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LOWER BOUNDS FOR THE SPACE COMPLEXITY OF CONTEXT-FREE RECOGNITION

Context-free languages are an important topic in practical as well as in theoretical computer science. Much effort was devoted to the construction of time - and (or) space - efficient recognition algorithms.

Early results were obtained by Lewis, Hartmanis and Stearns [6]: Every context-free language can be recognized by an off-line Turing machine in space $O(\log^2 n)$, where n is the length of the input. Recent results by Sudborough [18] and Monien [14] indicate that it is probably very hard to beat this bound. They showed that the existence of a general context-free recognition algorithm working in logarithmic space would imply the equality of deterministic and nondeterministic context-sensitive languages, i.e. a solution to Myhill's LBA-problem.

However, it is conceivable that large subclasses of the context-free languages (e.g. the deterministic languages) are recognizable in logarithmic space. First results in this direction were obtained by Ritchie and Springsteel [16], Hotz and Messerschmidt [10], Wrathall [19], Lynch [11] and Mehlhorn [12]. They showed that the Dycklanguages are recognizable in logarithmic space [16], by two-way one counter automata [10,19] and that parenthesis languages are recognizable in logarithmic space [11,12].

In this paper, we attack the problem from the other end, i.e. we attempt to prove *lower* bounds for the space complexity of context-free recognition. (cf. also Alt [1], Hotz [9] and Mehlhorn [13]). We show:

Thm.: Let $L \subset \sum^*$ be context-free and let exist words u,v,w,x,y such that $L \cap uv^*wx^*y$ is non-regular. Then the membership problem for either L or $\sum^* - L$ requires log n space infinitely often on a nondeterministic Turing machine.

We combine this result with a result of Stearns [17] and obtain.

Corollary: Let L be a non-regular deterministic context-free language. Then the membership problem for either L or $\sum^* - L$ requires log n space on a nondeterministic machine.

Thus we obtain lower bounds for entire families of languages, not just single languages. This is a new kind of lower bound result. Furthermore we have results about the minimal growth rate of tape constructable functions. (cf. also Seiferas [15]).

Finally we exhibit some examples which hint to the limits of our approach.

MAIN THEOREM

Our machine models are off-line Turing machines, i.e. TM with one read-only input tape and any number of read/write work tapes. The input is placed between endmarkers on the input tape. L(M) denotes the language accepted by Turing machine M. A deterministic TM M has space complexity S( ) if on every input x of length n at most S(n) cells of the work tapes are used. A nondeterministic TM M has space complexity S( ) if on every input $x \in L(M)$ of length n there is an accepting computation which uses at most S(n) cells of the work tapes. We write $L(M) \in SPACE(S(\ ))$ and $L(M) \in NSPACE(S(\ ))$ resp. in this case.

First we consider context-free subsets of a^*b^*, where $\{a,b\}$ is a two symbol alphabet.

Thm. 1: Let $L \subseteq a^*b^*$ be a non-regular context-free language and let $L, a^*b^* - L \in NSPACE(S(\ ))$. Then $\limsup_{n \to \infty} {}^{S(n)}/_{\log n} > 0$.

We note in passing that O(log n) space is sufficient to recognize context-free subsets of a^*b^*. This is an immediate consequence of Parikh's theorem. As a corollary we obtain.

Thm. 2: Let $L \subseteq \sum^*$ be context-free, let u,v,w,x,y be words with $L \cap uv^*wx^*y$ non-regular and let $L, \bar{L} \in NSPACE(S(\ ))$. Then $\limsup_{n \to \infty} {}^{S(n)}/_{\log n} > 0$. $\bar{L}$ denotes the set-theoretic complement $\sum^* - L$ of L.

Proof of Thm. 2:

Define $L' \subseteq a^*b^*$ by

$$L' = \{a^ib^j;\ uv^iwx^jy \in L\}$$

and $L'' \subseteq a^*b^*$ by

$$L'' = \{a^ib^j;\ uv^iwx^jy \in \bar{L}\}$$

Then $L' = a^*b^* - L''$. Let M be a NTM of space complexity S( ) which recognizes L ($\bar{L}$). It is easy to construct a NTM M' (M") recognizing L' (L") which on input a^ib^j uses at most space $S(|n| + i|v| + |w| + j|x| + |y|)$. Since L' is context-free and non-regular we infer from Theorem 1

$$\limsup_{n \to \infty} \frac{S(|n|+i|v|+|w|+j|x|+|y|)}{\log (i+j)} > 0$$

and hence $\limsup_{n \to \infty} \frac{S(n)}{n} > 0$.

q.e.d.

Proof of Thm. 1:

In the sequel we identify languages $L \subseteq a^*b^*$ with subsets of N_o^2, i.e. we identify L with $\{(n,m);\ a^nb^m \in L\} \subseteq \mathbb{N}_o^2$.

Definition 1:

a) Sets of the form

$L(\alpha;\beta_1,\ldots,\beta_k) = \{\alpha+n_1\beta_1+\ldots+n_k\beta_k;n_1,\ldots,n_k \in \mathbb{N}_o\}$

with $\alpha_k,\beta_1,\ldots,\beta_k \in \mathbb{N}_o^2$ are called *linear subsets* of $\mathbb{N}_o^2$.

A finite union of linear sets is a semilinear set.

b) Sets of the form

$C(\alpha;\beta_1,\ldots,\beta_k) = \{\alpha+q_1\beta_1+\ldots+q_k\beta_k;q_1,\ldots,q_k \in \mathbb{Q}_+\}$

with $\alpha,\beta_1,\ldots,\beta_k \in N_o^2$ are called *cones*.

($\mathbb{Q}_+$ denotes the set of non-negative rational numbers).

c) If $\beta_i = (b,0)$ ((0,b)) for some i and b > 0 then the linear set (cone) is *x-regular* (y-regular).

d) A cone $C(\alpha;\beta_1,\dots,\beta_k)$ is *nondegenerate* if there exist non-parallel β_i,β_j.

e) A *grid* $G(\alpha;r,s)$ is a linear subset of the form $L(\alpha,(r,0),(0,s))$. If $r,s > 0$ then the grid is *proper*.

Remarks: We refer the reader to Ginsburg [5] for additonal information on linear and semilinear sets. It is proved there that the class of semilinear sets is closed under set-theoretic complement. Cones are linear sets, though we will not make use of that fact here. Grids are regular sets; the grid $G((n,m);r,s)$ corresponds to the language $a^n(a^r)^*b^m(b^s)^*$. (See figure 1).

Definition 2: A language $L \subseteq \mathbb{N}_0^2$ *does not cling to the axes* if there is a polynomial $p(\ )$ such that $i + j \le p(\min(i,j))$ for infinitely many $(i,j) \in L$.

Lemma 1: Let $L \subset \mathbb{N}_0^2$ and $L \in NSPACE(S(\ ))$ with $\lim S(n)/\log n = 0$. If L does not cling to the axes then L contains a proper grid.
From lemma 1 we infer that nondeterministic recognizers for the languages $\{a^nb^n;\ n \in \mathbb{N}\}$ and $\{a^nb^m;\ m \ge n\}$ use log n space infinitely often.

Lemma 2: Let L be a linear subset of $\mathbb{N}_0^2$.
Then there is a nondegenerate x-regular cone C and grids $G_1,\dots,G_s$ $(s \geqslant 0)$ such that

$$L \cap C = C \cap (G_1 \cup \dots \cup G_s)$$

Lemma 2 states that in some nondegenerate cone linear sets are essentially regular. We extend lemma 2 to semilinear subsets.

Lemma 3: Let L be a semilinear subset of $\mathbb{N}_0^2$. Then there is a nondegenerate x-regular cone C and grids $G_1,\dots,G_s$ $(s \geqslant 0)$ such that

$$L \cap C = C \cap (G_1 \cup \dots \cup G_s)$$

Finally we need:

Lemma 4: Let $L \subseteq \mathbb{N}_0^2$ be context-free but not regular. Then L does not cling to the axes.

We combine these lemmas to a proof of theorem 1.

Let $L \subseteq a^*b^*$ be context-free but not regular, let $L, \bar{L}$ ($= a^*b^* - L$) $\in$ NSPACE(S( )) with $\lim S(n)/\log n = 0$. L is a semilinear set by Parikh's theorem. Because of lemma 3 there is a nondegenerate cone C and grids $G_1, \ldots, G_s$ $(s \geq 0)$ such that $L \cap C = C \cap (G_1 \cup \ldots \cup G_s)$

Let $G = G_1 \cup \ldots \cup G_s$. G is regular by the remark following definition 1. Let $L_1 = G - L = G \cap \bar{L}$ and let $L_2 = L - G$. Because of $L = (G - L_1) \cup L_2$ either L_1 or L_2 is non-regular. Furthermore $L_1, L_2 \in$ NSPACE(S( )). L_1 and L_2 are context-free by corollary 5.6.2. in [5].

Hence we can apply lemma 4 and therefore lemma 1 to either L_1 or L_2. Thus either L_1 or L_2 contains a proper grid. But neither L_1 nor L_2 can obtain a proper grid because of $L_1 \cap C = \emptyset = L_2 \cap C$ and the fact that every proper grid cuts every nondegenerate cone.

It remains to prove the lemmas.

Proof of lemma 1: Let M be an off-line NTM of space complexity S( ) which accepts L. M has k work tapes, tape alphabet Γ and state set Z. We consider M on inputs $a^i b^j \in L$. There is an accepting computation of M which uses $\leq S(i+j)$ cells of the work tapes. Hence M enters at most $|Z| \cdot (S(i+j) \cdot |\Gamma|^{S(i+j)})^k$ different configurations (= state, position of the heads on the work tapes, content of the work tapes) during this computation. Let

$$\#(n) = |Z| \cdot (S(n) \cdot |\Gamma|^{S(n)})^k$$

Since L does not cling to the axes there is a polynomial p( ) such that $i+j \leqslant p(\min(i,j))$ for infinitely many $(i,j) \in L$, thus $\log(i+j) \le C \cdot \log(\min(i,j))$ for some positive constant C. From $\lim S(n)/\log n = 0$ we infer

$$\log \#(n) = \log|Z| + k(\log S(n) + S(n) + \log|\Gamma|)$$

$$\le \log|Z| + k \cdot \log|\Gamma| + (k+1) \cdot S(n)$$

$$< {}^{1}/_{C} \cdot \log n$$

for all sufficiently large n. Hence there are infinitely many $(i,j) \in L$ with

$$\log \#(i+j) < {}^{1}/_{C} \cdot \log(i+j) \le \log(\min(i,j))$$

and therefore

$$\#(i+j) < \min(i,j)$$

Choose any $(i,j) \in L$ with $\#(i+j) < \min(i,j)$. On input $a^i b^j$ the number of configurations does not suffice to distinguish all positions within the a-portion and the b-portion of the input. This fact will allow us to establish the following claim.

Claim: M accepts $a^{i+s\cdot i!}\, b^{j+t\cdot j!}$ for all $(s,t) \in \mathbb{N}_0^2$.

We will construct an accepting computation from the accepting computation on input $a^i b^j$. The accepting computation on input $a^i b^j$ is a sequence $(p_1, C_1), (p_2, C_2), \ldots$ of pairs of (position of the head on the input tape, configuration). We split the computation into sections of maximal size such that within each section M reads either only a's or b's. A section of computation lies in one of the following eight categories.

1) M starts on the left endmarker, works on the a-portion of the tape without ever returning to the left endmarker and finally leaves the a-portion of the tape.

2) M starts on the left endmarker works on the a-portion of the tape and returns to the left endmarker.

⋮

We construct now the corresponding section of the accepting computation of M on input $a^{i+s\cdot i!}\ b^{j+t\cdot j!}$

We consider only cases 1) and 2), the remaining cases are treated analogously.

Case 1: Let M be in configuration C_k when it scans the kth cell for the first time in this section, $k = 1,2,\ldots,i$. Because of $\#(i+j) < \min(i,j) \leq i$ there exist $k_1, k_2 \neq 0$ with $C_{k_1} = C_{k_1+k_2}$. (See figure 2).

On input $a^{i+s\cdot i!}\ b^{j+t\cdot j!}$ M behaves the same as on input $a^i b^j$ until it reads cell k_1 the first time. It then cycles $1+s\cdot({}^{i!}/_{k_2})$ times through the sequence of moves which M used to advance the head from cell k_1 to cell k_1+k_2. This leaves M in configuration C_{k_1} scanning cell $k_1+s\cdot i!+k_2$. M now finishes the section as it does on input $a^i b^j$. (See figure 3).

Case 2: In this case M behaves the same on inputs $a^i b^j$ and $a^{i+s\cdot i!}\ b^{j+t\cdot j!}$.

So far we constructed inductively the computation of M on input $a^{i+s\cdot i!}\ b^{j+t\cdot j!}$. It is easily seen from the construction process that M enters exactly the same configurations on inputs $a^i b^j$ and $a^{i+s\cdot i!}\ b^{j+t\cdot j!}$. Therefore M accepts input $a^{i+s\cdot i!} b^{j+t\cdot j!}$.

end of proof of lemma 1.

We now turn to our lemmas on linear and semilinear sets.

Proof of lemma 2: Let $L = L(\alpha;\beta_1,\ldots,\beta_k)$ be a linear subset of $\mathbb{N}_o^2$. If L is not x-regular then $\mathbb{N}_o^2 - L$ contains a non-degenerate x-regular cone and the claim is obviously true (s=o).

If all β_i are parallel to the x-axis, we have a $c \in \mathbb{N}$ with: $(u,v) \in L \Rightarrow v < c$ and therefore $\mathbb{N}_o^2 - L$ contains a non-degenerate x-regular cone.

If all β_i $(i = 1,\ldots,k)$ are parallel to either one of the axes, then L is a finite union of grids $G_1,\ldots,G_s$ itself and the claim is obviously true with $C = C(\alpha;(0,1),(1,0))$.

The case remains, that there is one generating vector, say β_1, of the form (b_1,b_2) with $b_1,b_2 \neq 0$, and another, say β_2, of the form $(b_1',0)$. Let $r_1 = b_1b_1'$ $r_2 = b_2b_1'$.

We define $G := G(\alpha;\ r_1,\ r_2)$ and $C' := C(\alpha;\beta_1,\beta_2)$ (See figure 4).

Now we assign to each pair $(u,v) \in \mathbb{N}_0^2$ the point

$$p_{u,v} := \alpha + (ur_1, vr_2) \in \mathbb{N}_0^2$$

and the set

$$I_{u,v} := \{\delta \in [0:r_1-1]\times[0:r_2-1] \mid p_{u,v}+\delta \in L\}$$

($[0:t]$ denotes the set $\{0,\ldots,t\} \subset \mathbb{N}_0$).
i.e. the points of L within the mesh of G with corner $p_{u,v}$. (See figure 5).

Let $\gamma_1,\gamma_2 \in \mathbb{N}_0^2$. We say that γ_2 is *reachable* from γ_1 if there are natural numbers $n_1,\ldots,n_k$, such that $\gamma_2 = \gamma_1 + n_1\beta_1 + \ldots + n_k\beta_k$.

It is easy to verify that if $p_{u',v'}$ is reachable from $p_{u,v}$ then $I_{u',v'} \supseteq I_{u,v}$.

Since for all $u,v \in \mathbb{N}_0$ $I_{u,v} \subseteq [0:r_1-1]\times[0:r_2-1]$ there doesn't exist an infinite sequence $(u_i,v_i)_{i \in \mathbb{N}}$ with $I_{u_1,v_1} \subsetneq I_{u_2,v_2} \subsetneq \ldots$

It follows that there is a $(u_0,v_0) \in \mathbb{N}_0^2$ such that

$$I_{u,v} = I_{u_0,v_0}$$

for all $u,v \in \mathbb{N}_0$ with: $p_{u,v}$ is reachable from p_{u_0,v_0}.

That means: No mesh, whose corner is reachable from

p_{u_o,v_o} is fuller than I_{u_o,v_o}.

Let $I_{u_o,v_o} = \{x_1,\ldots,x_s\}$. Now we define

$$C := C(p_{u_o,v_o};\beta_1,\beta_2).$$

$$G_i := G(p_{u_o,v_o}+x_i;r_1,r_2) \quad \text{for } i=1,\ldots,s$$

First we make the following

Remark: If $\delta \in C$ and $p_{u,v}$ is the left lower corner of the mesh of G containing δ, then $p_{u,v} \in C$. This is because the border lines of C and the points of G are situated as in figure 4 , i.e. the left lower corners of the border meshes lie on the border lines of C.

Now we will show the following:

Claim: Each point of $G \cap C$ is reachable from p_{u_o,v_o}.

Proof : If $\delta \in G\cap C$, then there are $k_1,k_2 \in \mathbb{N}_o$ and $q_1,q_2 \in \mathbb{Q}_+$ with

$$\delta = \alpha + (k_1r_1,k_2r_2)$$

and
$$\delta = p_{u_o,v_o} + q_1\beta_1 + q_2\beta_2$$
$$= \alpha + (u_or_1,v_or_2) + q_1\beta_1 + q_2\beta_2$$

It follows

$$q_1\beta_1 + q_2\beta_2 = ((k_1-u_o)r_1,\ (k_2-v_o)r_2)$$

Comparing coefficients yields

$$q_1 = (k_2-v_o)b_1' \in \mathbb{Z}$$

and
$$q_2 = (k_1-u_o-k_2+v_o)b_1 \in \mathbb{Z}$$

Thus $q_1,q_2 \in \mathbb{Q}_+\cap \mathbb{Z} = \mathbb{N}_o$ and δ is reachable from q_{u_o,v_o}.

We will next show that:

$$L \cap C = C \cap (G_1 \cup \ldots \cup G_s)$$

$\subseteq$: Let $\delta \in L \cap C$. Let $p_{u,v}$ be the lower left corner of the mesh of G containing δ, i.e.

$$p_{u,v} - \delta \in [0:r_1-1] \times [0:r_2-1].$$

$p_{u,v} \in C$ by our remark and hence $p_{u,v}$ is reachable from p_{u_o,v_o} by the preceding claim. Thus

$$I_{u,v} = I_{u_o,v_o}$$

i.e. there exists an $i \in \{1,\ldots,s\}$ with $\delta - p_{u,v} = x_i$. It is easy to see now, that $\delta \in G_i$.

$\supseteq$: Let $\delta \in G_j \cap C$ for some $j \in \{1,\ldots,s\}$. It follows: There exist $k_1, k_2 \in \mathbb{N}_o$ with

$$\delta = p_{u_o,v_o} + x_j + (k_1 r_1, k_2 r_2)$$

$$= p_{u,v} + x_j$$

where $u = u_o + k_1$

$$v = v_o + k_2$$

Thus $p_{u,v}$ is the corner of the mesh containing δ. Since $\delta \in C$, we have by our remark above that $p_{u,v}$ is in C and hence reachable from p_{u_o,v_o}, i.e. there are $l_1,\ldots,l_k \in \mathbb{N}_o$ with:

$$\delta = p_{u_o,v_o} + l_1\beta_1 + \ldots + l_k\beta_k + x_j$$

From $p_{u_o,v_o} + x_j \in L$ we conclude $\delta \in L$.

Proof of lemma 3: Let $L = L_1 \cup \ldots \cup L_m$ be the decomposition of L into linear subsets, where

$L_1,\ldots, L_i$ are x-regular and non-degenerate

$L_{i+1},\dots,L_m$ are not x-regular or degenerate

Let $C_1,\dots,C_i$ be the non-degenerate x-regular cones attached to $L_1,\dots,L_i$, according to lemma 2. It is easy to see that the intersection of two and hence of finitely many non-degenerate x-regular cones contains again a non-degenerate x-regular cone. So we can find such a cone, say $C' \subseteq C_1 \cap \dots \cap C_i$.

Similarily we can find a non-degenerate x-regular cone $C'' \subset (\mathbb{N}_o^2 \setminus L_{i+1}) \cap \dots \cap (\mathbb{N}_o^2 \setminus L_m)$

Let C be a non-degenerate x-regular cone with $C \subset C' \cap C''$.

As $G_1,\dots,G_s$ we take all grids attached to one of the sets $L_1,\dots,L_i$ according to lemma 2.

Then within C each point of L is contained in one of these grids and vice versa. •

Proof of lemma 4: If $L = L_1 \cup \dots \cup L_m$ is a decomposition of L into linear sets and all generating vectors of $L_1,\dots,L_m$ have either the form (b,O) or the form (O,b) then obviously L is a finite union of grids and thus regular.

It follows that there has to exist at least one $L_i (1 \le i \le m)$ with $L_i = L(\alpha;\beta_1,\dots,\beta_k)$ with at least one of the β_j, say β_1 of the form $\beta_1 = (a,b)$ $a,b \neq 0$.

Since $\{\alpha + i(a,b) \mid i \in \mathbb{N}\} \subset L$ it doesn't cling to the axes.

APPLICATIONS

In this section we apply our results to obtain lower bounds for entire families of context-free languages and to obtain results about the growth rates of tape constructable functions.

Fact (Stearns [17]): Let L be a deterministic context-free language. Then L is regular if and only if for all $u,v,w,x,y \in \Sigma^*$ the language $L \cap uv^*wx^*y$ is regular.

We combine this fact with Thm. 2 and get:

Thm. 3: Let L be a non-regular deterministic context-free language and let $L, \Sigma^*-L \in NSPACE(S(\ ))$.

Then $\limsup_{n\to\infty} S(n)/\log n > 0$.

The following result was obtained jointly with Luc Boasson.

Definition 3: A context-free language is a full generator if the least full AFL containing L is the set of context-free languages.

Fact (Boasson [3]): Let L be a full generator. Then there are u,v,w,x,y with $L \cap uv^*wx^*y$ is non-regular.

Thm. 4: Let L be a full generator and $L, \Sigma^*-L \in NSPACE(S(\ ))$. Then $\limsup_{n\to\infty} S(n)/\log n > 0$.

Proof: Immediate consequence of Thm. 2 and the preceding fact.

Next we turn our attention to tape constructable functions.

Definition 4: A function $f: N \to \mathbb{N}$ is tape constructable if there is a deterministic off-line TM which for every n uses exactly f(n) work tape squares on input 0^n.

Thm. 5: Let f be tape constructable with $\liminf_{n\to\infty} f(n) = \infty$. Then $\liminf_{n\to\infty} f(n)/\log n > 0$, i.e. if the sequence $\{f(n)\}_{n=0}^{\infty}$ does not have a bounded subsequence then the sequence grows at least as fast as the logarithm.

Proof: Assume $\liminf_{n\to\infty} f(n)/\log n = 0$. Let $L = \{a^n b^m;\ m \geq f(n+m)\}$. Then $L \in SPACE(f(\ ))$. Furthermore, L does not cling to the axes and L does not contain a proper grid. However, the proof of Lemma 1 demonstrates the existence of a proper grid in L. Contradiction.

Thm. 5 was found independently by Seiferas [15].

HINTS TO THE LIMITS OF OUR APPROACH

In this section we investigate the question if the hypotheses of our main theorem (Thm. 2) are necessary.

Fact (Alt&Mehlhorn [2], Fredman&Ladner [4], Hartmanis&Berman [7]): There is a language L over a one symbol alphabet with L ∈ SPACE(log log n).

We conclude that the hypothesis "L context-free" cannot be dropped. Neither can the hypothesis "L,$\bar{L}$ ∈ NSPACE(S( ))" be replaced by "L ∈ NSPACE(S( ))".

Observation: $L = \{a^n b^m;\ n \neq m\} \in$ NSPACE(log log n).

Proof: The recognition algorithm is based on the following well known fact from number theory. For $n \neq m$ let f(n,m) be the smallest integer k with $n \not\equiv m \pmod{k}$. Then f(n,m) = O(log(n+m)). The following algorithm recognizes L.

$k \leftarrow 1;$

<u>while</u> $n \equiv m \pmod{k}$ <u>do</u> $k \leftarrow k+1;$

Accept

For a fixed value of k the test $n \equiv m \pmod{k}$ can be carried out in space log k and for pairs (n,m) with $n \neq m$ only values of $k \leq O(\log(n+m))$ have to be tested.

The proof of the observation heavily depends on our definition of nondeterministic space complexity: only accepted inputs are considered and only one accepting computation has to satisfy the space bound. In the literature (e.g. Hopcroft & Ullman [8]) a different definition is used sometimes: every computation has to satisfy the space bound.

Open Problem: Can we replace L, $\bar{L}$ ∈ NSPACE(S( )) by L ∈ NSPACE(S( )) with this definition of nondeterministic space complexity.

In Alt [1] a further result in the spirit of this paper will appear.

Thm.: Let L be a nonregular, bounded context-free language with L, Σ^*-L $\in$ NSPACE(S( )).

Then $\limsup_{n\to\infty} {}^{S(n)}/_{\log n} > 0$.

Acknowledgement: We want to thank G. Hotz and J. Messerschmidt for many stimulating discussions.

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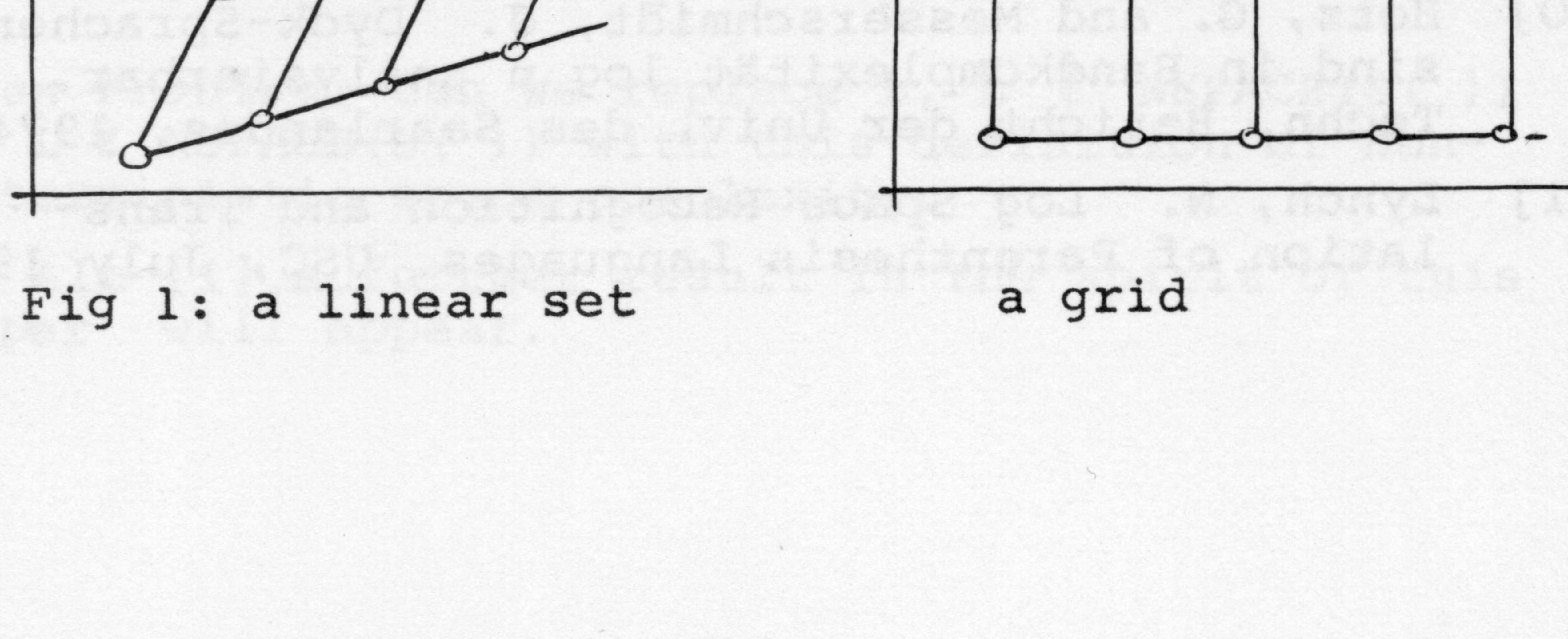

Fig 1: a linear set a grid

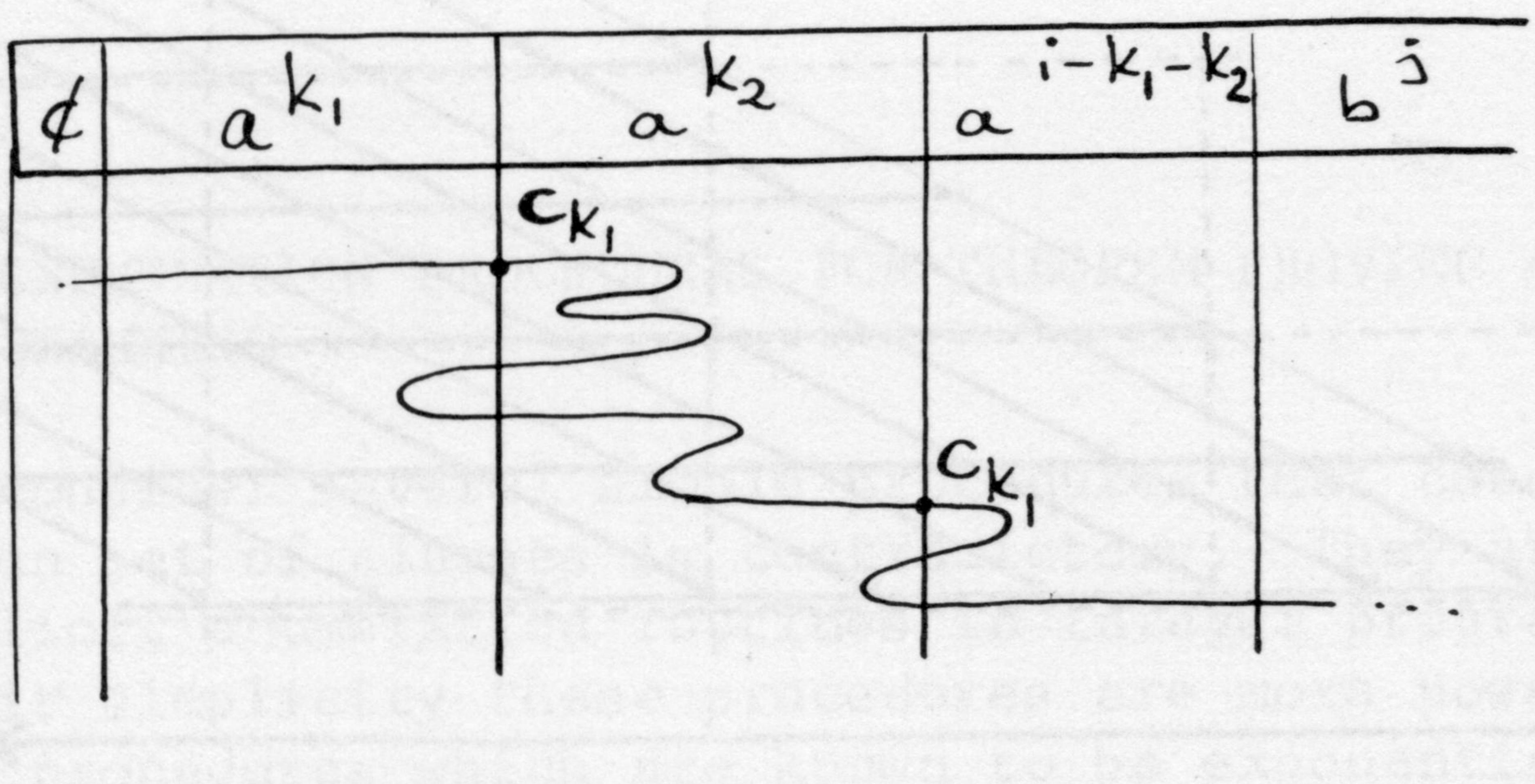

Fig 2: a segment of computation on $a^i b^j$

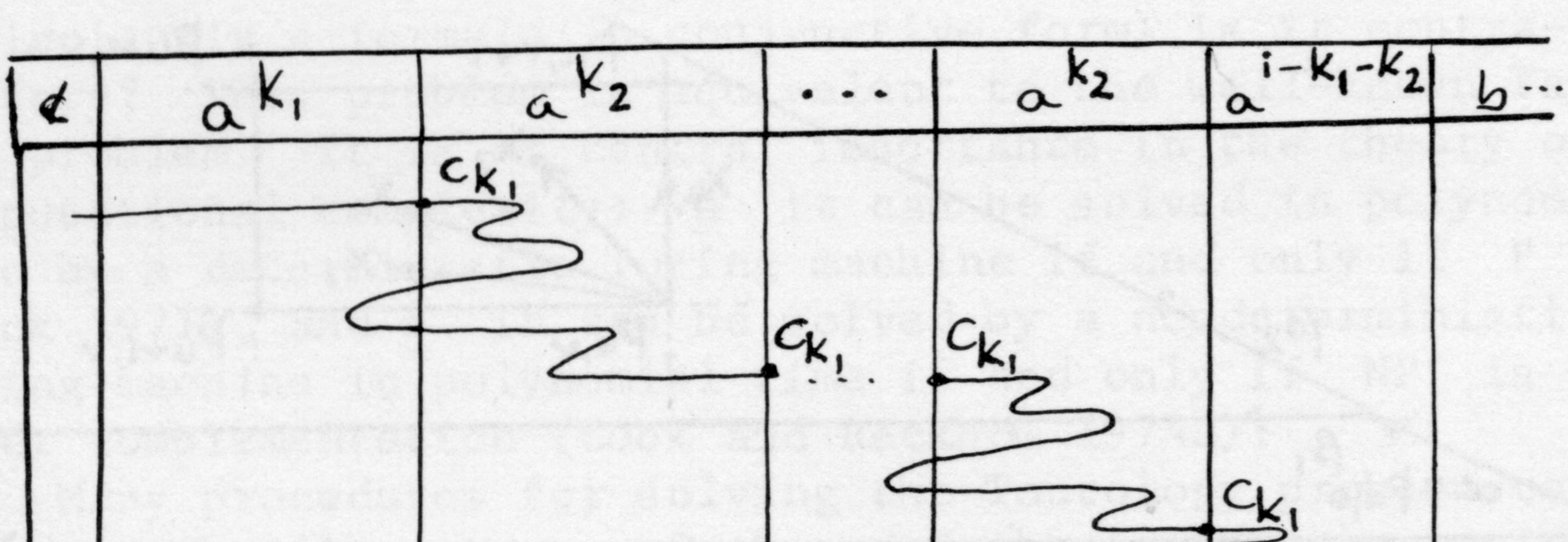

Fig 3: a segment of computation on $a^{i+si!}\ b^{j+tj!}$

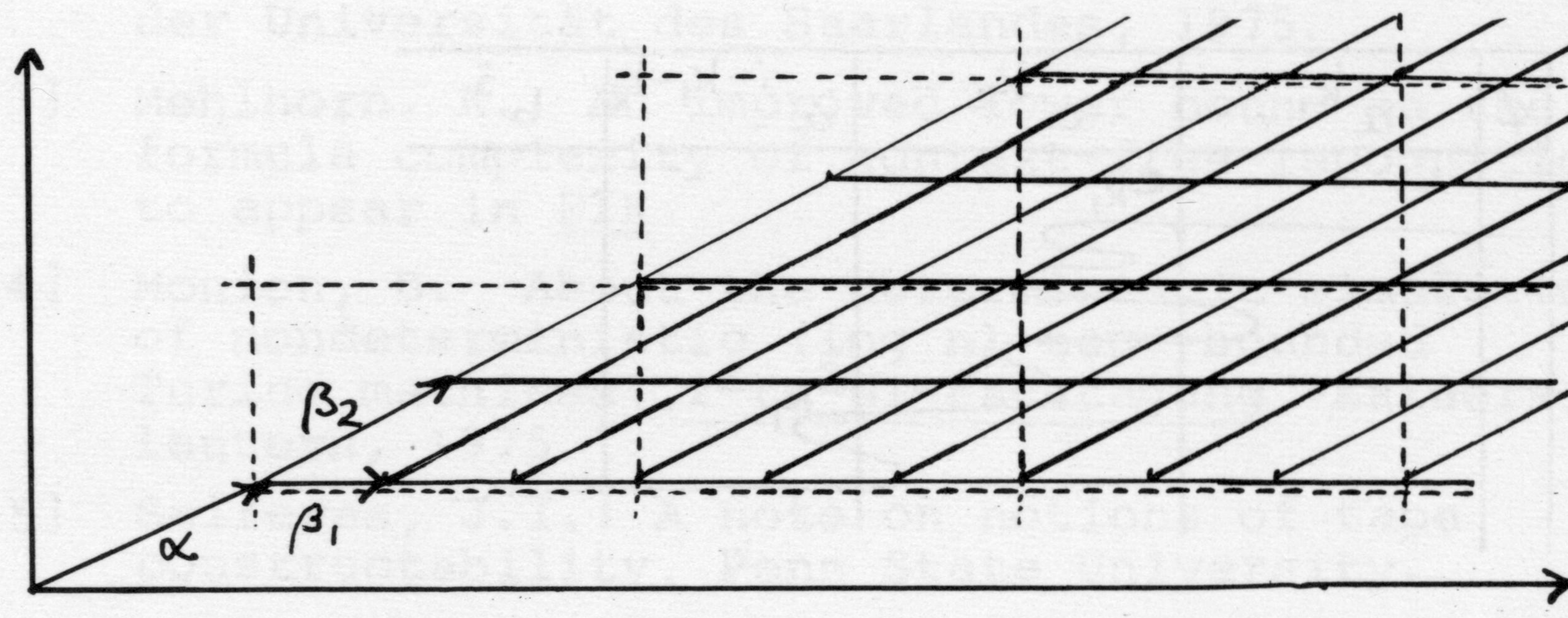

Fig 4: The grid G and the cone C' of lemma 2

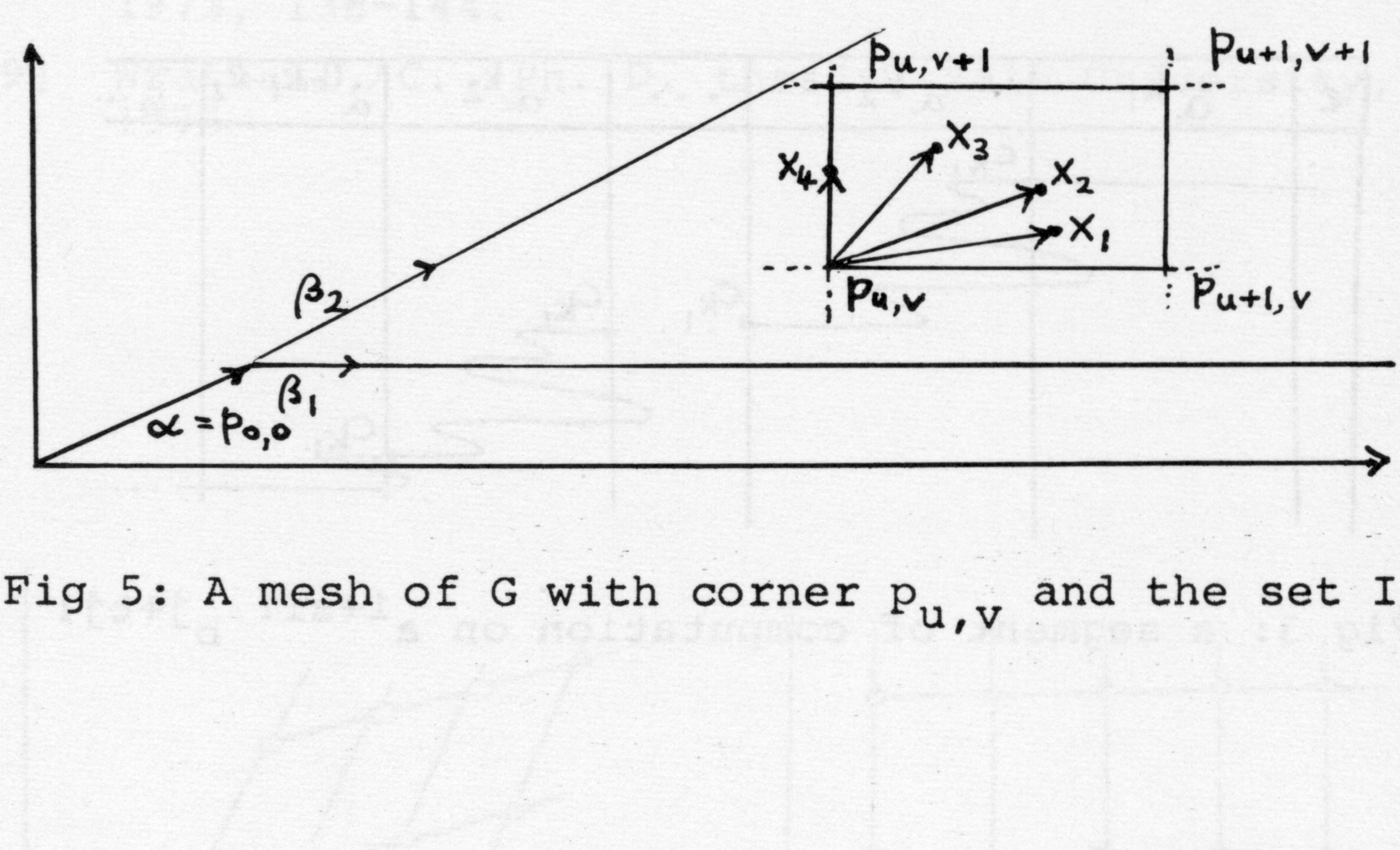

Fig 5: A mesh of G with corner $p_{u,v}$ and the set $I_{u,v}$