# Strongly Stable Matchings in Time $O(nm)$.

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**Abstract**

An instance of the stable marriage problem is an undirected bipartite graph $G = (X \dot\cup W, E)$ with linearly ordered adjacency lists; ties are allowed. A matching $M$ is a set of edges no two of which share an endpoint. An edge $e = (a,b) \in E \setminus M$ is a *blocking edge* for $M$ if $a$ is either unmatched or strictly prefers $b$ to his/her partner in $M$, and $b$ either is unmatched or strictly prefers $a$ to his/her partner in $M$ or is indifferent between them. A matching is strongly stable if there is no blocking edge with respect to it. We give an $O(nm)$ algorithm for computing strongly stable matchings. The previously best algorithm had running time $O(m^2)$.

## 1 Introduction

An instance of the *stable marriage problem* is an undirected bipartite graph $G = (X \dot\cup W, E)$ with linearly ordered adjacency lists; ties are allowed. As is customary, we call the nodes of the graph men and women, respectively[1]. If the edges $(a,b)$ and $(a,b')$ are tied, we say that $a$ is indifferent between $b$ and $b'$ and if the edge $(a,b)$ strictly precedes $(a,b')$, we say that $a$ prefers $b$ to $b'$. We use $n$ for the number of nodes and $m$ for the number of edges. A stable matching problem is called *complete* if there is an equal number of men and women and $G$ is the complete bipartite graph; thus $m = (n/2)^2$.

A matching $M$ is a set of edges no two of which share an endpoint. If $(a,b) \in M$ we call $b$ the partner of $a$ and $a$ the partner of $b$. An edge $e = (a,b) \in E \setminus M$ is a *blocking edge* for $M$ if $a$ is either unmatched or strictly prefers $b$ to his/her partner in $M$, and $b$ either is unmatched or strictly prefers $a$ to his/her partner in $M$ or is indifferent between them. In other words, if $a$ would prefer to match up with $b$ and $b$ would not object to the change. A matching is *strongly stable* if there is no blocking edge with respect to it. Figure 1 shows an instance of the stable marriage problem with no strongly stable solution.

Irving [Irv94] gave an $O(n^4)$ algorithm for computing strongly stable matchings in complete instances. In Manlove [Man99] and [IMS03] the algorithm was extended to incomplete bipartite graphs; the extended algorithm[2] has running time $O(m^2)$. In this paper we improve the running time to $O(nm)$. Our new algorithm can be viewed as a specialization of Irving's algorithm, i.e., every run of our algorithm is a run of his, but not vice versa.

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[1] We use $x$, $x'$, $x''$ to denote men and $w$, $w'$, $w''$ to denote women, and $a$, $a'$, $b$, $b'$ to denote persons of either sex.

[2] The extended algorithm actually solves the more general hospital-residents-problem. We come back to this point in Section 3.

$$\begin{array}{llcll} x_1: & w_1, w_2 & \qquad & w_1: & x_2, x_1 \\ x_2: & \{w_1, w_2\} & & w_2: & x_2, x_1 \end{array}$$

Figure 1: Both woman prefer $x_2$ to $x_1$. The men $x_1$ prefers $w_1$ to $w_2$ and $x_2$ is indifferent between the women. The instance is complete and hence any strongly stable matching must match all men and all woman. The matching $\{(x_1,w_1),(x_2,w_2)\}$ is not strongly stable since $w_1$ prefers $x_2$ to $x_1$ and $x_2$ is indifferent between $w_1$ and $w_2$. The matching $\{(x_1,w_2),(x_2,w_1)\}$ is not strongly stable since $w_2$ prefers $x_2$ to $x_1$ and $x_2$ is indifferent between $w_1$ and $w_2$.

Set phase number $i = 1$, $E' = E$ and $E_c = \emptyset$.
$M = \emptyset$
**repeat**
  **while** $\exists$ a free man $x$ **do**
    move all top choice edges $e = (x,w)$ of $x$ in $E'$ to $E_c$ and delete all edges $(x',w)$ from $E' \cup E_c$ which $w$ ranks strictly after $e$.
  **end while**
  Let $\mathcal{E}_i$ be the edges moved to $E_c$.
  **for all** free men $x$ w.r.t $M$ **do**
    **if** an alternating path from $x$ to a free woman exists **then**
      let $w$ be a free woman [*of maximal level*] reachable from $x$ by an alternating path and let $p$ be an alternating path from $x$ to $w$
      $M = M \oplus p$
    **else**
      let $Z$ be the set of men reachable from $x$ by alternating paths and let $N(Z)$ be the women adjacent to them in $E_c$;
      **for all** women $w \in N(Z)$ **do**
        delete all lowest ranked edges in $E_c \cup E'$ incident to $w$;
      **end for**
    **end if**
  **end for**
  $i = i + 1$
**until** (all free men have run out of proposals)
declare $M$ strongly stable if $M$ matches all non-isolated nodes of $G_c$, if no edge of $G$ connects two free nodes, and if every woman which was ever matched during the execution is also matched in $M$. Otherwise, there is no strongly stable matching

Figure 2: Two algorithms for strongly stable marriage. The algorithms differ by the phrase [*of maximal level*]. Without the phrase, the algorithm may augment the current matching along any augmenting path and the running time is $O(m^2)$. With the phrase, an augmenting path to a woman of maximal level (see Section 2.2) must be used. The running time improves to $O(nm)$.

## 2 The Algorithm

We review a variant of Irving's algorithm and then describe our modification. Figure 2 contains a concise write-up.

## 2.1 Irving's algorithm

We review a variant of Irving's algorithm for strongly stable matchings. The algorithm proceeds in phases and maintains two graphs $G'$ and $G_c$; $G'$ and $G_c$ are subgraphs of $G$. $G_c$ is the current graph in which we compute maximal matchings and $G'$ is the graph of edges not considered relevant yet. In each phase, a certain subset of the edges of $G'$ is moved to $G_c$. Also edges get deleted from $G'$ and $G_c$. We use $\mathcal{E}_i$ to denote the edges moved in phase $i$ and $\mathcal{E}_{\le i}$ to denote the edges moved in the first $i$ phases. Initially, we have $G' = G$ and $E_c = \emptyset$.

At the beginning of phase $i$, $E_c \subseteq \mathcal{E}_{<i}$ and we have a maximum matching $M$ in $G_c$. Also, if a man is free with respect to $M$ then no edges of $E_c$ are incident to it. Let $\mathcal{E}_i$ consist of the top choices[3] in $E'$ of each free man. We say, that every free man proposes to all women currently at the top of his list. When a woman receives a proposal from a man $x$, she deletes all strict successors of $x$ from $E'$ and $E_c$. This may also remove edges in $M$.

Observe, that the rules for adding and deleting edges guarantee that if $(a,b) \in E_c$ and $(a,b') \in E_c$ then $a$ is indifferent between $b$ and $b'$. For a free man $x$ all his top choices are moved to $E_c$ and hence edges in $E'$ go to strictly inferior woman. A woman keeps only the best proposals made to here and hence edges in $E'$ go either to strictly superior men or to men tied with her choices in $E_c$.

Next we extend $M$ to a maximum matching in $E_c$. During this process further edges may be deleted. We iterate over the free men in arbitrary order. Let $x$ be any free man. If there is an augmenting path starting at $x$, we use it to increase the cardinality of the matching. Otherwise, let $Z$ be the set of men reachable from $x$ by alternating path and let $N(Z)$ be the set of women adjacent to $Z$ in $E_c$. For each woman $w \in N(Z)$ we delete[4] all lowest ranked edges in $E_c \cup E'$ incident to it. This is at least one edge $(x,w) \in E_c$ and zero or more edges $(x',w) \in E'$.

At the end of the phase, we have a maximum matching in $E_c$. Also, every free man is isolated in $G_c$ since the edges incident to it where removed when we searched for an aumgenting path starting from it.

The algorithm terminates when all free man have run out of proposals. Let $M$ be the final matching and let $G_c$ be the final graph. Then $M$ is a maximum matching in $G_c$ and all free man are isolated in $G_c$ and $G'$. $M$ is a stable matching in $G$ if (1) no non-isolated node of $G_c$ is free with respect to $M$, (2) no edge of $G$ connects two free persons and and (3) no woman free with respect to $M$ was ever matched during execution of the algorithm[5].

The following Lemmas were essentially shown in [Irv94, Man99, IMS03]. We include them and their proofs for the sake of completeness.

**Lemma 1** *If an edge $e = (x,w)$ is deleted during the execution, it cannot block a stable matching consisting only of edges never deleted.*

**Proof:** Assume otherwise and let $M$ be a stable matching consisting only of edges never deleted. In $M$, $w$ is either unmatched or paired to a man it prefers to $x$. In the latter case $e$ is non-blocking. In the former case, $M$ is not stable since $x$ is either unmatched or matched to a woman which is either tied with $w$ or after $w$ in his preference list. ∎

[3] Recall that $E' \subseteq E$ and that adjacency lists are linearly ordered with ties allowed. The top choices for a man $x$ are the set of women tied for first place.

[4] It is here, where we slightly deviate from Irving's algorithm. We delete edges whenever we identify a free man which cannot be matched. Irving first computes a maximum matching in $E_c$ and then deletes edges.

[5] For complete instances, it is particularly easy to decide whether the final matching $M$ is stable. $M$ is stable if it is a perfect matching in $G$. Clearly, if $M$ is a perfect matching in $G$ the three conditions hold. On the other hand, if $M$ is not perfect, there is a free man $x$ and a free woman $w$. Since the instance is complete, $(x,w) \in E$. This contradicts (2).

**Lemma 2** *If the algorithm returns a matching, the matching is strongly stable.*

**Proof:** Assume otherwise, i.e., the algorithm returns a matching $M$ and yet a blocking pair $e = (x, w)$ exists. Then $e$ was never deleted during the execution by Lemma 1 and hence $e \in E_c \cup E'$ when the algorithm terminates. By the termination condition and since $M$ is declared strongly stable, $x$ must be matched in $M$. Since $e$ blocks $M$, $x$ cannot prefer his partner in $M$ to $w$, and since $e$ is not deleted, $x$ cannot prefer $w$ to his partner in $M$. Thus $x$ is indifferent between between $w$ and his partner in $M$. When $x$ proposed to his partner in $M$, he also proposed to $w$. At this time $(x, w)$ was added to $G_c$. Thus $w$ is a non-isolated node in the final current graph and hence matched in $M$ by condition (3). The partner of $w$ is tied with $x$ in $w$'s preference list and hence $x$ is non-blocking. ∎

Let us call an edge strongly stable, if it belongs to some strongly stable matching.

**Lemma 3** *No strongly stable edge is ever deleted during execution of the algorithm.*

**Proof:** Assume otherwise, and let $e = (x, w)$ be the first strongly stable edge deleted by the algorithm. Let $M$ be a strongly stable matching containing $e$. We distinguish cases according to why $e$ was deleted.

Assume first the $e$ is deleted because $w$ receives a better proposal, say from $x'$. Then $w$ strictly prefers $x'$ to $x$. When $x'$ proposes to $w$, all edges $(x', w')$ with $w'$ strictly preceding $w$ in $x'$'s preference list have been discarded already and hence none of them can be stable. We conclude that the partner of $x'$ in $M$ is either tied with $w$ or after $w$ in the preference list of $x'$ and $x'$ is before $x$ is $w$'s preference list. Thus $(x', w)$ blocks $M$, a contradiction.

Assume next that at some point of the execution we have a free man $x_0$ from which no augmenting path starts. Let $E_c$ be the edge set of the current graph when we search for augmenting paths from $x_0$ and let $M_c$ be the current matching. Also, let $E'$ be the edges not considered yet. Let $Z$ be the set of men reachable from $x_0$ by alternating paths and let $N(Z)$ be the women adjacent to them in $E_c$. Then $|Z| = |N(Z)| + 1$ and every woman in $N(Z)$ is matched with a man in $Z$ by $M_c$. Also $w \in N(Z)$ and $e$ is a lowest ranked edge incident to $w$. Let $\widetilde{E}$ be the set of lowest ranked edges incident to women in $N(Z)$, i.e., $\widetilde{E}$ is the set of edges which the algorithm removes after discovering the fact that no augmenting path starting at $x_0$ exists.

Consider $M \cap \widetilde{E}$, let $U'$ be their female endpoints and let $Z'$ be their male endpoints. Then $w \in U'$ and hence $U' \neq \emptyset$. Also, $M$ matches men in $Z'$ with women in $U'$. Thus $|Z'| \le |U'| \le |N(Z)| < |Z|$.

We claim the existence of an edge $e' = (x', w') \in E_c$ with $x' \in Z \setminus Z'$ and $w' \in U'$. In $M$, $x'$ is either unmatched or matched to a woman strictly after $w'$ in his preference list and $w'$ is matched to a man tied with $x'$ in her preference list. Thus $(x', w')$ blocks $M$, a contradiction.

It remains to prove the existence of an edge $e' \in E_c \cap (Z \setminus Z' \times U')$. Assume that $M_c$ contains no such edge. Then $M_c$ pairs the women in $U'$ with men in $Z'$ and hence $|Z'| \ge |U'|$. Thus $|Z'| = |U'|$ and $M_c$ pairs the men in $Z'$ with the women in $U'$. We conclude $x_0 \in Z \setminus Z'$. Consider the alternating path from $x_0$ to $w$ and let $e' = (a, b)$ be the first edge on this path with $b \in U' \cup Z'$. If $b \in Z'$, $e'$ is a matching edge and hence $a \in U'$. Thus $b \in U'$ and $a \in Z \setminus Z'$ and we have shown the existence of the desired edge. ∎

**Lemma 4** *Assume a strongly stable matching exists. Then the algorithm returns a strongly stable matching. Also all strongly stable matchings have the same cardinality and match the same set of men and women.*

**Proof:** Let $M_0$ be a strongly stable matching. By Lemma 3 no edge of $M_0$ is ever deleted and hence $M_0 \subseteq E_c \cup E'$ at termination. Let $M$ be the final matching computed by the algorithm. We show first that $|M| = |M_0|$ and that both matchings match the same set of men and women.

All men free in $M$ are isolated in $E_c \cup E'$ and hence are free in $M_0$. Thus every man matched by $M_0$ is also matched by $M$ and hence $|M_0| \leq |M|$. Assume that there is woman $w$ which is non-isolated in $G_c$ and free with respect to $M_0$. Let $(x,w) \in E_c$ be an edge incident to $w$. Then $(x,w)$ blocks $M_0$ since $x$ cannot strictly prefer its partner in $M_0$ to $w$. Thus $M_0$ matches all non-isolated woman of $G_c$ and $M$ certainly cannot match more women. Thus $|M| \leq |M_0|$. We conclude that both matchings have the same cardinality and match exactly the non-isolated nodes of $G_c$. This proves the second claim.

Assume next that the algorithm declares $M$ non-stable. By the above, $M$ matches all non-isolated nodes of $G_c$. Also there can be no edge of $G$ connecting a free man with a free woman, because such an edge would also block $M_0$. The last possibility remaining is the existence of a woman $w$ which is free in $M$, but was matched at some point of the execution. Then $w$ is also free in $M_0$. Let $e = (x,w)$ be the last matching edge incident to $w$ during the execution. Since $e$ was deleted, no edge in $E_c \cup E$ incident to $x$ is strictly before $e$ in the preference list of $x$. Thus $e$ blocks $M_0$. ∎

The algorithm runs in $O(m^2)$ time since there are at most $m$ phases, since the cost of a phase is $O(m \cdot (1 + \text{number of succesful augmenting path computations}))$ and since the number of successful augmenting path computations is at most $m$. The latter claim follows from the fact that a matched man becomes free only if the matching edge incident to it is deleted.

## 2.2 The New Algorithm

We show how to modify the algorithm so that it runs in time $O(nm)$. Our method maintains *level-maximal* matchings and uses level-maximal augmenting paths.

**Definition 1** *Let $\mathcal{E}_i$ be the edges added to $G_c$ in phase $i$ and define the level $l(e)$ of an edge to be the phase when this edge was first added to $G_c$. Edges never added to $G_c$ have no level assigned to them.*

So, the set of edges ever added to $G_c$ consists of the disjoint union $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \ldots \mathcal{E}_r$, where $r$ is the total number of phases in the algorithm. Note that $r$ can be as large as $m$.

**Definition 2** *Define the level $l(v)$ of a vertex $v$ to be the minimum level of the edges in $G_c$ incident to $v$. The level of an isolated vertex is undefined.*

**Definition 3** *The level $l(M)$ of a matching $M$ is the sum of the levels of the matched women. A matching $M$ is level-maximal if $l(M) \geq l(M')$ for any matching $M'$ which matches the same males.*

**Lemma 5** *For a male all incident edges in $G_c$ have the same level. All women adjacent to a male of level $i$ have level at most $i$. When a women looses an incident edge in $E_c$ she looses all her incident edges in $E_c$.*

**Proof:** Obvious ∎

**Lemma 6** *A matching $M$ is level-maximal iff there is no alternating path from a free female to a female of lower level.*

**Proof:** Observe that the endpoint of the path is a matched female. Augmentation increases the level of the matching and does not change the set of matched males.

For the converse, assume that $M$ is not level-maximal. Let $M'$ be level-maximal and matching the same males. Then $M \oplus M'$ is a set of alternating paths and cycles. Augmenting a cycle does not change the level sum. Thus there must be at least one path whose augmentation to $M$ increases the level sum. Since the degree of every male in $M \oplus M'$ is either zero or two, the path must connect two women, one free in $M$ and one free in $M'$. ∎

**Lemma 7** *If $M$ is level-maximal, $x$ is a free man with respect to $M$, $w$ is a woman of maximal level reachable from $x$ by an augmenting path and $p$ is an augmenting path from $x$ to $w$, then $N = M \oplus p$ is level-maximal.*

**Proof:** Assume that $N$ is not level-maximal. Then there is an alternating path $p'$ from a free woman $w'$ to a matched woman $w''$ (all with respect to $N$) with $l(w'') < l(w')$. Also, $w'$ is also free with respect to $M$ and $w''$ is either matched with respect to $M$ or equal to $w$. Let $N' = N \oplus p'$. Then

$$N' = N \oplus p' = (M \oplus p) \oplus p' = M \oplus (p \oplus p') \quad \text{and hence} \quad p \oplus p' = M \oplus N' \; .$$

$P = p \oplus p' = M \oplus N'$ decomposes into alternating paths and cycles with respect to $M$. In $P$ the degree of $x$ is one, the degree of all other males is even, the degree of $w'$ is one, the degrees of $w$ and $w''$ are odd if these nodes are distinct, and the degree of all other woman is even. One of the paths must be augmenting with respect to $M$ and hence must connect $x$ with either $w$ or $w'$.

Assume first that $P$ contains an augmenting path from $x$ to $w'$. Then $l(w') \le l(w)$ since $w$ is a level-maximal woman reachable by an augmenting path from $x$. Thus $l(w'') < l(w') \le l(w)$ and hence $w \neq w''$. If $w \neq w''$, both women have odd degree in $P$ and hence $P$ contains an alternating path connecting them. Also, $w$ is free in $M$ and $w''$ is matched in $M$ and hence $l(w'') < l(w)$ contradicts the level-maximality of $M$.

Assume next that $P$ contains a augmenting path connecting $x$ and $w$. Then $w \neq w''$ and hence $P$ also contains a path connecting $w'$ and $w''$. Thus $M$ is not level-maximal. ∎

**Lemma 8** *$M$ is a level-maximal matching at all times of the execution.*

**Proof:** We use induction on time. Initially, $M$ is empty and therefore level-maximal. For the induction step assume that $M$ is level-maximal at the beginning of phase $i$.

First, the free man propose to the women at the top of their lists. This introduces the edge set $\mathcal{E}_i$. The rank of non-isolated women does not change, the rank of women previously isolated and not isolated anymore is set to $i$. $M$ is still level-maximal. Assume otherwise, then there must be an alternating path from a free woman to a woman of lower rank. This path must use one of the new edges. The new edges are incident to free men, a contradiction.

Every women keeps only her best proposals. For a particular woman $w$ this has one of two effects: either she does not drop any incident edge or she keeps only edges in $\mathcal{E}_i$ (not necessarily all of them). The matching $M$ may be reduced in size. Let us use $M'$ to denote the resulting matching. We claim that it is level-maximal. Assume otherwise, then there must an alternating path $p$ from a free women to a women of lower level. It cannot use any of the new edges since new edges are incident to free man. Thus $p$ can use only old edges. Also $p$ cannot start at a woman of rank $i$ since only new edges are incident to such a woman. Thus $p$ starts at a woman of rank less than $i$ and hence the woman is free with respect to $M$. Since $M' \subseteq M$, $p$ is alternating with respect to $M$, a contradiction to the level-maximality of $M$.

Next, we consider the free men in turn and search for augmenting paths. Let $x$ be a free man.

If no augmenting path starting at $x$ exists, let $Z$ be the set of males reachable by alternating paths from $x$ and let $N(Z)$ be their neighbors. Then $|Z| > |N(Z)|$. We delete all lowest rank edges incident to the women in $N(Z)$. This may decrease the size of the matching. The matching clearly stays level-maximal.

If an augmenting path exists, let $p$ be an augmenting path to a woman of maximal rank. We use $p$ to increase the cardinality of the matching. By Lemma 7, the resulting matching is level-maximal. ∎

### 2.3 The Search for Augmenting Paths and the Analysis

We come to the implementation of the search for augmenting paths and the analysis.

Let $x$ be a free male. We need to determine a maximal rank free woman $w$ reachable from $x$ and an augmenting path from $x$ to $w$. Let $p$ be such an augmenting path. Then all women on this path have level at least $i = l(w)$ by Lemma 6. Also $i \leq l(x)$ since all edges incident to $x$ have level $l(x)$ and hence the women adjacent to $x$ have level at most $l(x)$. We organize the search in rounds $l(x)$, $l(x)-1$, $l(x)-2$, .... In round $j$, we explore all augmenting paths starting in $x$ and exploring only edges out of vertices of level $j$ or larger. We stop in round $j$ when a free women of level $j$ is reached by the search or if the Hungarian tree rooted at $x$ has reached its full size. In the former case, $j$ is the maximal level of a woman reachable from $x$. In the latter case, no free woman is reachable from $x$. If the search has not stopped yet, the frontier of the search consists of woman of level less than $j$. In the next round, we continue the search from all women of level $j-1$ in the frontier.

In order to find these women, we maintain an array $A$ of buckets (= linear lists) which implements a simple priority queue. All buckets are initially empty. At the beginning of round $j$, bucket $B_l$, $l \leq j$ contains the women of level $l$ in the frontier. We also keep an (unsorted) list of the non-empty buckets and the total number of woman contained in the buckets. We initialize the bucket structure by putting the neighbors of $x$ into the appropriate bucket and setting $j$ to $l(x)$. In round $j$, we continue the search from the women in bucket $j$. If the bucket is empty and the number of unexplored women is positive, we decrease $j$ by one. If the bucket is empty and the number of unexplored women is zero, we stop. There is no augmenting path starting at $x$ (failure). If the bucket is non-empty, let $w$ be a woman in the bucket. We remove $w$ from the bucket. If $w$ is free, we stop (success): $w$ is the highest ranked woman reachable from $x$. If $w$ is matched, we explore alternating paths from $w$ (starting with matched edges) until a woman of rank less than $j$ is reached. These women are then added to their appropriate buckets. When the search stops, we empty all buckets using the list of non-empty buckets.

Let $j(x)$ be the minimal bucket index from which we remove a woman. In the case of failure this is the minimal level of a woman reachable from $x$ and in the case of success this is the maximum level of a woman reachable from $x$.

The time for the search from $x$ is proportional to the number $k$ of edges explored in the search plus $l(x) - j(x) + 1$. We charge this cost as follows:

In the case of failure we charge one unit each to each edge deleted (this accounts for $k$) and we charge $l(x) - j(x) + 1$ to the minimum level woman $w$ reachable from $x$. The first kind of charges add up to $m$ since every edge is deleted at most once. The second kind of charge is less than the difference of the current level of $w$ and the next level of $w$. Thus for a single woman the total charge of the second kind is bounded by $m$. We conclude that the total cost of unsuccessful searches is $O(nm)$.

In the case of success, we charge both costs to $w$. Observe that all edges explored have level at least $l(w)$ $(= j(x))$ and at most $i$ (= the phase number) and that the level of $w$ jumps to at least $i+1$ if it ever becomes free again. Thus every edge can be assigned at most once to $w$. Also $l(x) - j(x) + 1$ is bounded by the difference between the current level of $w$ and the next level of $w$. Thus the total charge to $w$ is bounded by $m$. The total cost of successful searches is therefore bounded by $O(nm)$.

**Theorem 1** *Strongly stable matchings can be computed in time $O(nm)$.*

# 3 Conclusion

We have given an $O(nm)$ algorithm to compute strongly stable matchings. The algorithms is a variant of a previous algorithm with running time $O(m^2)$. We close with two open problems.

Firstly, it is conceivable that Irving's algorithm runs in time $O(nm)$ and that only the analysis is not tight enough to show this. Improve the analysis or give a family of instances which force the algorithm to spend time $\Omega(m^2)$.

Secondly, the algorithm in [IMS03] solves a more general problem, the so-called hospital-residents-problem. This problem is also given by a bipartite graph in which the adjacency lists are linearly ordered with ties allowed. Also, every hospital $h$ has an integral capacity $c_h$. In the stable matching problem considered in this paper all capacities are equal to one. The goal is to determine a strongly stable set $M$ of edges such that at most one edge is incident to any resident and at most $c_h$ edges are incident to any hospital. Again a set $M$ is strongly stable if there is no blocking pair and the notion of blocking pair is the natural generalization of the corresponding notion in the stable matching problem. The algorithm in [IMS03] has running time $O(m^2)$. Can our approach be generalized to the hospital-residents-problem?

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