A Simple Test on 2-Vertex- and 2-Edge-Connectivity

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Abstract

Testing a graph on 2-vertex- and 2-edge-connectivity are two fundamental algorithmic graph problems. For both problems, different linear-time algorithms with simple implementations are known. Here, an even simpler linear-time algorithm is presented that computes a structure from which both the 2-vertex- and 2-edge-connectivity of a graph can be easily "read off". The algorithm computes all bridges and cut vertices of the input graph in the same time.

1 Introduction

Testing a graph on 2-connectivity (i. e., 2-vertex-connectivity) and on 2-edge-connectivity are fundamental algorithmic graph problems. Tarjan presented the first linear-time algorithms for these problems, respectively [11, 12]. Since then, many linear-time algorithms have been given (e. g., [2, 3, 4, 5, 6, 13, 14, 15]) that compute structures which inherently characterize either the 2- or 2-edge-connectivity of a graph. Examples include *open ear decompositions* [8, 16], *block-cut trees* [7], *bipolar orientations* [2] and *s-t-numberings* [2] (all of which can be used to determine 2-connectivity) and *ear decompositions* [8] (the existence of which determines 2-edge-connectivity).

Most of the mentioned algorithms use a depth-first search-tree (DFS-tree) and compute so-called *low-point* values, which are defined in terms of a DFS-tree (see [11] for a definition of low-points). This is a concept Tarjan introduced in his first algorithms and that has been applied successfully to many graph problems later on. However, low-points do not always provide the most natural solution: Brandes [2] and Gabow [6] gave considerably simpler algorithms for computing most of the above-mentioned structures (and testing 2-connectivity) by using simple path-generating rules instead of low-points; they call these algorithms *path-based*.

The aim of this paper is a self-contained exposition of an even simpler linear-time algorithm that tests both the 2- and 2-edge-connectivity of a graph. It is suitable for teaching in introductory courses on algorithms. While Tarjan's two algorithms are currently the most popular ones used for teaching (see [6] for a list of 11 text books in which they appear), in my teaching experience, undergraduate students have difficulties with the details of using low-points.

The algorithm presented here uses a very natural path-based approach instead of low-points; similar approaches have been presented by Ramachandran [10] and Tsin [14] in the context of parallel and distributed algorithms, respectively. The approach is related to ear decompositions; in fact, it computes an (open) ear decomposition if the input graph has appropriate connectivity.

Notation. We use standard graph-theoretic terminology from [1]. Let $\delta(G)$ be the minimum degree of a graph G. A *cut vertex* is a vertex in a connected graph that disconnects the graph upon deletion. Similarly, a *bridge* is an edge in a connected graph that disconnects the graph upon deletion. A graph is 2-connected if it is connected and contains at least 3 vertices, but no cut vertex. A graph is 2-edge-connected if it is connected and contains at least 2 vertices, but no bridge. Note that for very small graphs, different definitions of (edge)connectivity are used in literature; here, we chose the common definition that ensures consistency with Menger's Theorem [9]. It is easy to see that every 2-connected graph is 2-edge-connected, as otherwise any bridge in this graph on at least 3 vertices would have an end point that is a cut vertex.

2 Decomposition into Chains

We will decompose the input graph into a set of paths and cycles, each of which will be called a *chain*. Some easy-to-check properties on these chains will then characterize both the 2- and 2-edge-connectivity of the graph. Let $G = (V, E)$ be the input graph and assume for convenience that G is simple and that $|V| \geq 3$.

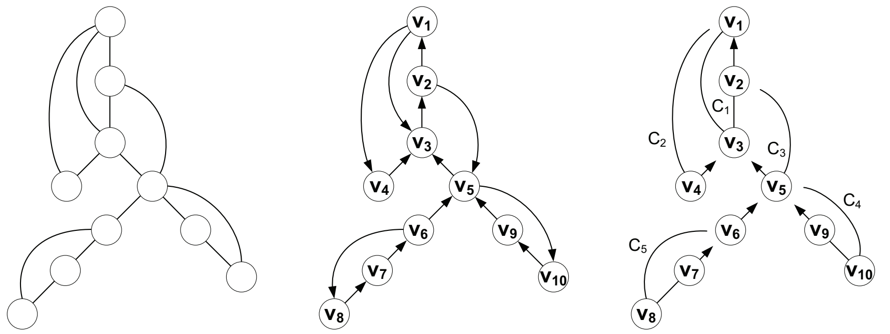
We first perform a depth-first search on G. This implicitly checks G on being connected. If G is connected, we get a DFS-tree T that is rooted on a vertex r; otherwise, we stop, as G is neither 2- nor 2-edge-connected. The DFS assigns a *depth-first index* (DFI) to every vertex. We assume that all *tree edges* (i. e., edges in T) are oriented towards r and all *backedges* (i. e., edges that are in G but not in T) are oriented away from r. Thus, every backedge lies in exactly one *directed cycle* $C(e)$. Let every vertex be marked as *unvisited*.

We now decompose G into *chains* by applying the following procedure for each vertex v in ascending DFI-order: For every backedge e that starts at v, we traverse $C(e)$, beginning with v, and stop at the first vertex that is marked as visited. During such a traversal, every traversed vertex is marked as *visited*. Thus, a traversal stops at the latest at v and forms either a directed path or cycle, beginning with v; we call this path or cycle a *chain* and identify it with the list of vertices and edges in the order in which they were visited. The ith chain found by this procedure is referred to as C_i.

The chain C_1, if exists, is a cycle, as every vertex is unvisited at the beginning (note C_1 does not have to contain r). There are $|E| - |V| + 1$ chains, as every of the $|E| - |V| + 1$ backedges creates exactly one chain. We call the set $C = \{C_1, \ldots, C_{|E|-|V|+1}\}$ a *chain decomposition*; see Figure 1 for an example.

Clearly, a chain decomposition can be computed in linear time. This almost concludes the algorithmic part; we now state easy-to-check conditions on C that characterize 2- and 2-edge-connectivity. All proofs will be given in the next section.

Theorem 1. *Let C be a chain decomposition of a simple connected graph G. Then G is 2-edge-connected if and only if the chains in C partition E.*



(a) An input graph G.

(b) A DFS-tree of G (depicted with straight-lines) and the edge-orientation it imposes. There are $|E| - |V| + 1 = 5$ backedges.

(c) A chain decomposition $C = \{C_1, \ldots, C_5\}$ of G. The chains C_2 and C_3 are paths; all other chains are cycles. The edge v_6v_5 is not contained in any chain and therefore a bridge. Since $\delta(G) \geq 2$ and $C \setminus C_1$ contains a cycle, G contains a cut vertex (in fact, v_5 and v_6 are cut vertices).

Figure 1: A graph G, its DFS-tree and a chain decomposition of G.

Theorem 2. *Let C be a chain decomposition of a simple 2-edge-connected graph G. Then G is 2-connected if and only if C_1 is the only cycle in C.*

The properties in Theorems 1 and 2 can be efficiently tested: In order to check whether C partitions E, we mark every edge that is traversed by the chain decomposition. If G is 2-edge-connected, every C_i can be checked on forming a cycle by comparing its first and last vertex on identity. For pseudo-code, see Algorithm 1.

Algorithm 1 Check(graph G) $\quad\triangleright$ G is simple and connected with $|V| \geq 3$

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Compute a DFS-tree T of G
Compute a chain decomposition C; mark every visited edge
if G contains an unvisited edge then
    output "NOT 2-EDGE-CONNECTED"
else if there is a cycle in C different from C_1 then
    output "2-EDGE-CONNECTED BUT NOT 2-CONNECTED"
else
    output "2-CONNECTED"
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We state a variant of Theorem 2, which does not rely on edge-connectivity. Its proof is very similar to the one of Theorem 2.

Theorem 3. *Let C be a chain decomposition of a simple connected graph G. Then G is 2-connected if and only if $\delta(G) \geq 2$ and C_1 is the only cycle in C.*

3 Proofs

It remains to give the proofs of Theorems 1 and 2. For a tree T rooted at r and a vertex x in T, let $T(x)$ be the subtree of T that consists of x and all descendants of x (independent of the edge orientations of T). Theorem 1 is immediately implied by the following lemma.

Lemma 4. *Let C be a chain decomposition of a simple connected graph G. An edge e in G is a bridge if and only if e is not contained in any chain in C.*

Proof. Let e be a bridge and assume to the contrary that e is contained in a chain whose first edge (i. e., whose backedge) is b. The bridge e is not contained in any cycle of G, as otherwise the end points of e would still be connected when deleting e, contradicting that e is a bridge. This contradicts the fact that e is contained in the cycle $C(b)$.

Let e be an edge that is not contained in any chain in C. Let T be the DFS-tree that was used for computing C and let x be the end point of e that is farthest away from the root r of T, in particular $x \neq r$. Then e is a tree-edge, as otherwise e would be contained in a chain. For the same reason, there is no backedge with exactly one end point in $T(x)$. Deleting e therefore disconnects all vertices in $T(x)$ from r. Hence, e is a bridge. □

The following lemma implies Theorem 2, as every 2-edge-connected graph has minimum degree 2.

Lemma 5. *Let C be a chain decomposition of a simple connected graph G with $\delta(G) \geq 2$. A vertex v in G is a cut vertex if and only if v is incident to a bridge or v is the first vertex of a cycle in $C \setminus C_1$.*

Proof. Let v be a cut vertex in G; we may assume that v is not incident to a bridge. Let X and Y be connected components of $G \setminus v$. Then X and Y have to contain at least two neighbors of v in G, respectively. Let X^{+v} and Y^{+v} denote the subgraphs of G that are induced by $X \cup v$ and $Y \cup v$, respectively. Both X^{+v} and Y^{+v} contain a cycle through v, as both X and Y are connected. It follows that C_1 exists; assume w. l. o. g. that $C_1 \notin X^{+v}$. Then there is at least one backedge in X^{+v} that starts at v. When the first such backedge is traversed in the chain decomposition, every vertex in X is still unvisited. The traversal therefore closes a cycle that starts at v and is different from C_1, as $C_1 \notin X^{+v}$.

If v is incident to a bridge, $\delta(G) \geq 2$ implies that v is a cut vertex. Let v be the first vertex of a cycle $C_i \neq C_1$ in C. If v is the root r of the DFS-tree T that was used for computing C, both cycles C_1 and C_i end at v. Thus, v has at least two children in T and v must be a cut vertex. Otherwise $v \neq r$; let wv be the last edge in C_i. Then no backedge starts at a vertex with smaller DFI than v and ends at a vertex in $T(w)$, as otherwise vw would not be contained in C_i. Thus, deleting v separates r from all vertices in $T(w)$ and v is a cut vertex. □

4 Extensions

We state how some additional structures can be computed from a chain decomposition. Note that Lemmas 4 and 5 can be used to compute all bridges and cut vertices of G in linear time. Having these, the 2-*connected components* (i. e., the maximal 2-connected subgraphs) of G and the 2-edge-connected components (i. e., the maximal 2-edge-connected subgraphs) of G can be easily obtained. This gives the so-called *block-cut tree* [7] of G, which represents the dependence of the 2-connected components and cut vertices in G in a tree (it gives also the corresponding tree representing the 2-edge-connected components and bridges of G).

Additionally, the set of chains C computed by our algorithm is an ear decomposition if G is 2-edge-connected and an open ear decomposition if G is 2-connected. Note that C is not an arbitrary (open) ear decomposition, as it depends on the DFS-tree. The existence of these ear decompositions characterize the 2-(edge-)connectivity of a graph [8, 16]; Brandes [2] gives a simple linear-time transformation that computes a bipolar orientation and a s-t-numbering from such an open ear decomposition.

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A Appendix

We omitted the proof of Theorem 3, as it is very similar to the one of Theorem 2. For completeness, we give the proof here.

Proof of Theorem 3: Let T be the DFS-tree that was used for computing C and let r be its root. First, let G be 2-connected. Clearly, this implies $\delta(G) \geq 2$. Moreover, r has exactly one child, as otherwise r would be a cut vertex. Thus, r is incident to a backedge, which implies that C_1 exists and is a cycle that starts at r. Assume to the contrary that v is the first vertex of a cycle $C_i \neq C_1$. If $v = r$, both cycles C_1 and C_i end at v. Thus, v has at least two children in T. This implies that v is a cut vertex, which contradicts the 2-connectivity of G. If $v \neq r$, let wv be the last edge in C_i. There is no backedge that starts at a vertex with smaller DFI than v and ends at a vertex in $T(w)$, as otherwise wv would be contained in a chain C_j with $j < i$. Thus, deleting v disconnects r from all vertices in $T(w)$, which contradicts the 2-connectivity of G.

Let $\delta(G) \geq 2$ and C_1 be the only cycle in C and assume to the contrary that G is not 2-connected. Then G contains a cut vertex v, as $\delta(G) \geq 2$ implies $|V| \geq 3$. Clearly, C_1 can intersect with at most one connected component of $G \setminus v$. Let X be a connected component of $G \setminus v$ that does not contain any vertex of C_1. Let X^{+v} be the subgraph of G that is induced by $X \cup v$. There must be a cycle in X^{+v}, as otherwise X^{+v} would be a tree, whose leafs would contradict $\delta(G) \geq 2$. Hence, X^{+v} contains at least one backedge; let b be the first backedge in X^{+v} that is traversed by the chain decomposition. As $r \notin X$, all vertices in $D(b)$ except the start point w of b have greater DFIs than w. Thus, the traversal on b computes a chain $C_i \subset X^{+v}$ that is a cycle and that is distinct from C_1, as X does not contain any vertex of C_1. $\square$