Every DFS Tree of a 3-Connected Graph Contains a Contractible Edge

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Abstract

Let G be a 3-connected graph on more than 4 vertices. We show that every depth-first-search tree of G contains a contractible edge. Moreover, we show that if G is 3-regular or does not have two disjoint pairs of adjacent degree-3 vertices, every spanning tree of G contains a contractible edge.

1 Introduction

A graph G is *connected* if there exists a path between every two of its vertices. For $k > 1$, G is k*-connected* if deleting any $k-1$ of its vertices leaves a connected graph. In a k-connected graph every vertex has degree at least k. An edge in a 3-connected graph is *contractible* (also called 3-*contractible*) if its contraction results in a 3-connected graph.

Over 40 years ago, Tutte [Tut61] proved the fundamental result that every 3-connected graph on more than 4 vertices contains a contractible edge. Since then, the distribution of contractible edges in 3-connected graphs has been intensively studied. Many papers establish lower bounds on the number of contractible edges [AES87, Ota88], or on entire contractible subgraphs [Kri08]. See [Kri02] for an excellent survey. Bounds on the number of *removable* edges in 3-connected graphs [HJSW90, KWL07] have also been proved; an edge is removable if its removal leaves a 3-connected graph.

In this paper, we strengthen Tutte's result by showing that every depth-first-search tree of a 3-connected graph contains a contractible edge. We also exhibit 3-connected graphs with a depth-first-search tree containing exactly one contractible edge, and 3-connected graphs with a spanning tree containing no contractible edge. We call a 3-connected graph a *fox* if it has a spanning tree containing no contractible edge. We present infinite families of foxes and give conditions under which a 3-connected graph is not a fox.

A certifying algorithm for 3-connectivity returns a proof (a *certificate*) for the 3-connectivity of the input graph, which can be verified efficiently. See [KMMS06] and [MNU99, Section 2.14] for a general discussion of certifying algorithms. In [EMS10] we exploit the existence of a contractible edge in every depth-first-search tree to establish a linear-time

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certifying algorithm for 3-connectivity of Hamiltonian graphs. The best certifying algorithm for 3-connectivity [Sch10] of general graphs runs in time $O(n^2)$ and the best non-certifying decision algorithms [HT73, MR92] run in time $O(n+m)$, where n is the number of vertices and m is the number of edges of the underlying graph.

Notations

Let $G=(V,E)$ be an undirected graph with $n:=|V|$ and $m:=|E|$. For a vertex $x\in V$, let $\deg(x)$ be its degree in G. We denote an edge between vertices x and y by xy. If $xy\in E(G)$, we say that x and y are neighbors in G.

For any subset of vertices $V'\subset V$, let $G\setminus V'$ denote the graph resulting from G by the removal of the vertices in V' and all their incident edges. A set of vertices whose removal disconnects the graph is called a *vertex cut*. If V' is a vertex cut of G, the connected components of $G\setminus V'$ are called the *split components* (or *separation classes*) with respect to V'. Vertex cuts of size one, two and three are called *separation vertices*, *separation pairs* and *separation triples*, respectively. Analogously, an *edge cut* of a graph G is a set of edges whose removal disconnects G.

Let xy be an edge of G. The contraction of xy results in a graph $G'=G/xy$ with vertex set $V(G')=V(G)\setminus\{x,y\}\cup\{v_{xy}\}$, where v_{xy} is a new vertex. The edge xy is removed and for all edges having exactly one endpoint in $\{x,y\}$, this vertex is replaced by v_{xy}. Finally, only one edge of each set of parallel edges is kept.

Preliminaries

We use the following known results in our proofs.

Fact 1.1 *An edge xy in a 3-connected graph with $n>4$ is contractible if and only if no separation triple containing x and y exists.*

Theorem 1.2 (Tutte [Tut61]) *Every 3-connected graph with $n>4$ contains a contractible edge.*

Lemma 1.3 (Halin [Hal69]) *In a 3-connected graph with $n>4$, every vertex of degree 3 has an incident contractible edge.*

Lemma 1.4 (Ota [Ota88]) *Let v be a vertex of degree 3 in a 3-connected graph G with $n>4$ and let x, y, and z be its neighbors. If $xy\in E(G)$, then vz is contractible.*

2 Separation Triples and Split Components

We establish some properties of separation triples and split components. We shall use them later to prove our main results.

Lemma 2.1 *Let $ST=\{x,y,z\}$ be a separation triple in a 3-connected graph G. Let D be one of the split components of $G\setminus ST$. Then, every vertex in ST has a neighbor in D.*

Proof. Assume otherwise, say z has no neighbor in D. Then, D is a split component of $G \setminus \{x, y\}$, a contradiction to G being 3-connected. $\square$

Lemma 2.2 *Let $ST = \{x, y, z\}$ be a separation triple in a 3-connected graph G, and let D be one of the split components of $G \setminus ST$. If $ST' = \{x', y', z'\}$ is a separation triple in G with $ST' \neq ST$ and $ST' \subset V(D) \cup ST$, then there is a split component of $G \setminus ST'$ properly contained in D.*

Proof. Let $D, D_1, \ldots, D_j$ be the split components of $G \setminus ST$. Consider the components D_i, where $1 \leq i \leq j$. Every D_i is connected in $G \setminus (V(D) \cup ST)$, and hence connected in $G \setminus ST'$. Moreover, according to Lemma 2.1, any vertex in $ST \setminus ST'$ has a neighbor in D_i. It follows that $(ST \setminus ST') \cup V(D_i)$ is contained in a split component of $G \setminus ST'$. Since $ST \setminus ST'$ is non-empty, $(ST \setminus ST') \cup \bigcup_{1 \leq i \leq j} V(D_i)$ is contained in a split component of $G \setminus ST'$. Any other split component of $G \setminus ST'$ (there must be at least one) is contained in $V(G) \setminus ((ST \setminus ST') \cup \bigcup_{1 \leq i \leq j} V(D_i) \cup ST')$, and hence properly contained in D. $\square$

Lemma 2.3 *Let G be a 3-connected graph, and let $\{x, y, z\}$ and $\{v, y, w\}$ be two separation triples in G intersecting exactly in y. Then, v and w are contained in the same split component of $G \setminus \{x, y, z\}$ if and only if x and z are contained in the same split component of $G \setminus \{v, y, w\}$. Moreover, if v and w belong to distinct split components, then each of $G \setminus \{x, y, z\}$ and $G \setminus \{v, y, w\}$ has exactly two split components.*

Proof. Assume that v and w are contained in the same split component of $G \setminus \{x, y, z\}$. Then, there is a split component S of $G \setminus \{x, y, z\}$ that contains a neighbor of x and a neighbor of z, but neither v nor w. As $S \cup \{x, z\}$ is connected in $G \setminus \{v, y, w\}$, x and z belong to the same split component of $G \setminus \{v, y, w\}$. Conversely, if x and z belong to the same split component of $G \setminus \{v, y, w\}$, then v and w belong to the same split component of $G \setminus \{x, y, z\}$ for the same reason. This proves the first claim.

Assume that there are more than two split components of $G \setminus \{x, y, z\}$. Then, among these split components there is a component containing neither v nor w. It follows that x and z belong to the same split component of $G \setminus \{v, y, w\}$, and in accordance v and w belong to the same split component of $G \setminus \{x, y, z\}$. The same arguments apply if there are more than two split components of $G \setminus \{v, y, w\}$. $\square$

We call two separation triples $\{x, y, z\}$ and $\{v, y, w\}$ *crossing*, if they intersect in exactly one vertex and if v and w belong to distinct components of $G \setminus \{x, y, z\}$. Then, x and z belong to distinct components of $G \setminus \{v, y, w\}$ by Lemma 2.3. In addition, $G \setminus \{x, y, z\}$ and $G \setminus \{v, y, w\}$ both have exactly two split components.

Lemma 2.4 *Let G be a 3-connected graph, let $\{x, y, z\}$ and $\{v, y, w\}$ be two crossing separation triples in G, let D be the split component of $G \setminus \{x, y, z\}$ containing v, and let X and Z be the split components of $G \setminus \{v, y, w\}$ containing x and z, respectively. Then, either $X \cap D = \emptyset$ or $\{x, y, v\}$ is a separation triple. Also, either $Z \cap D = \emptyset$ or $\{z, y, v\}$ is a separation triple.*

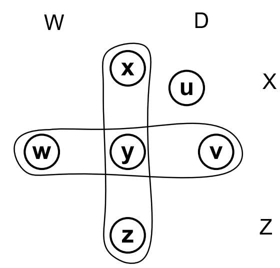


Figure 1: Two crossing separation triples.

Proof. Assume $X \cap D \neq \emptyset$. Consider any edge $ur \in E(G)$ with $u \in V(X \cap D)$ and $r \notin V(X \cap D)$. See Figure 1. Then, $r \in \{x, y, z, v, w\}$. However, $r \neq z$ because $\{v, y, w\}$ separates X from z, and $r \neq w$ because $\{x, y, z\}$ separates D from w. It follows that $\{x, y, v\}$ separates $X \cap D$ from the rest of G. Analogously, if $Z \cap D \neq \emptyset$, then $\{z, y, v\}$ separates $Z \cap D$ from the rest of G. □

Lemma 2.5 *Let G be a 3-connected graph, let $ST = \{v, y, w\}$ be a separation triple in G, and let X be a split component of $G \setminus ST$. If $G \setminus X$ is not 2-connected and a is a separation vertex of $G \setminus X$, then $a \notin ST$ and one of the vertices in ST has a as its only neighbor in $G \setminus X$ (and hence is a split component of $G \setminus (V(X) \cup \{a\})$). Conversely, if each vertex in ST has at least two neighbors in $G \setminus X$, then $G \setminus X$ is 2-connected.*

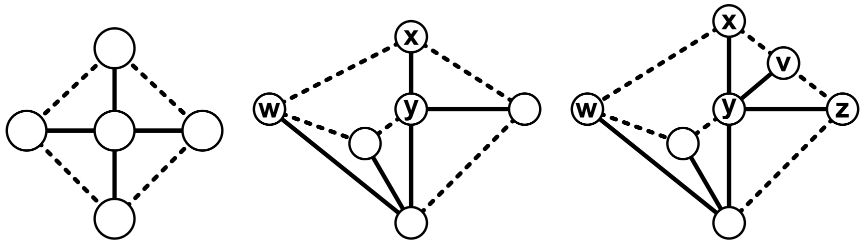
Proof. Assume that $G \setminus X$ is not 2-connected. Then, there is a separation vertex a that splits $G \setminus X$. If one of the split components of $G \setminus (V(X) \cup \{a\})$ does not contain a vertex from ST, then a is a separation vertex in G, contradicting G being 3-connected. It follows that every split component of $G \setminus (V(X) \cup \{a\})$ contains at least one vertex from ST. If $a \in ST$, say $a = y$, then $G \setminus (V(X) \cup \{a\})$ has exactly two split components one containing v and one containing w. Since ST is a separation triple in G, there are vertices in $G \setminus X$ other than those in ST. It follows that one of the components of $G \setminus (V(X) \cup \{a\})$ must have at least two vertices, say the component containing w. Then, $\{y, w\}$ splits G, a contradiction to G being 3-connected. Therefore, $a \notin ST$.

The vertices of ST cannot all lie in one split component of $G \setminus (V(X) \cup \{a\})$. Hence, at least one of these split components, say S, contains exactly one vertex from ST, say w. If w has a neighbor in $G \setminus X$ other than a, then $|V(S)| > 1$ and $S \setminus \{w\}$ is a split component of $G \setminus \{a, w\}$, contradicting G being 3-connected. □

3 Contractible Edges and Spanning Trees

We next give a sufficient condition for every spanning tree of a 3-connected graph to contain a contractible edge.

Lemma 3.1 *Let G be a 3-connected graph with $n > 4$, and let F be an edge cut of G. If every edge e in F has an end vertex x, where $\deg(x) = 3$ and x has two neighbors in $G \setminus F$ adjacent to each other, then G is not a fox.*



(a) W_5 (b) $n = 6$ (c) $n = 7$

Figure 2: The solid edges are non-contractible and form a spanning tree.

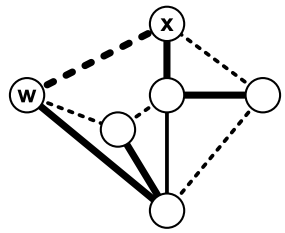


Figure 3: A depth-first-search tree (thick edges) with only one contractible edge, namely xw.

Proof. Let v and w be the two neighbors of x in $G \setminus F$. Since v and w are adjacent, Lemma 1.4 implies that e is contractible. Therefore, every edge in F is contractible, and hence every spanning tree of G contains at least one contractible edge. □

Examples: There are arbitrary large foxes; the wheel graphs W_i, $i \geq 5$, with the spokes as the spanning tree form an infinite family, see Figure 2(a). Figure 2(b) shows the base graph of another infinite family of examples. In this graph, the vertices x, y, and w play a special role. The next larger graph in this family is obtained as follows: Let v be the neighbor of x that is neither y nor w in the smaller graph, subdivide xv by one vertex and connect the new vertex with y; see Figure 2(c).

We will show that every depth-first-search tree of a 3-connected graph contains a contractible edge. The graph on 6 vertices of Figure 3 shows that this bound is tight. However, we are not aware of any graph on more than 6 vertices that admits a depth-first-search tree containing exactly one contractible edge.

Consider a 3-connected graph G with $n > 4$. Assume that G is a fox and let T be a spanning tree of G containing no contractible edge. It follows that for every edge $xy \in E(T)$ there exists a vertex $z \in V(G)$ such that $\{x, y, z\}$ is a separation triple. We call $\{x, y, z\}$ a T-separation triple. Split components that result from the removal of a T-separation triple are called T-split components. A T-split component is minimal if there is no T-split component properly contained in it.

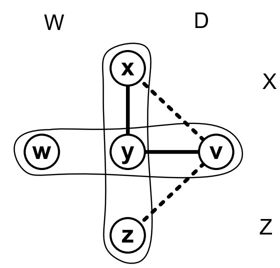


Figure 4: The T-minimal split component D.

Lemma 3.2 *Let G be a 3-connected graph with $n > 4$. Assume that G is a fox and let T be a spanning tree of G containing no contractible edge. Then, every T-minimal split component consists of exactly one vertex, say v. This vertex has degree 3 and is incident to exactly one edge of T. More precisely, if the neighbors of v in G are x, y, and z with $xy \in E(T)$, then $vz \notin E(T)$, and either $vx \in E(T)$ or $vy \in E(T)$.*

Proof. Let D be a T-minimal split component, and let $\{x, y, z\}$ with $xy \in E(T)$ be the associated separation triple. Since T is a spanning tree, there exists a vertex $v \in V(D)$ that is a neighbor of x, y, or z in T. We show that D has only one vertex, namely v.

If $vz \in E(T)$, then vz is non-contractible, and hence a separation triple $\{v, z, w\}$ exists. Since $xy \in E(G)$, either $w \in \{x, y\}$ or both x and y are in the same split component of $G \setminus \{v, z, w\}$. Consequently, there exists a split component S of $G \setminus \{v, z, w\}$ such that $x, y \notin V(S)$. By Lemma 2.1, v has a neighbor, say u, in S. Since $u \notin \{x, y, z\}$, u is in the same split component of $G \setminus \{x, y, z\}$ as v, i.e. $u \in V(D)$. It follows that every vertex in S is also in D. Since $v \notin V(S)$, S is properly contained in D, a contradiction to D being minimal. It follows that $vz \notin E(T)$. Accordingly, either $vx \in E(T)$ or $vy \in E(T)$.

Assume w.l.o.g. that $vy \in E(T)$. See Figure 4. Therefore, vy is non-contractible and a separation triple $\{v, y, w\}$ exists. If there is a split component of $G \setminus \{v, y, w\}$ containing neither x nor z, the arguments of the preceding paragraph indicate that the T-split component D is not T-minimal. It follows that $\{v, y, w\}$ splits G into exactly two components, one containing x and one containing z. Call the former component X and the latter Z. We show next that both $X \cap D$ and $Z \cap D$ must be empty.

If $X \cap D \neq \emptyset$, Lemma 2.4 implies that $\{x, y, v\}$ separates $X \cap D$ from the rest of G, a contradiction to D being minimal. This implies that $X \cap D = \emptyset$. Analogously, $Z \cap D = \emptyset$.

We have thus shown that, assuming $v \in V(G)$ is in a T-minimal split component, there exists a separation triple $\{x, y, z\}$ with $xy \in E(T)$, such that $vx, vy, vz \in E(G)$, $\deg(v) = 3$, $vz \notin E(T)$ and w.l.o.g. $vy \in E(T)$. □

It is interesting to note that although foxes must have some degree-3 vertices as indicated by the previous lemma, not all vertices of a fox can be of degree 3.

Theorem 3.3 *If G is a 3-connected 3-regular graph with $n > 4$, then G is not a fox.*

Proof. Assume that G has a spanning tree T containing no contractible edge. According to Lemma 3.2, there are vertices $v, x, y, z \in V(G)$, such that $vx, vy, vz \in E(G)$, $xy, vy \in$

$E(T)$ but $vz \notin E(T)$. Because G is 3-regular, $deg(x) = deg(y) = 3$. As T is a spanning tree of G, either the third edge incident to x, say xr, or the third edge incident to y, say ys, is a tree edge. Since $vy \in E(G)$, xr is contractible by Lemma 1.4. Since $xy, vy \in E(T)$, both edges are non-contractible by assumption. Accordingly, ys is contractible by Lemma 1.3. This contradicts the assumption that T contains no contractible edge. □

Consider a 3-connected graph G with $n > 4$. Assume that G is a fox and let T be a spanning tree of G containing no contractible edge. Let v be a T-minimal split component in G, and let vy be the only tree edge incident to v. We call a T-separation triple $\{v, y, w\}$ a *special T-separation triple*. Split components that result from the removal of a special T-separation triple are called *special T-split components*. A special T-split component is minimal if there is no special T-split component properly contained in it.

Lemma 3.4 *Let G be a 3-connected graph with $n > 4$. Assume that G is a fox and let T be a spanning tree of G containing no contractible edge. Then, every special T-minimal split component consists of exactly one vertex and has a neighbor that is also a special T-minimal split component. Let v and v' be such a pair of special T-minimal split components with $vv' \in E(G)$. Then, there exists a vertex y such that $vy, v'y \in E(T)$.*

Proof. Let X be a minimal special T-split component; it is split off by the special T-separation triple $ST = \{v, y, w\}$ with v being a T-minimal split component and $vy \in E(T)$. By Lemma 2.2, no other special T-separation triple has its three vertices in $V(X) \cup ST$. Since ST is a T-separation triple, there exists a T-minimal split component $v' \in V(X)$; v' belongs to a special T-separation triple $ST' = \{v', y', w'\}$ with $v'y' \in E(T)$, where $y' \in V(X) \cup ST$ and $w' \notin V(X) \cup ST$ (otherwise, X would not be minimal).

Assume first that $y' \in V(X)$. Then, w' must split $G \setminus X$, and Lemma 2.5 implies that one of the vertices in ST has w' as its only neighbor in $G \setminus X$. Since $vy \in E(G)$, such vertex must be w. We next show that all neighbors of w are contained in ST', and hence w has degree 3. Assume to the contrary that w has a neighbor $u' \notin ST'$. Then, u' and w belong to the same split component of $G \setminus ST'$. Every path from u' to any vertex in a different split component of $G \setminus ST'$ must pass through either v', y' or w. Hence, $\{v', y', w\}$ is a special T-separation triple contained in $V(X) \cup ST$. But such possibility is ruled out in the previous paragraph because of the minimality of X. It follows that w has degree 3, its neighbors are precisely the vertices in ST', and w is a T-minimal split component. By Lemma 1.4, ww' is contractible, and accordingly does not belong to T. Also, $wv' \notin E(T)$ since $v'y' \in E(T)$ and v' has only one incident tree edge. Hence, $wy' \in E(T)$. Let z' be the third neighbor of v' besides y' and w. Then, $\{w, y', z'\}$ is a special T-separation triple that separates v' from the rest of G, see Figure 5. This again contradicts our choice of X being minimal. We conclude that $y' \notin V(X)$, and hence $y' \in ST$.

Since v' and w' are in different split components of $G \setminus ST$, using Lemma 2.3, the triples ST and ST' cross. Hence, the vertices of $ST \setminus \{y'\}$ must belong to different split components of $G \setminus ST'$. Since $vy \in E(G)$, this excludes the possibility that $y' = w$. Also, $y' \neq v$, since otherwise v would be incident to two tree edges, namely vy and $v'y'$. It must then be the case that $y = y'$. If $|V(X)| > 1$, Lemma 2.4 implies that either $\{v', y, v\}$ or $\{v', y, w\}$ is a special T-separation triple. Such a triple has a split component properly

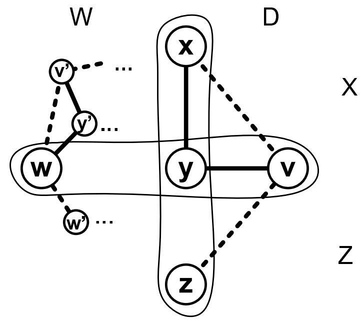


Figure 5: A case contradicting the minimality of X

contained in X, a contradiction to the minimality of X. It follows that v' is the only vertex in X. Let w'' be the third neighbor of v besides v' and y. Then, $\{v', y, w''\}$ is a special T-separation triple that separates v from the rest of G. We conclude that v and v' are both special T-minimal split components, $vv' \in E(G)$ and $vy, v'y \in E(T)$. $\square$

Theorem 3.5 *Let G be a 3-connected graph with $n > 4$. Assume that G is a fox and let T be a spanning tree of G containing no contractible edge. Then, there exist two edges in G such that their four end vertices are distinct special T-minimal split components.*

Proof. Let v and v' be adjacent special T-minimal split components as in Lemma 3.4; v is split off by $ST' = \{v', y, w'\}$ and v' is split off by $ST = \{v, y, w\}$.

Assume first that there is a special T-minimal split component in $V(G) \backslash \{v, v', y, w, w'\}$. Call it z, and let z' be the adjacent special T-minimal split component. Then, $z' \notin \{v, v'\}$, and hence (v, v') and (z, z') are the desired pairs.

Otherwise, any special T-minimal split component of G is contained in $\{v, v', y, w, w'\}$. Let W' be the component of $G \setminus ST$ containing w', and let W be the component of $G \setminus ST'$ containing w. Both W' and W are special T-split components, and hence contain special T-minimal split components. These components must be w for W and w' for W'. Then, (v, w') and (v', w) are the desired pairs. $\square$

Next, we use Theorem 3.5 to prove our main result.

Theorem 3.6 *Consider a 3-connected graph G with $n > 4$. Every depth-first-search tree of G contains a contractible edge.*

Proof. Let T be a depth-first-search tree of G, and assume that T contains no contractible edge. By Theorem 3.5, there exist two pairs of distinct degree-3 vertices, each vertex is a T-minimal split component, such that the vertices of each pair are adjacent in G. By Lemma 3.2, every T-minimal split component is a degree-3 vertex that is either the root or a leaf in T. Accordingly, there exists a pair of vertices that are leaves in T while being adjacent in G, a contradiction to the fact that T is a depth-first-search tree. $\square$

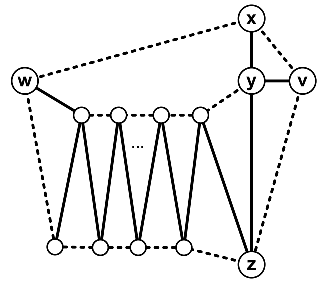


Figure 6: A fox with exactly four degree-3 vertices.

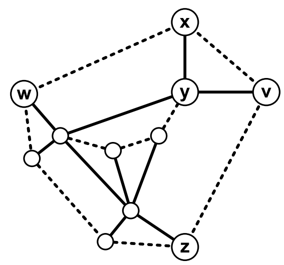


Figure 7: A fox with $m = 2n - 3$.

We remark that there are arbitrarily large foxes having exactly four vertices of degree 3, see Figure 6.

4 Conclusions

Our main result is that every depth-first-search tree of a 3-connected graph contains a contractible edge. However, not every spanning tree of a 3-connected graph contains a contractible edge. We hope that our positive result will lead to a linear-time certifying algorithm for three-connectivity.

Foxes, i.e. 3-connected graphs with a spanning tree that contains no contractible edge, are interesting from a combinatorial view. We wonder if there is an inductive characterization of foxes. All Wheel graphs, as well as the foxes in Figure 2, satisfy the equation $m = 2n - 2$. Figure 7 depicts a fox with $m = 2n - 3$.

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