Exercise 1. (total: 6 points)
(a) (2 points) In the lecture, we say that when a buyer has a CES utility function of the format
\[ u(y) = \left[ a_1 \cdot (y_1)^\rho + a_2 \cdot (y_2)^\rho + \cdots + a_n \cdot (y_n)^\rho \right]^{1/\rho}, \]
for some \( \rho > 0 \), then at price vector \( p \), her demand \( y \) is unique, and is given by:
\[ y_j = m \cdot \frac{(a_j)^{1-c} \cdot (p_j)^{c-1}}{\sum_{i=1}^n (a_i)^{1-c} \cdot (p_i)^c}, \]
where \( c := \rho / (\rho - 1) \). Show steps to derive the above equality.

(b) (4 points) In the lecture, we say that in a Fisher market with all buyers having CES utility functions with \( \rho \) parameters between 0 and \( \rho < 1 \), the income and demand elasticity satisfy
\[ E_I^{\text{min}} = E_I^{\text{max}} = 1 \quad \text{and} \quad E_D = 1 / (1 - p). \]
Explain why the above equalities hold.

Exercise 2. (total: 8 points)
In this exercise we show that Linear Arrow Debreau Markets are more general than Linear Fisher Markets.

(a) (3 points) Consider a Linear Arrow-Debreau market with \( n \) agents and \( n \) goods, such that

- Agent \( i \) owns only good \( i \) completely (i.e., no other agent owns any amount of good \( i \)).
- Let \( \hat{B} \subset [n] \) be a subset of agents and \( \hat{G} \subset [n] \) be a subset of goods. Let \( V_{ij} \) denote the utility of buyer \( i \) by having one unit of good \( j \).
- The utility that every buyer in \( \hat{B} \), derives from one unit of good \( j \in \hat{G} \), is \( c_j \) for some positive constant \( c_j \). (In other words, \( V_{ij} = c_j \) for every \( i \in \hat{B} \) and \( j \in \hat{G} \). This suggests that \( \hat{B} \) is a set of identical buyers — all of them have same valuation for the goods in \( \hat{G} \)).
- Additionally \( V_{ij} = 0 \) for every \( i \in [n] \setminus \hat{B} \) and \( j \in \hat{G} \), i.e., no other agent outside \( \hat{B} \) is interested in \( \hat{G} \).

Let \( p \) be a market equilibrium. Show that for every \( j \in \hat{G} \), \( p_j = c_j \cdot k \) for some positive constant \( k \).

(b) (5 points) Now consider a Linear Fisher Market instance with \( \ell \) buyers and \( n \) goods, a utility matrix \( V \in \mathbb{R}^{\ell \times n} \) and a budget vector \( m \in \mathbb{R}^\ell \) of the buyers. Let \( p \in \mathbb{R}^n \) denote the market equilibrium of this Fisher market. Construct a Linear Arrow-Debreau Market instance with \( n + \ell \) buyers and \( n + \ell \) goods and a utility matrix \( \hat{V} \in \mathbb{R}^{(n+\ell) \times (n+\ell)} \), such that we can determine \( p \) from \( \hat{p} \) in time \( O(n + \ell) \), where \( \hat{p} \) is the market equilibrium of the Arrow-Debreau Market.

Exercise 3. (9 points)
In the lecture note concerning Orlin’s weakly polynomial-time algorithm for computing market equilibrium of Linear Fisher Markets, there are six claims. Prove them.
Exercise 4. (total: 6 points)
In the lecture note concerning Orlin’s weakly polynomial-time algorithm for computing market equilibrium of Linear Fisher Markets, there is a proposition stating that each equilibrium price can be written as a positive rational number of the form \( a/b \), with \( a \leq \ell \cdot U^{n^2+1} \) and \( b \leq (n + 1) \cdot U^{n^2} \).

(a) (3 points) Prove the proposition.
(b) (3 points) Prove that every equilibrium price is at least \( \frac{1}{U^{(n-1)+1}} \).

Exercise 5 (*). (total: 10 points)
In this exercise, we will present you some Linear Fisher Markets in which the denominator of one of its equilibrium prices has to be large.

Let \( U \geq 4 \) be a positive even integer. Let \( \mathcal{P}(U) \) denote the set of all prime numbers which are strictly larger than \( U/2 \) but strictly less than \( U \). By some standard results in analytic number theory, it is known that for all sufficiently large \( U \),

- the product of all primes in \( \mathcal{P}(U) \) is more than \( 2^{U/2} \); and
- the cardinality of \( \mathcal{P}(U) \) is more than \( \frac{U}{3 \log U} \), but less than \( \frac{2U}{\log U} \).

You can use the above results.

Let \( d \) denote the cardinality of \( \mathcal{P}(U) \). We construct a LFM with \( d \) buyers and \( d + 1 \) goods as below:

- For each prime \( q \in \mathcal{P}(U) \), we have one distinct buyer denote by \( B_q \), and we have one distinct good denoted by \( G_q \). This constructs \( d \) goods; the remaining good is called \( G_\# \).
- Each buyer \( B_q \) has budget of amount 1.
- Each buyer \( B_q \) is interested in two goods only, namely \( G_q \) and \( G_\# \). In the utility function of the buyer, the coefficient of good \( G_q \) is \( \frac{q+1}{2} \), while the coefficient of good \( G_\# \) is \( q \). All other coefficients in the utility function are zero.

(a) (6 points) Prove that for all sufficiently large \( U \), in the market constructed above, when the equilibrium price of good \( G_\# \) is written as a rational number \( a/b \), where \( a, b \) are positive integers, then \( b \geq \Omega(2^{U/2}) = U^{\Omega(d)} \).

Note for (a): If you have some other construction of LFM that achieves similar or better bound, i.e., when there are \( n \) goods in the market, the lower bound is of the format \( U^{\Omega(n)} \), you are most welcomed to submit it.

(b) (4 points) Can you construct a family of LFM such that the denominator of one of its equilibrium prices has to be \( U^{\Omega(n^2)} \), where \( n \) is the number of goods?

Note for (b): If you can find any number theory result from literature which will help you with such construction, state it with citation, then feel free to use it.