10, so that she can still win the item, but she pays much less to the auctioneer.

Now, we look at the second-price auction. The highest bidder wins the item, still. But her payment is the value of the second-highest bid. It is a truthful mechanism. For simplicity, here we write $v_i \equiv v_i(G)$. To see why this mechanism is truthful, see the followings. Let $\tilde{b}$ denote the highest bid among all bidders other than bidder $i$.

Case 1: $v_i \geq \tilde{b}$. Then by reporting truthfully, bidder $i$ wins the item and pays $\tilde{b}$. Bidder $i$’s benefit is $v_i - \tilde{b}$.

If bidder $i$ changes her bid to another value which remains larger than or equal to $\tilde{b}$, then she still wins the item and pays $\tilde{b}$, so the outcome is indifferent to her.

If bidder $i$ changes her bid to another value which is less than $\tilde{b}$, then she is allocated nothing and she pays nothing, so her benefit is zero, which is not better than $v_i - \tilde{b}$.

Case 2: $v_i < \tilde{b}$. Then by reporting truthfully, bidder $i$ is allocated nothing and she pays nothing, so her benefit is zero.

If she wants to change her bid to make a difference to the outcome, she must outbid all other bidders, i.e., she must bid a value larger than or equal to $\tilde{b}$. Then she wins the item and pays $\tilde{b}$. However, her benefit in this case is only $v_i - \tilde{b}$, which is negative.

3 Vickrey-Clarke-Groves Mechanism

In Example 3 we see a simple mechanism for one-item auction which has the favourable truthfulness property. In this section, we introduce a truthful mechanism, the Vickrey-Clarke-Groves (VCG) mechanism, for general combinatorial auction.

Let $M$ denote the set of bidders. For any non-empty subset of bidders $M' \subset M$, define the following optimization problem:

Find a feasible allocation $\{G_i\}_{i \in M'}$ to bidders in $M'$ which maximizes $\sum_{i \in M'} v_i(G_i)$.

Let $\text{OPT}(M')$ denote the optimal value, and let $\text{ALLOC}(M')$ denote a feasible allocation that leads to the

$$v_i(G_i(b_1, \cdots, v_i, \cdots, b_m)) - q_i(b_1, \cdots, v_i, \cdots, b_m) \geq v_i(G_i(b_1, \cdots, b'_i, \cdots, b_m)) - q_i(b_1, \cdots, b'_i, \cdots, b_m).$$

It is so simple that some high-school students can understand it.
optimal value. Note that in general, finding the optimal value can be computationally expensive. However, if the bidders’ valuation functions are additive or unit-demand, the optimal value can be computed in polynomial time. In this section, we ignore the computational complexity issue.

We are ready to introduce VCG mechanism. The mechanism receives bids $b_1, b_2, \cdots, b_m$, then

- Compute $\text{ALLOC}(M)$ and $\text{OPT}(M)$. Let $\text{ALLOC}(M)$ be $G_1, G_2, \cdots, G_m$.
- For each bidder $i$, compute $\text{OPT}(M \setminus \{i\})$.
  Bidder $i$ is required to pay $\text{OPT}(M \setminus \{i\}) - \sum_{k \neq i} b_k(G_k)$ to the auctioneer.

There is a more general format of VCG mechanism, as we shall discuss next; the mechanism presented above is a special case. We present this special case since it guarantees the payment from any bidder to the auctioneer is always non-negative, i.e., auctioneer never needs to pay money to any bidder. We leave you as an exercise to see why this favourable property holds.

**Theorem 1: Truthfulness of Vickrey-Clarke-Groves Mechanism**

The Vickrey-Clarke-Groves mechanism is truthful.

**Proof of Theorem 1:** The proof of truthfulness is remarkably short. The benefit of bidder $i$ is

$$v_i(G_i) - \left( \text{OPT}(M \setminus \{i\}) - \sum_{k \neq i} b_k(G_k) \right) = v_i(G_i) + \sum_{k \neq i} b_k(G_k) - \text{OPT}(M \setminus \{i\}).$$

Note that

- the blue term is computed by ignoring bidder $i$, therefore its value is independent of the bid of bidder $i$;
- the bid of bidder $i$ does not change $v_i$ (her true valuation function) and all $b_k$’s.

Thus, bidder $i$’s benefit is maximized by reporting a bid that yields an allocation $G_1, \cdots, G_m$ which maximizes

$$v_i(G_i) + \sum_{k \neq i} b_k(G_k),$$

while the bids of all other bidders $k \neq i$ are fixed.

Suppose that if bidder $i$ reports a bid $b'_i$, then the allocation becomes $G'_1, \cdots, G'_m$. By the definition of the function $\text{ALLOC}$ (think when bidder $i$ bids $v_i$),

$$v_i(G'_i) + \sum_{k \neq i} b_k(G'_k) \leq v_i(G_i) + \sum_{k \neq i} b_k(G_k).$$

In other words, by reporting any other bid $b'_i$, the benefit of bidder $i$ will never exceed her benefit when she reports truthfully.

We finish this lecture with two remarks. First, in VCG mechanism, the introduction of term $\sum_{k \neq i} b_k(G_k)$ is crucial, since it aligns the selfish interest of bidder $i$ to the global interest (which is computed by the functions $\text{OPT}$ and $\text{ALLOC}$). Second, the truthfulness property still holds when the blue term $\text{OPT}(M \setminus \{i\})$ is replaced by any other function which is independent from the bid of bidder $i$. 

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