A Lower Bound for Area–Universal Graphs

G. Bilardi S. Chaudhuri D. Dubhashi
K. Mehlhorn

MPI–I–93–144 Oktober 1993

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Gianfranco Bilardi[†] Shiva Chaudhuri[‡] Devdatt Dubhashi[§]
K. Mehlhorn[¶]

Abstract

We establish a lower bound on the efficiency of area–universal circuits. The area A_u of every graph H that can host any graph G of area (at most) A with dilation d, and congestion $c \leq \sqrt{A}/\log\log A$ satisfies the tradeoff

$$A_u = \Omega(A \log A/(c^2 \log(2d))).$$

In particular, if $A_u = O(A)$ then $\max(c, d) = \Omega(\sqrt{\log A}/\log\log A)$.

1 Introduction

Bay and Bilardi [2] showed that there is a graph H which can be laid out in area $O(A)$ and into which any graph G of area at most A can be embedded with load 1, and dilation and congestion $O(\log A)$. As a consequence, they showed the existence of an area $O(A)$ VLSI circuit that can simulate any area A circuit with a slowdown of $O(\log A)$. This note explores the feasibility of more efficient embeddings.

Our main result is Theorem 5 which establishes a limitation relating the area of a universal graph to the parameters of the embedding. Informally, it asserts that any circuit which is universal for a family of graphs of area A, and itself has area $O(A)$, must incur a slowdown of $\Omega(\sqrt{\log A}/\log\log A)$. To prove it, we consider a family of graphs, each a collection of expanders, and show that each graph poses constraints on the number of edges of the universal graph in different edge-length ranges.

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*The work of G. Bilardi was supported in part by the Italian National Research Council and the Italian Ministry of Research. The work of K. Mehlhorn was supported in part by ESPRIT II Basic Research Actions Program of the EC under contract No. 7141 (project ALCOM II) and by the BMFT (Förderungskennzeichen ITS 9103).

[†]Dipartimento di Elettronica e Informatica, Università di Padova, 35131 Padova, ITALY

[‡]Max–Planck–Institut für Informatik, Im Stadtwald, D-6600 Saarbrücken, Germany

[§]Max–Planck–Institut für Informatik, Im Stadtwald, D-6600 Saarbrücken, Germany

[¶]Max–Planck–Institut für Informatik and Fachbereich Informatik, Universität des Saarlandes, Im Stadtwald, D-6600 Saarbrücken, Germany

2 Definitions

An *embedding* of a graph G into a graph H is a mapping of the nodes of G to the nodes of H and of the edges of G to paths in H connecting the images of the endpoints of the edge.

For an embedding,

- the *dilation* is the maximal length of a path used to realize an edge,
- the *load* is the maximal number of nodes of G mapped to any single node of H, and
- the *congestion* is the maximal number of paths realizing edges of G that pass through the a fixed edge of H.

For $c > 0$, an l, c-*layout* of a graph G is a special kind of embedding: the host graph H is the two–dimensional grid, the load is l and the congestion is c. In other words, at most l vertices of G are mapped onto a single grid point and edges are mapped into grid paths, and at most c different paths can use any given grid edge. A $1, 1$-layout is simply called a *layout*. Note that in the layout of a graph, each pair of distinct edges is mapped onto edge disjoint grid paths. Layouts play an important role in VLSI theory [5, 4].

The *area* of a layout is the area of the smallest rectangle containing the layout. If a graph has a layout of area A then it is called a *graph of area A*. A graph containing a vertex with degree greater than 4 has no layout, thus, the area of a graph is defined only for graphs with maximum degree 4.

A graph H is called (A, l, d, c)–*universal* if every graph of area A can be embedded into H with load l, dilation d, and congestion c.

Using this notation, we may phrase the result of Bay and Bilardi [2] as showing the existence of an $(A, 1, O(\log A), O(\log A))$–universal graph of area $A_u = O(A)$. The well–known tree–of–meshes with A leaves is an $(A, 1, O(\sqrt{A}), 1)$–universal graph of area $O(A \log^2 A)$ [3].

We will use the following definition of an *expander* graph from [1].

A k-regular bipartite graph $G = ((U, V)E)$ is an (n, β) expander if $|U| = |V| = n/2$ and $\forall X \subset U,\ |X| \leq n/4, |\Gamma(X)| \geq (1 + \beta)|X|$, where $\Gamma(X) \subseteq V$ is the neighbourhood of X.

Ajtai gave a construction of 3-regular (n, β) expanders for a constant $\beta > 0$, for sufficiently large n [1]. For a sufficiently large value of n, where "sufficiently large" will be clear from the context, we will denote by G_n the 3-regular expander with $\beta > 0$. For a positive integer r, let $G_{r,n}$ denote the graph consisting of r disjoint copies of such an expander G_n.

3 Layouts of Expanders

The following is an easy observation on layouts of graphs: it is actually well–known (see, for instance, Exercise 5.6 in [4]), but we reproduce the proof here for completeness.

Proposition 1 *Any graph on n vertices with maximum degree 4 has a layout with area $(4n)^2$.*

Proof: Let G be the graph. We demonstrate a layout of G in a square grid of area $(4n)^2$. Index the columns of the grid by 0 through $4n-1$ from left to right and the rows similarly from top to bottom. Then, each grid point may be identified by (i,j), $i,j \in \{0,\ldots,4n-1\}$. Let the vertices of G be $v_0,\ldots,v_{n-1}$. We locate v_i at grid point $(4i+2,4i+1)$. Say that rows and columns $4i,4i+1,4i+2,4i+3$ *belong* to v_i, and initially, mark each row and column as *free.*

We may now greedily lay out the edges. Consider an edge (v_i,v_j). Without loss of generality, let $i<j$. Pick a free row a, belonging to v_i and a free column b, belonging to v_j, and mark these *used.* Since the degree of each vertex is at most 4, such a free row and column will always exist. We then lay out this edge along the path $(a,a) \rightarrow (a,b) \rightarrow (b,b)$. We have still to connect $(4i+2,4i+1)$ to (a,a) and (b,b) to $(4j+2,,4j+1)$, but we will do this later. Notice that all the paths laid out in this manner do not share any edges and they only pass through points (x,y) such that $x \leq y$. Once we have accomplished this for all edges, it only remains to connect points $(4i,4i),(4i+1,4i+1),(4i+2,4i+2)$ and $(4i+3,4i+3)$ to $(4i+2,4i+1)$, for each i. It is easy to do this with paths that use only vertices (x,y) such that $x \geq y$, and $x,y \in \{4i,4i+1,4i+2\}$ and do not use the same edge. Clearly these paths cannot use any of the edges that were allocated earlier. This shows that G can be laid out in area $(4n)^2$. ∎

The following fact about layouts of expanders is now immediate.

Fact 2

- *There is a positive constant α, such that G_n can be laid out in area αn^2.*
- *For α as above, the graph $G_{r,n}$ can be laid out in area $\alpha r n^2$.*

In the following, $\beta > 0$ will be the expansion factor of G_n.

Proposition 3 *For $l \leq \beta n/4$, $c \leq \beta n/64$, in any l,c-layout of G_n there are at least $f_1 n$ edges of G_n whose layouts have length at least $f_2 n/c$, for some constants $f_1, f_2 > 0$.*

Proof: We will use a bisection width argument as introduced by Thompson [5]. We first review the argument that there exist cuts of a certain shape that have the property that at least $(n-l)/2$ vertices of G_n lie on either side of the cut.

Consider a rectangle enclosing the c-layout of G_n. Let $x < n$ be a nonnegative integer. Then there exists a cut consisting of two vertical line segments joined by a horizontal line segment of length 1 or 0, such that there are at least $x - l/2$ vertices of G_n to the left of the cut and at least $n - x - l/2$ vertices to the right. To see this consider sweeping two vertical lines, m_1 and m_2, separated by one unit, across the rectangle. Let i be the rightmost column such that if we place m_1 between columns i and $i+1$, then there are at most $x - l/2$ vertices to the left of m_1. If there are exactly $x - l/2$ vertices to the left, then m_1 is the desired cut. Otherwise, let $x' < x$ be the number of vertices to the left of m_1 and $x'' > x$ be the number of vertices to the left of m_2. Now sweep a horizontal line segment of length 1 between m_1 and m_2, from bottom to top, till the first position such that there are at least $x - l/2 - x'$ vertices in column $i+1$ below the segment. Since, at the previous position, there were less than $x - l/2 - x'$ vertices below the segment, and each grid vertex can have at most l vertices of G_n, there are at most $x + l/2 - x'$ vertices of G_n below the segment. Then, the upper part of m_1, the horizontal segment and the lower part of m_2 give the desired cut.

Let C_0 be the cut, as above, such that there are at least $(n-l)/2$ vertices on both sides. For a given positive integer i, consider the vertical line m_1 that is i units to the left of the leftmost vertical segment of C_0 and the vertical line m_2 that is i units to the right of the rightmost vertical segment of C_0. Let y and z be the number of vertices to the left of m_1 and the right of m_2 respectively. Note that one of $y, z \leq n/2$. Without loss of generality, $y \leq n/2$. Let $x = n/2 - y$. By an argument similar to the one above, we can show that there exists a cut C, consisting of two horizontal segments joined by a vertical segment of length at most 1 joining m_1 and m_2 such that the number of vertices between m_1 and m_2 that are below the cut is between $x - l/2$ and $x + l/2$. We denote by C_i the cut consisting of the upper part of m_1, the horizontal part C and the lower part of m_2. Observe that C_i has at least $(n-l)/2$ vertices on both sides. We shall refer to the upper part of m_1 and the lower part of m_2 together as the *vertical* part of C_i, and the remaining part as the horizontal part.

Consider a c-layout of $G_n = ((U,V),E)$ and a cut C_i such that exactly $n/2$ vertices are to the left. Without loss of generality we may assume that there are at least $n/4$ vertices from U to the left of C_i and at least $n/4 - l/2$ vertices of V to the right. By the expansion property, there are at least $\beta n/4 - l/2$ edges connecting vertices of U on the left to vertices of V on the right. Thus, the layout of at least $\beta n/4 - l/2$ edges of G_n cross the cut C_i.

The number of grid edges that cross the horizontal part of C_i is at most $2i+2$. Since each of these edges may be part of the layout of at most c edges of G_n, at most $c(2i+2)$ edges of G_n can cross C_i without crossing the vertical part. Let $w_i \geq \beta n/4 - l/2 - c(2i+2)$ be the number of edges of G_n that cross the vertical part of C_i. In the range $0 \leq i \leq$

$\beta n/64c$, with the assumptions $l \leq \beta n/4$ and $c \leq \beta n/64$, we have $w_i \geq \beta n/16$. Hence

$$\sum_{i=0}^{\beta n/64c} w_i \geq \beta^2 n^2/1024c.$$

The cuts $C_0, C_1, \ldots$ were chosen so that an edge whose layout has length q can cross the vertical part of at most q cuts, and therefore can contribute at most q to the above sum. Also, any edge can contribute at most 1 to each w_i, so no edge can contribute more than $\beta n/64c$ to the above sum. If x is the fraction of edges whose layouts have length at most $f_2 n/c$, then by upperbounding the contribution of each edge, we have

$$x 3n f_2 n/c + (1-x)3n\beta n/64c \geq \beta^2 n^2/1024c.$$

and so

$$x(3f_2 - 3\beta/64) \geq \beta(\beta/16 - 3)/64.$$

Choosing $f_2 = \beta^2/6144$, and substituting, we get $x \leq (48-\beta)/(48-\beta/2) < 1$.

Choosing $f_1 = 3(1-x)$, and the above value for f_2 suffice to prove the proposition. ∎

It now follows that

Corollary 4 *For $l \leq \beta n/4$, $c \leq \beta n/64$, in any l, c-layout of $G_{r,n}$, there are at least $r f_1 n$ edges of $G_{r,n}$ whose layouts have length at least $f_2 n/c$ (for constants f_1, f_2 as above).*

4 Constraints on Embeddings

Let H be a (A, l, c, d)-universal graph and denote the area of an optimal (minimal area) layout of H by A_u. Let m_h denote the number of edges in H which have lengths in the interval $[2^h, 2^{h+1})$. Clearly,

$$A_u \geq \sum_{h \geq 0} m_h 2^h. \tag{1}$$

We focus attention on the family of graphs $\{G_{A/\alpha 4^i, 2^i}\}$ for $0 \leq i \leq L := \frac{1}{2}\log(A/\alpha)$. Note that each of these graphs can be laid out in area A by Fact 2. Since H is (A, l, c, d)-universal, it embeds each of the graphs in this family with load l, congestion c and dilation d (at most). This embedding, composed with the layout of H, yields a l, c-layout of each graph in this family.

Fix i, $0 \leq i \leq L$, let $n := 2^i$, and concentrate on $G := G_{A/\alpha 4^i, 2^i}$. For any edge e of G, let $P(e)$ denote the set of edges of H realising it in the embedding of G in H. For an edge e of G or H, let $L(e)$ denote the grid path realising e in the c-layout of G or the layout of H respectively, and let $l(e)$ denote the (grid) length of the realisation. For an edge e, of G, $L(e)$ is simply the concatenation of the grid paths realising the edges in $P(e)$ in the layout of H. Thus, $l(e) = \sum_{\bar{e} \in P(e)} l(\bar{e})$.

From Corollary 4, there are at least $f_1 A/\alpha n$ edges of length at least $f_2 n/c$. For any edge e of G, let $P'(e) \subseteq P(e)$ denote the subset of edges of length at least $f_2 n/2cd$. Then, $\sum_{\bar{e} \in P(e) \setminus P'(e)} l(\bar{e}) \leq f_2 n/2c$ since $P(e)$ consists of at most d edges and $l(\bar{e}) < f_2 n/2cd$ for any $\bar{e} \in P(e) \setminus P'(e)$. Altogether then, we deduce that there are $f_1 A/\alpha n$ edges of G with $\sum_{\bar{e} \in P'(e)} l(\bar{e}) \geq f_2 n/2c$. Clearly then, there are $f_1 A/\alpha n$ edges e of G with $\sum_{\bar{e} \in P'(e)} \min(l(\bar{e}), f_2 n/2c) \geq f_2 n/2c$ [1]. Hence,

$$\begin{aligned} f_1 f_2 A/2\alpha c &\leq \sum_{e \in G} \sum_{\bar{e} \in P'(e)} \min(l(\bar{e}), f_2 n/2c) \\ &\leq \sum_{\bar{e} \in H, l(\bar{e}) \geq f_2 n/2cd} c \cdot \min(l(\bar{e}), f_2 n/2c) \end{aligned}$$

where the last inequality follows from the facts that for any edge $\bar{e}$ of H, there can be at most c edges of G with $\bar{e} \in P(e)$, and any edge in $P'(e)$ has length at least $f_2 n/2cd$.

In terms of m_h, the number of edges of H in the range $[2^h, 2^{h+1})$, we can write the previous inequality as:

$$\sum_{\log(f_2 n/2cd) \leq h \leq \log(f_2 n/2c)} m_h 2^{h+1} + \sum_{h > \log(f_2 n/2c)} m_h f_2 n/2c \geq f_1 f_2 A/2\alpha c^2 \tag{2}$$

Recall now that $n := 2^i$ and that i was arbitrarily chosen in the range between 0 and $L := \frac{1}{2}\log(A/\alpha)$. Summing over this range of i's and rearranging terms yields:

$$\sum_h m_h [\sum_{i \in I_h} f_2 2^i/2c + \sum_{i \in J_h} 2^{h+1}] \geq f_1 f_2 AL/2\alpha c^2 \tag{3}$$

where $I_h := \{i : \log(f_2 2^i/2c) < h\} = \{i : f_2 2^i/2c < 2^h\}$ and $J_h := \{i : \log(f_2 2^i/2cd) \leq h \leq \log(f_2 2^i/2c)\}$. Clearly, $|J_h| \leq 1 + \log d$ and $\sum_{i \in I_h} f_2 2^i/2c \leq 2^{h+1}$. The value of the square bracket is therefore bounded above by $2^{h+1}(2 + \log d)$ and so we get:

$$\sum_h m_h 2^{h+1}(2 + \log d) \geq f_1 f_2 AL/2\alpha c^2 \tag{4}$$

Combining inequalities proves We can now prove

Theorem 5 *The area A_u of every (A, l, d, c)–universal graph, when $c \leq \sqrt{A}/\log\log A$, satisfies the tradeoff*

$$A_u = \Omega(A \log A/(c^2 \log(2d))).$$

[1] Indeed if $\min(l(\bar{e}), f_2 n/2c) = f_2 n/2c$ ever holds, then trivially the inequality holds as well; otherwise the inequality holds from the previous statement.

Proof: Consider any graph that requires area A. Proposition 1 implies that the number of vertices in the graph is $n \geq \sqrt{A}/4$. Then, we have $c \leq \beta n/64$, for sufficiently large A. Note that $l \leq 4c$ always holds, since, if a vertex has more than $4c$ vertices mapped to it, then some edge will have congestion more than c. This yields $l \leq \beta n/16$. Thus the conditions of Corollary 4 hold, and inequalities 1 and 4 are valid. Combining the two yields the theorem. ∎

Corollary 6 *In particular, if* $A_u = O(A)$ *then* $\max(c, d) = \Omega(\sqrt{\log A}/\log\log A)$.

Proof: Observe that $A_u = O(A)$ implies that $c^2 \log d = \Omega(\log A)$. Subject to this constraint, $\max(c, d)$ is minimized for $c, d := \Omega(\sqrt{\log A}/\log\log A)$. ∎

5 Conclusions and Open Problems

We close with three open problems.

- Is there a $(A, 1, O(1), O(1))$–universal graph of area $O(A \log A)$?
- Is the $\log d$ term in the lower bound really necessary?
- Can the c^2 term in the lower bound be replaced by something smaller?

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