In the context of combinatorial auction, we have introduced the simple (concerning its description, not its running time) VCG truthful mechanism. Recall that it involves a monetary transfer from bidders to the auctioneer, and the key to truthfulness is introducing a term of the form \( \sum_{k \neq i} v_k(G_k) \) in the payment.

One might argue that is the flexibility to setting payment permits the simple structure of VCG and its truthfulness.

However, in many practical contexts, monetary transfer is infeasible:

- Suppose that the goods are computing resources (like CPU time, RAM, etc.) within a company and bidders are employees of the company. It does not make sense to ask employees to pay for resources which are used for the company’s purpose.

- Some public facility (e.g., hospital) is going to be built and citizens report their preferred locations. It is infeasible, and perhaps also immoral, to permit citizens to pay in exchange for a more preferred facility location.

- The same as above holds when “facility” is replaced by “congress representative” and “prime minister”. This draws a more explicit link between “mechanism without monetary transfer” and “election”. A somewhat independent subject named Social Choice Theory (in which a fundamental result is the Arrow’s Impossibility Theorem) is concerned.

Without the flexibility of deriving a payment scheme, achieving truthfulness requires new ideas. In this lecture, we present a number of truthful mechanisms in the context when the goods are divisible and the bidders have normed-linear utility functions.

1 Model

There are \( m \) bidders and \( n \) divisible items, each normalized to be one unit. Each bidder \( i \) has a normed-linear utility function of the format

\[
    u_i(x) = \sum_{j \in G} u_{ij} \cdot x_j, \quad \text{where each } u_{ij} \geq 0 \text{ and } \sum_{j \in G} u_{ij} = 1.
\]

In this lecture, we confine ourselves to the above utility functions. However, keep in mind that in some scenarios, the sum of coefficients of a bidder might be higher than that of another bidder, so as to reflect the distinct levels of importance of different bidders.

Each bidder \( i \) bids a normed vector \( b_i = (b_{i1}, b_{i2}, \ldots, b_{in}) \), such that each \( b_{ij} \geq 0 \) and \( \sum_{j \in G} b_{ij} = 1 \). A mechanism takes the bids \( b_1, b_2, \ldots, b_m \) as inputs, and output a feasible allocation \( \{x_{ij}\}_{i \in B, j \in G} \) such that each \( x_{ij} \geq 0 \) and for each good \( j \), \( \sum_{i \in B} x_{ij} \leq 1 \).

We are currently at the final lecture of this course, thus you should be able to write down the formal definition of a truthful mechanism without monetary transfer by yourself. The following lemma permits interpolation of truthful mechanisms while preserving truthfulness; its proof is left as your exercise.

**Proposition 1: Interpolation of Truthful Mechanisms**

Let \( \mathcal{M}^Q \) and \( \mathcal{M}^R \) be two truthful mechanisms. Upon bids \( B = (b_1, b_2, \ldots, b_m) \), suppose the allocations by \( \mathcal{M}^Q \) is \( \{x^Q_{ij}(B)\}_{i \in B, j \in G} \). \( \{x^R_{ij}(B)\}_{i \in B, j \in G} \) is defined similarly.

Let \( \mathcal{M}^S \) be a mechanism such that for some real numbers \( \alpha, \beta \geq 0 \) and \( \alpha + \beta \leq 1 \),

\[
    \forall i \in B, j \in G, \quad x^S_{ij}(B) = \alpha \cdot x^Q_{ij}(B) + \beta \cdot x^R_{ij}(B).
\]

Then \( \mathcal{M}^S \) is truthful. We say \( \mathcal{M}^S \) is the \((\alpha, \beta)\)-interpolation of \( \mathcal{M}^Q \) and \( \mathcal{M}^R \).
2 Some Trivial Truthful Mechanisms

The first trivial truthful mechanism is: allocate nothing to every bidder, no matter what. Since the bids do not affect the allocation, the mechanism is trivially truthful, but is also trivially ineffective and undesirable.

Another trivial truthful mechanism is: for each good, allocate evenly among all bidders. Again, since the bids do not affect the allocation, the mechanism is trivially truthful, and in a naive common sense, the mechanism looks fair. However, if some buyers desire highly on some goods and do not desire on other goods, such allocation remain somewhat ineffective.

In literature, there are quantitative measures of different effectiveness or fairness. But in this lecture, we choose not to introduce such quantitative measures, and let ourselves focus on truthfulness.

3 Swap-Dictatorial Truthful Mechanisms

In this section, we focus on the case when \( m = 2 \). This permits us to give a simple presentation of swap-dictatorial mechanisms. But keep in mind that simple generalizations exist for general \( m \).

A dictatorial mechanism is determined by a set \( \mathcal{A} \) which collects vectors of the form 
\[
(y_1, y_2, \cdots, y_n)
\]
such that each \( y_j \in [0, 1] \).

Bidder 1 is permitted to choose an allocation from \( \mathcal{A} \), then Bidder 2 takes the rest of the goods. Clearly, Bidder 1 will choose 
\[
\arg \max_{y \in \mathcal{A}} u_1(y),
\]
and clearly she should report her utility truthfully. Since the bid of Bidder 2 has no effect on her own allocation, Bidder 2 does not have motivation to report non-truthfully, and hence the SD mechanism is truthful. This is why its name is “dictatorial”, since Bidder 1’s bid dictates the outcome.

A swap-dictatorial (SD) mechanism is the \((1/2, 1/2)\) interpolation of a dictatorial mechanism, and the same dictatorial mechanism but with the roles of Bidders 1 and 2 swapped. By Proposition 1, a SD mechanism is truthful.

There is a huge flexibility of choosing the set \( \mathcal{A} \). A simple way of defining the set \( \mathcal{A} \) is Increasing-Price (IP) mechanism. Each bidder is hypothetically endowed with one unit of money. First, for each good \( j \), there is an increasing price function \( P_j : [0, 1] \to \mathbb{R}^+ \). \( P_j(y) \) represents the marginal price for purchasing good \( j \) when the dictator has already purchased \( y \) unit of good \( j \). Mathematically,
\[
\mathcal{A} = \left\{ (y_1, y_2, \cdots, y_n) \mid \forall j \in G, \; y_j \in [0, 1] \quad \text{and} \quad \sum_{j \in G} \int_0^{y_j} P_j(t) \, dt \leq 1 \right\}.
\]

The advantage of IP mechanism is by introducing hypothetical prices, it motivates the dictator to purchase more of the goods which she desires more. This apparently has tendency to improve efficiency. Unfortunately, a more rigorous analysis show that as \( n \) increases, the performance of SD (and hence also IP) mechanism has to deteriorate w.r.t. to some natural effectiveness measure.

4 Partial-Allocation Truthful Mechanism

Partial-Allocation (PA) mechanism is a truthful mechanism without monetary transfer, but is resonant with the VCG mechanism which permits monetary transfer. For any subset of bidders \( B' \), consider the following Eisenberg-Gale convex program:
\[
\max \prod_{i \in B'} u_i(x_{i1}, x_{i2}, \cdots, x_{in})
\]
such that \( \forall j \in G, \sum_{i \in B'} x_{ij} \leq 1 \)
\[
\forall i \in B', \; j \in G, \quad x_{ij} \geq 0.
\]

1In literature, there are different formats of IP mechanism, but the qualitative motivation behind them is identical.
Let $x^*(B')$ denote the optimal solution to the above convex program, and for each bidder $i \in B'$, let $x_i^*(B')$ denote the allocation to bidder $i$ at the optimal solution, while we let $u_i^*(B')$ denote the utility attained by bidder $i$ at the optimal solution.

We are now ready to present the PA mechanism (there are more general formats of PA mechanisms, but for simplicity we present the canonical one):

- Compute $x^*(B)$, and for each bidder $i$, compute $x_i^*(B)$ and $u_i^*(B)$.
- For each bidder $i$, compute
  \[ \prod_{k \in B \setminus \{i\}} u_k^*(B) \quad \text{and} \quad \prod_{k \in B \setminus \{i\}} u_k^*(B \setminus \{i\}). \]

Bidder $i$ is allocated a fraction
\[ \frac{\prod_{k \in B \setminus \{i\}} u_k^*(B)}{\prod_{k \in B \setminus \{i\}} u_k^*(B \setminus \{i\})}. \tag{2} \]

Similar to VCG, it is not hard to prove that for each $i \in B$, in the fraction (2), the numerator is always less than or equal to the denominator, i.e., the fraction is always between zero and one.

**Theorem 1: Partial-Allocation Mechanism is Truthful**

The PA mechanism is feasible and truthful.

**Proof:** Feasibility is trivial after showing that the fraction in (2) is always between zero and one.

Without loss of generality, assume that every bidder other than Bidder $i$ bids truthfully. If Bidder $i$ also bids truthfully, her attained utility is
\[
\begin{align*}
    u_i \left( \frac{\prod_{k \in B \setminus \{i\}} u_k^*(B)}{\prod_{k \in B \setminus \{i\}} u_k^*(B \setminus \{i\})} \cdot x_i^*(B) \right) &= \frac{\prod_{k \in B \setminus \{i\}} u_k^*(B)}{\prod_{k \in B \setminus \{i\}} u_k^*(B \setminus \{i\})} \cdot u_i(x_i^*(B)) \\
    &= \frac{\prod_{k \in B \setminus \{i\}} u_k^*(B)}{\prod_{k \in B \setminus \{i\}} u_k^*(B \setminus \{i\})} \cdot u_i^*(B) \\
    &= \frac{\prod_{k \in B} u_k^*(B)}{\prod_{k \in B \setminus \{i\}} u_k^*(B \setminus \{i\})}.
\end{align*}
\] (3)

The first equality holds due to linearity of the function $u_i$; the second equality holds due to definition of $u_i^*(B)$.

Now, note that in (3), the denominator is independent of the bid of Bidder $i$. Thus, Bidder $i$ wants to make a bid that maximizes the numerator in (3). Following the same logic for proving VCG is truthful, we are done.

PA mechanisms (or their deviants) have been found to provide good effectiveness guarantees. This might be a bit surprising, since a PA mechanism in general will reserve a part of every good not to be allocated to any bidder.

## 5 Dynamic-Increasing-Price Truthful Mechanism

Finally, we present a variant of IP mechanism, named Dynamic-Increasing-Price (DIP) mechanism. Instead of having a single set $A$, we have a set $A_i$ for each Bidder $i$. 
For each Bidder $i$, and for each good $j$, there is an increasing price function $P_{ij} : [0, 1] \rightarrow \mathbb{R}^+$. In DIP mechanism, each price function $P_{ij}$ depends on the bids of all bidders other than Bidder $i$. So technically, we should write it as $P_{ij, b_{-i}}$. Then

$$A_i = \left\{ (y_1, y_2, \ldots, y_n) \mid \forall j \in G, y_j \in [0, 1] \quad \text{and} \quad \sum_{j \in G} \int_0^{y_j} P_{ij, b_{-i}}(t) \, dt \leq 1 \right\}. \quad (4)$$

Note that $A_i$ does not depend on the bid of Bidder $i$. Thus, following the same logic for IP mechanism, it is easy to prove that DIP mechanism is truthful. However, feasibility is a delicate issue. If the prices are set too low, it is quite possible that goods are over-allocated (i.e., the sum of allocations of a good $j$ to all bidders is more than one). When the prices are set very high, feasibility can be guaranteed, yet effectiveness will be low. Yet, DIP provides a rich family of truthful mechanisms which awaits exploration.

DIP mechanisms have been found to provide very good guarantee for some effectiveness measures. For instance, if all goods are enforced to be allocated completely and if there are only two bidders, then the optimal competitive mechanism, w.r.t. social welfare effectiveness measure, is found to be a DIP mechanism.

5.1 Motivating DIP Mechanism

A motivation for letting price function $P_{ij}$ depend on the bids of all bidders other than Bidder $i$ is market, which we have discussed in some prior lectures. Implicitly, the price of a good serves the purpose of reflecting how desirable the good is among the buyers/bidders. In economics, there are fundamental results showing a market is somewhat effective (see Fundamental Theorems of Welfare Economics), indicating that equilibrium prices serve as coordination signals for effective allocation. By letting $P_{ij}$ depend on bids of other bidders, it can serve as a coordination signal while also achieving the favourable truthfulness property.