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FURTHEST SITE ABSTRACT VORONOI DIAGRAMS

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ABSTRACT

Abstract Voronoi diagrams were introduced by R. Klein as a unifying approach to Voronoi diagrams. In this paper we study furthest site abstract Voronoi diagrams and give a unified mathematical and algorithmic treatment for them. In particular, we show that furthest site abstract Voronoi diagrams are trees, have linear size, and that, given a set of n sites, the furthest site abstract Voronoi diagram can be computed by a randomized algorithm in expected time $O(n \log n)$.

Keywords: furthest site Voronoi diagram, randomized incremental construction, topological properties

1. Introduction

Voronoi diagrams are among the structures most frequently investigated in Computational Geometry. Because of their wide range of applications, cf. Leven and Sharir[11] or Aurenhammer[1], many different kinds of diagrams have been considered. Different kinds of diagrams are obtained by varying the shape of the sites, e.g., points, line segments, circles, and the distance function. A unifying approach to Voronoi diagrams has been proposed recently by Klein[9], cf. Ref. [7] for a related

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approach. Klein's approach is based on the notion of bisecting curves instead of the concept of distance. For each pair p and q of sites the existence of a bisector dividing the plane into a p-region and a q-region is postulated. The Voronoi region of site p is then obtained by intersecting all p-regions generated by the sites different from p. The abstract Voronoi diagram is formed by the boundaries induced by the Voronoi regions. Klein investigated the topological properties of abstract Voronoi diagrams and showed that two natural assumptions, namely that Voronoi regions are connected and that every point of the plane belongs to a Voronoi region, suffice to derive many properties of Voronoi diagrams. We review some of these properties in Section 2. Abstract Voronoi diagrams encompass a large number of specific diagrams, e.g., diagrams for point, disjoint line segment, and disjoint circle sites under any L_p-norm $(1 < p < \infty)$.

Abstract Voronoi diagrams separate topological and geometric properties of Voronoi diagrams by ignoring the latter. It is a pleasing fact that many important properties of Voronoi diagrams, e.g., the existance of $O(n \log n)$ construction algorithms, follow from topological reasoning alone. It is this separation of concerns that makes abstract Voronoi diagrams useful.

In his monograph[9] Klein gave an $O(n \log n)$ deterministic divide-and-conquer algorithm for a subclass of his abstract diagrams. Mehlhorn, Meiser, and O'Dúnlaing[12] obtained an $O(n \log n)$ randomized algorithm for all abstract diagrams provided a certain general position assumption is satisfied. Finally, Klein, Mehlhorn and Meiser[10] removed the general position assumption. The algorithms of Ref. [12,10] are both instances of Clarkson and Shor's randomized incremental constructions (Ref. [5]) in the history graph version introduced in Ref.[2].

In this paper we study furthest site abstract Voronoi diagrams and thus give a unified treatment of a large class of furthest site diagrams. See Figure 3 for an example of a nearest and a furthest site Voronoi diagram. In section 2, we derive the basic topological properties of the furthest site abstract Voronoi diagram. In particular, we show that the diagram is a tree, i.e., a connected planar graph with no bounded face, and that, although the Voronoi region of a site may consist of more than one face, the total number of faces is linear. In section 4, we give a randomized algorithm which constructs the furthest site abstract Voronoi diagram of n sites in time $O(n \log n)$. Previously, an $O(n \log n)$ algorithm has been known only for a few cases of furthest site Voronoi diagrams, cf. Ref. [3,6,13].

The main features of our algorithm are its generality, as it applies to all abstract Voronoi diagrams, its modularity, as only the basic operation depends on the particular kind of diagram, and its simplicity. The algorithm is an instance of Clarkson and Shor's randomized incremental constructions[5]. The connection to RICs is made in Section 3.

We use the following notation. For a subset X of $\mathbb{R}^2$ we use $\operatorname{cl} X$, $\operatorname{int} X$, and $\operatorname{bd} X$ to denote the closure, interior and boundary of X under the standard topology, respectively. We use $\subseteq$ and $\subset$ to denote set inclusion and proper set inclusion.

2. Abstract Voronoi Diagrams

2.1. Admissible and semi-admissible dominance systems

In this section we define nearest and furthest site abstract Voronoi diagrams, review basic properties of nearest site diagrams as established in Ref.[9,10], and show that furthest site diagrams share these properties.

Let $n \in \mathbb{N}$ and $S = \{1, \ldots, n-1\}$. A family $\mathcal{D} = \{D(p,q) \mid 1 \le p \ne q < n\}$ of subsets of the plane is called a *dominance system* over S, if the following conditions are satisfied for all p and q with $1 \le p \ne q < n$:

0. $D(p,q)$ is a non-empty open subset of the plane.
1. $D(p,q) \cap D(q,p) = \emptyset$ and $\operatorname{bd} D(p,q) = \operatorname{bd} D(q,p)$.
2. $J(p,q) = \operatorname{bd} D(p,q)$ is homeomorphic to the open interval $(0,1)$.

Clearly, $J(p,q) = J(q,p)$. We call the elements of S *sites*, the curve $J(p,q)$ the *bisector* of sites p and q and $D(p,q)$ the *region of dominance* of p over q. Following Klein[9], the abstract nearest site Voronoi diagram is now defined as follows:

Definition 1. Let $S = \{1, \ldots, n-1\}$ and let $<$ be a linear order on S. Let

$$\begin{aligned} R_<(p,q) &\overset{def}{=} \begin{cases} D(p,q) \cup J(p,q) & \text{if } p < q \\ D(p,q) & \text{if } p > q \end{cases}, \\ VR_<(p,S) &\overset{def}{=} \bigcap_{\substack{q \in S \\ q \ne p}} R_<(p,q), \\ V_<(S) &\overset{def}{=} \bigcup_{p \in S} \operatorname{bd} VR_<(p,S). \end{aligned}$$

We call $\operatorname{int} VR_<(p,S)$ the *nearest site Voronoi region* of p w.r.t. to S and $<$, $VR_<(p,S)$ the *extended nearest site Voronoi region* of p w.r.t. to S and $<$, and $V_<(S)$ the *nearest site Voronoi diagram* of S with respect to $<$.

A dominance system is called *admissible*, if it satisfies the following additional properties:

3. Any two bisecting curves intersect in only a finite number of connected components.
4. For all non-empty subsets S' of S and all orderings $<$ of S:
   (A) $VR_<(p,S')$ is path-connected and has a nonempty interior for every $p \in S'$,
   (B) $\mathbb{R}^2 = \bigcup_{p \in S'} VR_<(p,S')$.

A dominance system is called *semi-admissible* if it satisfies properties 3 and 4B (but not necessarily 4A). The following alternative characterization of property 4B is useful. For a point $x \in \mathbb{R}^2$ and distinct sites p and q let $p <_x q$ if and only if $x \in R_<(p,q)$.

Fact 1 (Ref. 9) *Let $\mathcal{D}$ be a dominance system. Then the following three conditions are equivalent:*

a. *$\mathcal{D}$ satisfies* 4B.

b. *$<_x$ is a linear order on the sites for all $x \in \mathbb{R}^2$.*

c. *$R_<(p,q) \cap R_<(q,r) \subseteq R_<(p,r)$ for all triples p, q, r of distinct sites.*

Moreover, $x \in VR_<(p,S)$ if and only if $p <_x q$ for all $q \in S \backslash \{p\}$.

Intuitively, $<_x$ orders the sites according to increasing distance from x and the extended nearest site Voronoi region of a site p consists of all points x having p as their closest site. It is now natural to also consider furthest site diagrams.

Definition 2. Let $<$ be a linear order on S and $p \in S$. Let

$$\begin{aligned} VR^*_<(p,S) &\stackrel{def}{=} \{\, x \in \mathbb{R}^2 \mid q <_x p \text{ for all } q \in S \setminus \{p\} \,\}, \\ V^*_<(S) &\stackrel{def}{=} \bigcup_{p \in S} \operatorname{bd} VR^*_<(p,S)\,. \end{aligned}$$

In analogy to Definition 1 we call $\operatorname{int} VR^*_<(p,S)$ the *furthest site Voronoi region* of p or *p-region* w.r.t. S and $<$, $VR^*_<(p,S)$ the *extended furthest site Voronoi region* of p w.r.t. S and $<$, and $V^*_<(S)$ the *furthest site Voronoi diagram* of S w.r.t. $<$.

As we will see next the furthest site abstract Voronoi diagram can also be obtained by "reversing" the dominance relations and the linear order $<$. To make this intuition more precise we define the dual of a dominance system and consider the reverse order of the linear order on S. The *dual* $\mathcal{D}^* = \{D^*(p,q) \mid 1 \le p \ne q < n\}$ of a dominance system $\mathcal{D}$ is defined by $D^*(p,q) = D(q,p)$ for all $p, q \in S$ with $p \ne q$. For a linear order $<$ on S the *reverse order* $<^*$ is obtained by $p <^* q \Leftrightarrow q < p$ for all $p, q \in S$ with $p \ne q$.

Lemma 1 *Let $\mathcal{D}$ be a semi-admissible dominance system and let $\mathcal{D}^*$ be its dual.*

a. *$\mathcal{D}^*$ is semi-admissible.*

b. *Let $<$ be a linear order on S and let $<^*$ be the reverse order of $<$. Then $VR^*_<(p,S)$ is equal to the extended nearest site Voronoi region of p w.r.t. S, the linear order $<^*$, and the dominance system $\mathcal{D}^*$.*

c. *$V^*_<(S)$ for $\mathcal{D}$ is equal to $V_{<^*}(S)$ for $\mathcal{D}^*$.*

Proof: Let $<$ be a linear order on S. For sites p and q, define $p <^*_x q$ if either $x \in D^*(p,q)$ or $x \in J(p,q)$ and $p <^* q$. Then $p <_x q$ if and only if $q <^*_x p$. Parts (a) and (b) now follow from Fact 1. Part (c) is an immediate consequence of part (b). □

Lemma 1 implies that the furthest site abstract Voronoi diagram can again be defined by means of a dominance system, namely the dual of the given dominance system. Thus the following results on nearest site abstract Voronoi diagrams from Ref.[9,10] are valid in both contexts of nearest and furthest site abstract Voronoi diagrams.

Fact 2 (Ref. 10) *Let $\mathcal{D}$ be semi-admissible and let $<_1$ and $<_2$ be two linear orders on S.*

a. $\operatorname{int} VR_{<_1}(p,S) = \operatorname{int} VR_{<_2}(p,S)$ *for all* $p \in S$,

b. $V_{<_1}(S) = V_{<_2}(S)$.

Fact 2 states that the Voronoi diagram and the Voronoi regions do not depend on the particular linear order imposed on S. Only for points in $V_<(S)$ does the linear order $<$ decide to which Voronoi region they belong. In the light of Fact 2 we write $V(S)$ instead of $V_<(S)$ and $\operatorname{int} VR(p,S)$ instead of $\operatorname{int} VR_<(p,S)$ from now on. We will also write $VR(p,S)$, resp. $R(p,q)$, instead of $VR_<(p,S)$, resp. $R_<(p,q)$, when the linear order $<$ is clear from the context. In this way, the omission of the symbol $<$ also applies to $V^*_<(S)$, $\operatorname{int} VR^*_<(p,S)$, and $VR^*_<(p,S)$ which are replaced by $V^*(S)$, $\operatorname{int} VR^*(p,S)$, and $VR^*(p,S)$, respectively.

The extended Voronoi region of a site p may also include points which are not contained in $\operatorname{cl}\operatorname{int} VR(p,S)$. Depending on the particular linear order imposed on S the extended Voronoi region can have long tentacles, i.e., $VR(p,S)$ may include points of the Voronoi diagram which do not belong to the boundary of the Voronoi region $\operatorname{int} VR(p,S)$. See Figure 1 for an illustration.

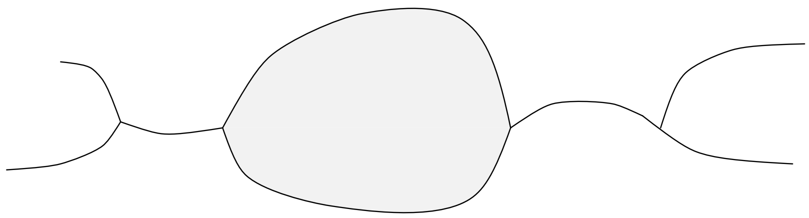


Fig. 1. A nearest site Voronoi region with tentacles

Definition 3. An *edge* e of $V(S)$ is a maximal connected subset of $V(S)$ such that every point $x \in e$ lies on $\operatorname{bd}\operatorname{int} VR(p,S)$ for exactly two sites p of S. The edge is said to *separate* the regions of these two sites. A *vertex* v of $V(S)$ is a point $x \in V(S)$ which lies on $\operatorname{bd}\operatorname{int} VR(p,S)$ for at least three sites p of S. A *face* of $V(S)$ is a maximal connected subset of $\operatorname{int} VR(p,S)$ for some $p \in S$.

In the case of a semi-admissible dominance system a Voronoi region $\operatorname{int} VR(p,S)$ may consist of zero or more faces. In the case of admissible systems each Voronoi region consists, by Property 4A, of exactly one face.

Fact 3 *Let $\mathcal{D}$ be semi-admissible and let $<$ be a linear order on S.*

a. *All but finitely many points of $V(S)$ belong to an edge of $V(S)$.*

b. *Every face of $V(S)$ is homeomorphic to an open disc and its boundary is a simple curve.*

c. *For each point $x \in V(S)$ there are arbitrarily small neighborhoods U of x having the following properties: $V(S) \cap \operatorname{bd} U$ is finite. Let $w_1, \ldots, w_h$ be the points in $V(S) \cap \operatorname{bd} U$ as encountered in a clockwise traversal of $\operatorname{bd} U$.*

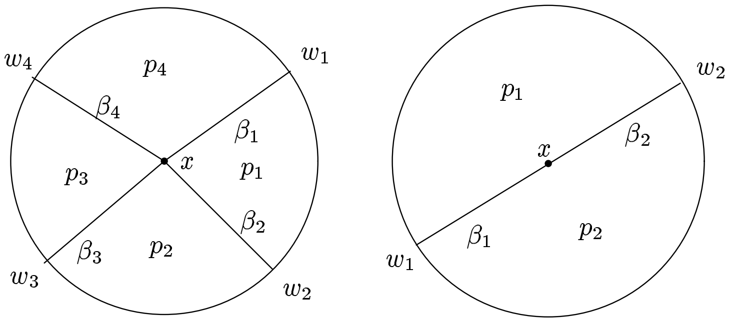


Fig. 2. Illustration of Fact 3

Then $h \geq 2$ *and* $V(S) \cap U$ *is the union of curve segments* $\beta_1, \ldots, \beta_h$ *where* β_i *connects* x *to* w_i *and the* β_i*'s are disjoint except at their common endpoint* x. *For each* i, $1 \leq i \leq h$, *there is a site* $p_i \in S$ *such that the open "piece of pie" bordered by* β_i, β_{i+1} *(read indices mod* h*) is contained in* $\operatorname{int} VR(p_i, S)$ *with* $p_i \neq p_{i+1}$ *for all* i. *Also, there is a site* $q_i \in S$ *such that* $\beta_i \setminus \{x\} \subseteq VR(q_i, S)$. *We have* $q_i \leq \min\{p_{i-1}, p_i\}$. *The point* x *belongs to* $VR(p, S)$, *where* $p = \min\{p_1, \ldots, p_h, q_1, \ldots, q_h\}$.

d. *If* $\mathcal{D}$ *is admissible then* $p_i \neq p_j$ *for all* $i \neq j$ *in part* (c) *and only site* p *can occur more than once among* $p_1, \ldots, p_h, q_1, \ldots, q_h$.

Figure 2 illustrates Fact 3. Fact 3 is a consequence of Theorem 2.3.5 of Ref.[9]. For admissible systems this was observed in Fact 1 and Fact 2 in Ref.[10]. For semi-admissible systems, the argument is as follows. Theorem 2.3.5 in Ref.[9] is proved for admissible systems; cf. pages 31 to 51 in Ref.[9]. However, Property 4A is used only twice in the proof of Theorem 2.3.5. The first use is in Lemma 2.3.3 to show that $V(S)$ contains no isolated points. This use of Property 4A is unnecessary, as we show next. Assume that there is an isolated point v in $V(S)$, i.e., $v \in VR(p, S)$ for some site p and there is a neighbourhood U of v such that $U \setminus \{v\} \subseteq VR(q, S)$ for some site q different from p. Then this situation arises even for $S' = \{p, q\}$, a contradiction to Properties 1 and 2. Thus Lemma 2.3.3 holds even for semi-admissible systems. The only other use is to prove the last sentence of Theorem 2.3.5. Thus all but the last sentence of that theorem already hold for semi-admissible dominance systems. This justifies parts (a) and (c). Part (b) follows from Lemma 2.2.4 of Ref.[9].

From now on, we proceed on the assumption that $\mathcal{D}$ is the primal admissible dominance system and that $\mathcal{D}^*$ is its dual. So $\mathcal{D}$ determines $V(S)$ and $\mathcal{D}^*$ determines $V^*(S)$. Note, however, that by Lemma 1 the dual system $\mathcal{D}^*$ is only guaranteed to be semi-admissible.

An example of a nearest and a furthest site Voronoi diagram is given in Figure 3.

2.2. Properties of the furthest site abstract Voronoi diagram

We characterize the furthest site Voronoi diagram. The furthest site diagram can be represented as an embedded planar graph in a natural way. Vertices, edges

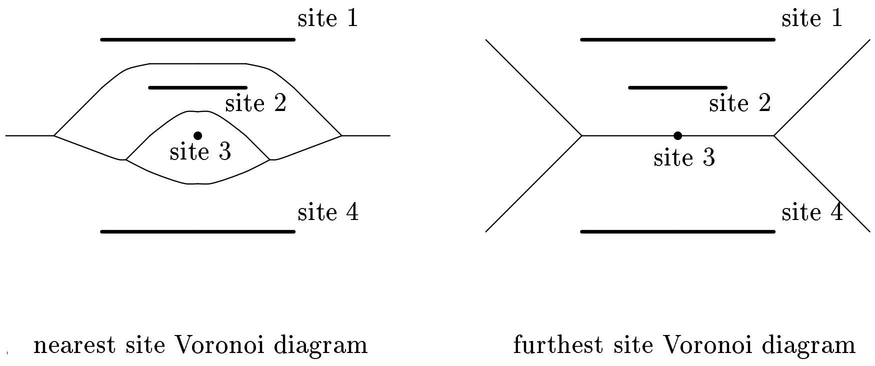


Fig. 3. The nearest and furthest site Voronoi diagram of three line segment sites (sites 1, 2, and 4) and one point site (site 3) under the Euclidean metric. In the furthest site diagram the region of site 2 is empty and the region of site 3 has two faces.

and faces of $V^*(S)$ are in one-to-one correspondence to the vertices, edges and faces of this graph so that we use $V^*(S)$ to denote this graph, too.

Lemma 2 *The furthest site Voronoi diagram $V^*(S)$ is a tree.*

Proof: We show first that $V^*(S)$ is connected and then that it has no bounded face.

Claim 1: $V^*(S)$ is a connected set.

Proof: We show that if $V^*(S)$ is not connected, there is a site p whose region in the primal diagram is empty. So let us assume that $V^*(S)$ consists of at least two connected components. Since the faces of $V^*(S)$ are homeomorphic to open discs there must be a simple curve C disjoint from $V^*(S)$ which splits the plane into two unbounded domains h_1 and h_2 both of which contain at least one component of $V^*(S)$. Let $p \in S$ be such that $C \subseteq \operatorname{int} VR^*(p, S)$. By assumption there are points $x_i \in h_i$ and sites $q_i \neq p$ such that $x_i \in VR^*(q_i, S)$ for $i = 1, 2$. The bisector $J(p, q_i)$ does not intersect C (since $C \subseteq \operatorname{int} VR^*(p, S)$) and hence is completely contained in either h_1 or h_2. From $x_i \in VR^*(q_i, S) \cap h_i$ we conclude that $J(p, q_i)$ is completely contained in h_i and therefore $D(p, q_i) = D^*(q_i, p) \subseteq h_i$. Thus $\operatorname{int} VR(p, S) \subseteq h_1 \cap h_2 = \emptyset$, a contradiction to Property 4A. $\square$

Claim 2: All faces of $V^*(S)$ are unbounded faces.

Proof: We show that if $V^*(S)$ has a bounded face, there are two sites for which the bisector is not a simple curve. So let us assume that there is a bounded face $f \subseteq VR^*(p, S)$ for some site $p \in S$. W.l.o.g. we can suppose that S is a minimal set having this property. If $|S| = 2$, then a contradiction is immediate since the only bisector would have to cross itself. So let us assume that $|S| \geq 3$ and that all faces of $V^*(S')$ for $\emptyset \subset S' \subset S$ are unbounded. We may also assume that $p < q$ for all $q \in S \setminus \{p\}$ in the primal linear order imposed on S. Then $R(p, q) = \operatorname{cl} D(p, q)$ for all $q \in S \setminus \{p\}$.

Let $\overline{x} \in \operatorname{int} VR(p, S)$ be a point in the interior of the Voronoi region of p; $\overline{x}$ exists according to Property 4A. Also, for each pair $q', q'' \in S \setminus \{p\}$ with $q' \neq q''$ the set $VR(p, \{p, q', q''\}) = R(p, q') \cap R(p, q'')$ is path-connected and contains $\overline{x}$ by Property 4A. Let

$$K = \bigcup_{\substack{q', q'' \in S \setminus \{p\} \\ q' \neq q''}} R(p, q') \cap R(p, q'').$$

Assume first that there is a simple closed curve $C \subseteq K$ which contains f in its inner domain. Let $q' \in S \setminus \{p\}$ be arbitrary and consider $VR^*(p, S \setminus \{q'\})$. Clearly, $f \subseteq VR^*(p, S \setminus \{q'\})$. On the other hand we have $C \cap VR^*(p, S \setminus \{q'\}) = \emptyset$ since $x \in C$ implies $x \in R(p, q'')$ for some $q'' \in S \setminus \{p, q'\}$ and hence $x \notin VR^*(p, S \setminus \{q'\})$. Consequently, there would be a bounded face in $V^*(S \setminus \{q'\})$, a contradiction to the minimality of S.

Thus there is no simple closed curve $C \subseteq K$ which contains f in its inner domain and hence there is a (topological) ray with $(K \cup f) \cap r = \emptyset$ having its endpoint on $\operatorname{bd} f$ and going to infinity.

Since $\mathbb{R}^2 \setminus K$ and f are open sets we may assume that the endpoint of r lies on an edge, say e, of $\operatorname{bd} f$. Let q be such that e separates $VR^*(p, S)$ and $VR^*(q, S)$. Note that $e \subseteq J(p, q)$. Let x_1 and x_2 be the two endpoints of e and let q_1 and q_2 be sites different from p and q (but $q_1 = q_2$ is possible) such that $x_i \in J(p, q) \cap J(p, q_i) \subseteq R(p, q) \cap R(p, q_i)$ for $i = 1, 2$. Thus there is a path $P_i \subseteq R(p, q) \cap R(p, q_i) \subseteq K$ connecting x_i and $\overline{x}$ for $i = 1, 2$. The concatenation of e, P_1, and P_2 is contained in $R(p, q)$ and partitions the plane into a number of domains; since the ray r is disjoint (except for its endpoint) from e, P_1, and P_2, we conclude that f is contained in one of the bounded domains. Thus $\operatorname{int} VR(p, \{p, q\}) = D(p, q)$ is not homeomorphic to an open disc, a contradiction to Property 4A. □

This completes the proof of Lemma 2. □

Lemma 3 *Suppose that f_1 and f_2 are two distinct faces with $f_1, f_2 \subseteq VR^*(p, S)$ for some $p \in S$. Then $\operatorname{cl} f_1 \cap \operatorname{cl} f_2 = \emptyset$.*

Proof: Assume that $\operatorname{cl} f_1 \cap \operatorname{cl} f_2 \neq \emptyset$. Since $V^*(S)$ is a tree $\operatorname{cl} f_1 \cap \operatorname{cl} f_2$ consists either of an edge of $V^*(S)$ together with its endpoints or of a single vertex of $V^*(S)$. The intersection cannot be an edge because this edge would disappear from $V^*(S)$ for an appropriate order of the sites, a contradiction to Fact 2. Thus the intersection must be a single vertex v of $V^*(S)$. Since f_1 and f_2 are unbounded and $V^*(S)$ is a tree, the set $\mathbb{R}^2 \setminus (\{v\} \cup f_1 \cup f_2)$ consists of exactly two connected components h_1 and h_2 for which $h_1 \cap h_2 = \emptyset$. See also Figure 4.

Next observe that $\operatorname{bd} f_1, \operatorname{bd} f_2 \subseteq \bigcup_{q \in S \setminus \{p\}} J(p, q)$. So it is possible to select a bisector $J(p, q)$ contributing to the boundary of h_1 and f_1. Obviously, $J(p, q) \cap (f_1 \cup f_2) = \emptyset$. Also, we have $h_2 \cap J(p, q) = \emptyset$, because otherwise $D(p, q) \cap f_2 = D^*(q, p) \cap f_2 \neq \emptyset$. Thus $D(p, q) \subseteq h_1$ and hence $\operatorname{int} VR(p, S) \subseteq h_1$. By a symmetric argument we have $\operatorname{int} VR(p, S) \subseteq h_2$. Recalling $h_1 \cap h_2 = \emptyset$ it follows that $\operatorname{int} VR(p, S) = \emptyset$, a contradiction to Property 4A. □

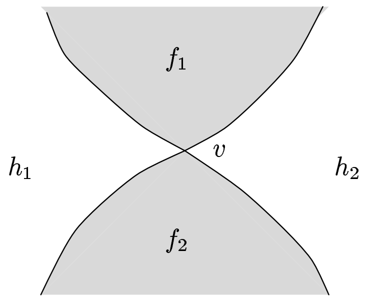


Fig. 4. f_1, f_2, h_1, h_2 decompose $\mathbb{R}^2$

For the sequel, it is helpful to restrict attention to the "finite part" of $V^*(S)$. Let Γ be a simple closed curve such that in the outer domain of Γ any two bisectors are either disjoint or identical and such that each bisector $J(p,q)$ intersects Γ exactly twice. We may also assume that if two bisectors are identical outside Γ then they meet before leaving the inner domain of Γ. That is, if the intersection of Γ and two bisectors $J(p,q)$ and $J(p',q')$ contains a point x then $J(p,q) \cap U_x = J(p',q') \cap U_x$ for sufficiently small neighbourhoods of x.

We add a site 0 to S, define $J(p,0) = J(0,p) = \Gamma$ for all p, $1 \leq p < n$, and $D(0,p)$ to be the inner domain of Γ for each p. Then $VR^*(0,S)$ is the outer domain of Γ and $\Gamma = \operatorname{bd} VR^*(0,S)$. Also, the choice of Γ ensures that every vertex of $V^*(S)$ on Γ has a degree of exactly three.

From now on, $V^*(S)$ denotes the diagram including the site 0.

We have seen above (cf. Figure 3) that furthest site diagrams may contain more than one face for a site. Next, we bound the number of faces.

Lemma 4 a. *Let $e_1, \ldots, e_{m+1}$ with $e_{m+1} = e_1$ be the cyclic list of edges of $V^*(S) \cap \Gamma$ and let $p_i \in S \setminus \{0\}$ be such that e_i lies on the boundary of $\operatorname{int} VR^*(p_i, S)$. Then there are no four indices $1 \leq i < j < k < h \leq m$ such that $p_i = p_k$ and $p_j = p_h$.*

b. *$V^*(S)$ has at most $2n-2$ faces, at most $6n-12$ edges, and at most $4n-8$ vertices.*

Proof: (a) Assume that there are four such indices. Let $p = p_i = p_k$ and $q = p_j = p_h$. Observe that $V^*(\{p,q\}) = V(\{p,q\})$ since $\operatorname{int} VR^*(p,\{p,q\}) = \operatorname{int} VR(q,\{p,q\})$ and $\operatorname{int} VR^*(q,\{p,q\}) = \operatorname{int} VR(p,\{p,q\})$. Also, $VR^*(p,S) \subseteq VR^*(p,\{p,q\})$. Thus the situation described in part (a) arises even in $V^*(\{p,q\})$ and hence in $V(\{p,q\})$. But this is a contradiction to Property 4A.
(b) By part (a), the sequence $p_1, \ldots, p_m$ is a Davenport-Schinzel sequence[8] of order 2 over an alphabet of size $n-1$. Thus $m \leq 2(n-1)-1 = 2n-3$. Thus $V^*(S)$ has at most $2n-2$ faces, one for site 0 and m for the sites 1 to $n-1$. $V^*(S) \setminus \Gamma$ is a tree with at most $2n-3$ vertices of degree 1. Also, there are no vertices of degree 2 in $V^*(S)$. Thus $V^*(S)$ has at most $2n-3+2n-5 = 4n-8$ vertices and at most $4n-9+2n-3 = 6n-12$ edges. $\square$

Next, we focus on the Voronoi vertices.

Definition 4. Let p, q, r be three sites of S and let f_p, f_q, f_r be faces of $VR^*(p,S)$, $VR^*(q,S), VR^*(r,S)$, respectively. A vertex $v \in V^*(S)$ is called a (p,q,r)-*vertex* iff v is located on $\operatorname{bd} f_p \cap \operatorname{bd} f_q \cap \operatorname{bd} f_r$ and there exists a clockwise tour around v encountering f_p, f_q, f_r in this order.

Lemma 5 *Let p, q, and r be three distinct sites in S. Then $V^*(S)$ contains at most one (p,q,r)-vertex and at most one edge separating p- and q-region incident to that vertex.*

Proof: Assume for the sake of a contradiction that there are two distinct (p,q,r)-vertices v and w in $V^*(S)$. There must be faces f_p^v, $f_p^w \subseteq VR^*(p,S)$, f_q^v, $f_q^w \subseteq VR^*(q,S)$, f_r^v, $f_r^w \subseteq VR^*(r,S)$ such that v lies simultaneously on $\operatorname{bd} f_p^v$, $\operatorname{bd} f_q^v$, $\operatorname{bd} f_r^v$ and w lies simultaneously on $\operatorname{bd} f_p^w$, $\operatorname{bd} f_q^w$, $\operatorname{bd} f_r^w$. We have to distinguish two cases:

Assume first that one of the three sites, say p, is equal to 0: This implies $v, w \in \Gamma$ and $f_q^v \neq f_q^w$ and $f_r^v \neq f_r^w$ since v and w can be connected by a path in $V^*(S)\backslash\Gamma$. In a clockwise tour on Γ starting at v the faces f_q^v, f_r^w, f_q^w, f_r^v will be encountered in this order. This contradicts Lemma 4.

Assume next that $0 \notin \{p,q,r\}$: According to Lemma 2 each of the six faces touches Γ and hence $|\{f_p^v, f_p^w, f_q^v, f_q^w, f_r^v, f_r^w\}| \leq 4$ according to Lemma 4. $|\{f_p^v, f_p^w, f_q^v, f_q^w, f_r^v, f_r^w\}| = 3$ would imply $v = w$ and hence exactly two of the associated pairs of faces collapse, say $f_p^v = f_p^w$ and $f_q^v = f_q^w$. But now it is possible to connect v and w by paths $P_p \subseteq VR^*(p,S)$ and $P_q \subseteq VR^*(q,S)$. Glueing together both paths we obtain a circle containing either f_r^v or f_r^w. Thus either f_r^v or f_r^w is a bounded face, a contradiction to Lemma 2.

So there is at most one (p,q,r)-vertex in $V^*(S)$, say v. The vertex v can be incident to at most one edge separating p- and q-region because otherwise Lemma 3 would be violated. $\square$

Note that Lemma 5 does not exclude the possibility of more than one edge separating the p- and the q-region. It only states that such edges have no common endpoints.

2.3. Addition of a site

This section prepares the ground for the incremental construction scheme used to compute $V^*(S)$. Suppose from now on that $R \subseteq S$ and $|R| \geq 3$. Throughout this section we also assume that $0 \in R$. Note that the last condition implies that all edges of $V^*(R)$ are bounded. We consider the case when a new site $t \in S \setminus R$ is to be inserted.

Let $VR_o^*(p,R)$ denote $\operatorname{cl}\operatorname{int} VR^*(p,R)$, i.e., the closure of the p-region.

Lemma 6 $T \overset{def}{=} (V^*(R)\backslash\Gamma) \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset$ *if and only if* $VR_o^*(t, R \cup \{t\}) \neq \emptyset$.

Proof: If $T = \emptyset$ and $VR_o^*(t, R \cup \{t\}) \neq \emptyset$ then the boundary of each face $f \subseteq VR^*(t, R \cup \{t\})$ is completely contained in a face of $V^*(R)$ or is located on Γ. $|R \setminus \{0\}| \geq 2$ ensures that $V^*(R) \setminus \Gamma \neq \emptyset$. Now consider $V^*(R \setminus \{0\} \cup \{t\})$. Since outside Γ no Voronoi vertices can occur, $V^*(R \setminus \{0\} \cup \{t\})$ consists of at least two components, a contradiction to Lemma 2. The converse direction is trivial. $\square$

Lemma 7 *Let $VR_o^*(t, R \cup \{t\}) \neq \emptyset$, let f be a face of $VR^*(t, R \cup \{t\})$, and let $T_f \stackrel{def}{=} (V^*(R) \backslash \Gamma) \cap \mathrm{cl}\, f$. Then:*

a. *T_f is nonempty.*

b. *T_f is a connected set.*

c. *T_f is not just a single point.*

Proof: Part (a). This was already shown in the proof of Lemma 6.
Part (b). Assume that T_f consists of at least two components. Then we can choose two endpoints, say x and y, of distinct components of T_f such that x and y can be connected by a path $P \subseteq (V^*(R) \setminus \Gamma) \setminus \mathrm{cl}\, f$. P does not touch $\mathrm{cl}\, f$ except at its endpoints x and y. On the other hand there must be a path $Q \subseteq (\mathrm{bd}\, f) \setminus \Gamma$ connecting x and y. P and Q are disjoint except for their common endpoints, i.e., $P \circ Q$ is a simple curve. Path Q is contained in $V^*(R \cup \{t\}) \setminus \Gamma$. We next construct a path $P' \subseteq V^*(R \cup \{t\}) \setminus \Gamma$ from P which also connects x and y and which is disjoint from Q, i.e., $P' \circ Q$ is a simple cycle contained in $V^*(R \cup \{t\}) \setminus \Gamma$. This contradicts Lemma 2.

To construct P' path P is decomposed in subpaths $P_1 \circ P_2 \circ \ldots \circ P_h$ such that P_i, for i even, is a maximal subpath of P contained in $VR_o^*(t, R \cup \{t\})$. For each even i there is a face $f_i \subseteq VR^*(t, R \cup \{t\})$ different from f with $P_i \subseteq \mathrm{cl}\, f_i$ (since, by Lemma 3, the closures of any two faces of $VR^*(t, R \cup \{t\})$ are disjoint). Let $P_i' \subseteq (\mathrm{bd}\, f_i) \setminus \Gamma$ be the path connecting the two endpoints of P_i. Then $P' = P_1 \circ P_2' \circ P_3 \circ P_4' \circ \ldots \circ P_h$ is a path contained in $V^*(R \cup \{t\}) \setminus \Gamma$ and disjoint from Q by Lemma 3. Figure 5 illustrates the definition of P'.

Part (c). At this point we know already that T_f is a nonempty connected set. Assume now that T_f is a single point. This point, say x, is a vertex of $V^*(R)$ or lies on an edge of $V^*(R)$. In either case f splits a face $\overline{f}$ of $V^*(R)$ into two new ones, say $\overline{f}_1$ and $\overline{f}_2$. Recall that f must touch Γ according to Lemma 2. We conclude that $\mathrm{cl}\, \overline{f}_1 \cap \mathrm{cl}\, \overline{f}_2 = \{x\}$, a contradiction to Lemma 3. □

Note that although $(V^*(R) \setminus \Gamma) \cap \mathrm{cl}\, f$ is connected, this is not necessarily true for $V^*(R) \cap \mathrm{cl}\, f$. We therefore distinguish two types of faces:

Definition 5. A face $f \subseteq VR^*(t, R \cup \{t\})$ is called *rooted* if $T_f^+ \stackrel{def}{=} V^*(R) \cap \mathrm{cl}\, f$ is connected and *unrooted* otherwise.

If t gives rise to unrooted faces, we can prove stronger properties of $VR^*(t, R \cup \{t\})$:

Lemma 8 *Let f be a face of $VR^*(t, R \cup \{t\})$.*

a. *Let v be a Voronoi vertex of $V^*(R)$ located on Γ and let e be the unique edge in $V^*(R) \backslash \Gamma$ incident to v. If $v \in \mathrm{cl}\, f$ then $U_v \cap e \cap \mathrm{cl}\, f \neq \emptyset$ for all neighbourhoods U_v of v.*

b. *f is unrooted if and only if $\mathrm{cl}\, f$ does not include a Voronoi vertex of $V^*(R)$ located on Γ.*

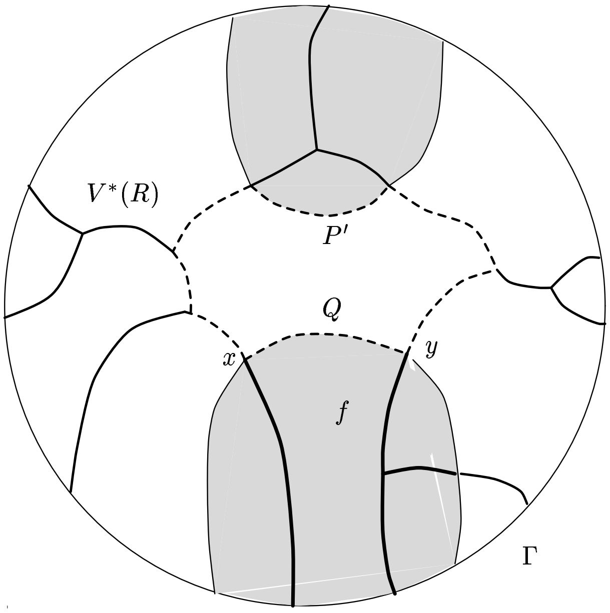


Fig. 5. Illustration of paths P' and Q

Proof: Part a). First note that there can be no face $f' \subseteq VR^*(t, R \cup \{t\})$ with $f \neq f'$ and $v \in \operatorname{cl} f'$ according to Lemma 3. If $U_v \cap e \cap \operatorname{cl} f = \emptyset$ for some neighbourhood U_v then v is also a Voronoi vertex in $V^*(R \cup \{t\})$. Moreover, in $V^*(R \cup \{t\})$ there are four Voronoi regions meeting at v, namely the t-region and the three Voronoi regions meeting at v before site t has been inserted. Thus v is incident to four Voronoi edges in $V^*(R \cup \{t\})$, a contradiction to the choice of Γ.

Part b). ($\Rightarrow$) If f is an unrooted face $T_f^+ = V^*(R) \cap \operatorname{cl} f$ consists of exactly two components: T_f and $\Gamma \cap \operatorname{cl} f$. For the sake of a contradiction assume that $\Gamma \cap \operatorname{cl} f$ contains a Voronoi vertex v of $V^*(R)$. Now let e be the unique edge in $V^*(R) \setminus \Gamma$ incident to v. By part a) we have $U_v \cap e \cap \operatorname{cl} f \neq \emptyset$ for all neighbourhoods U_v of v. Since $U_v \cap e \cap \operatorname{cl} f \subseteq T_f$, it follows that T_f and $\Gamma \cap \operatorname{cl} f$ are connected via v, a contradiction.

($\Leftarrow$) To show the converse, suppose that $\Gamma \cap \operatorname{cl} f$ does not include a Voronoi vertex of $V^*(R)$. T_f and $\Gamma \cap \operatorname{cl} f$ are nonempty sets according to Lemma 7 and Lemma 2, respectively. Now observe that any path inside $V^*(R)$ which runs from T_f to $\Gamma \cap \operatorname{cl} f$ must pass through a Voronoi vertex on Γ. Thus T_f^+ is not connected and the claim follows. □

Lemma 9 *If $VR^*(t, R \cup \{t\})$ has an unrooted face then $VR^*(t, R \cup \{t\})$ consist of a single face.*

Proof: Let f be an unrooted face of $VR^*(t, R \cup \{t\})$. Lemma 8 shows that $\Gamma \cap \operatorname{cl} f$ contains no Voronoi vertex of $V^*(R)$. Consequently, there is a site p such that $VR^*(p, R \cup \{t\})$ is the clockwise and counterclockwise neighbour of f on Γ. Thus f must be the only face of $VR^*(t, R \cup \{t\})$ by Lemma 4. □

The following observation is also helpful.

Lemma 10 *Let f be a face of $VR^*(t, R \cup \{t\})$ and let e be an edge in $V^*(R) \setminus \Gamma$. Then $e \cap \mathrm{cl}\, f$ has at most one component.*

Proof: We only need to notice that $V^*(R \cup \{t\})$ is also a tree. $\square$

3. Descriptions and Conflicts

Our algorithm for furthest site abstract Voronoi diagrams is an instance of the randomized incremental construction paradigm introduced by Clarkson and Shor[5]; cf. also Ref.[2,4]. We briefly review the paradigm.

Let S be a set with $|S| = n$ *objects*, let b be an integer, let $\mathcal{F}(S) \subseteq S^b$ be a subset of the b-tuples over S and let $C \subseteq S \times \mathcal{F}(S)$ be a relation (the so-called *conflict relation*). It is assumed that $(s, (s_1, \ldots, s_b)) \in C$ implies $s \neq s_i$ for $1 \leq i \leq b$. Let $\mathcal{F}_o(S) = \{D \in \mathcal{F}(S) \mid$ there is no $s \in S$ with $(s, D) \in C\}$. Clarkson and Shor have analyzed the incremental construction of $\mathcal{F}_o(S)$. In the general step, $\mathcal{F}_o(R)$ for some subset $R \subseteq S$ is already available, a random object $t \in S \setminus R$ is chosen, and $\mathcal{F}_o(R \cup \{t\})$ is constructed from $\mathcal{F}_o(R)$.

In order to apply the paradigm we need to interpret S, $\mathcal{F}(S)$ and C. S is just our set $\{0, \ldots, n-1\}$ of sites. For $\mathcal{F}(S)$ and C the situation is more difficult. Intuitively, we want $\mathcal{F}_o(R)$ to be the set of edges of $V^*(R)$, formally $\mathcal{F}(R)$ and hence $\mathcal{F}_o(R)$ has to be a set of b-tuples of sites for some integer b. We resolve this dilemma as follows: We identify edges with certain 6-tuples of sites; for example, the edge e in Figure 6 will be identified with the 6-tuple $(p, q, r_q, r_p, r'_p, r'_q)$, i.e., the description of an edge involves the sites whose Voronoi regions are separated by the edge e and sites owning neighbouring faces. We will now give the precise definition of $\mathcal{F}(R)$.

Throughout this section the set R need not necessarily contain the site 0. However, $|R| \geq 3$ is supposed.

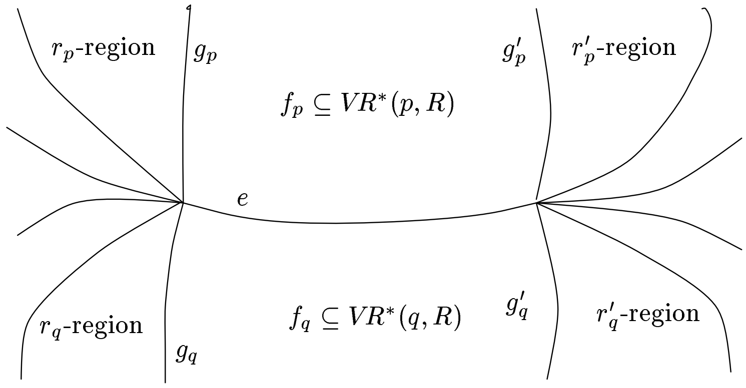


Fig. 6. The description of e is $D_R(e) = \{(p, q, r_q, r_p), (q, p, r'_p, r'_q)\}$

Definition 6. A set $D = \{(p, q, r_1, r_2), (q, p, r_3, r_4)\}$ is called a *description* over R iff $\{p, q, r_1, r_2, r_3, r_4\} \subseteq R$ and $\{p\}, \{q\}, \{r_1, r_2, r_3, r_4\}$ are pairwise disjoint. For a description D let $set(D) \stackrel{def}{=} \{p, q, r_1, r_2, r_3, r_4\}$.

Remark: A description $D = \{(p, q, r_1, r_2), (q, p, r_3, r_4)\}$ may also be written as a 6-tuple $(p, q, r_1, r_2, r_3, r_4)$, i.e., the set of descriptions can be viewed as a subset of $\mathbb{N}^6$. We prefer the notation of Definition 3 because it allows a natural interpretation which we give next.

A bounded edge e of $V^*(R)$ is mapped to a description in the following way (see Figure 6): Let e separate faces $f_p \subseteq VR^*(p, R)$ and $f_q \subseteq VR^*(q, R)$. Let g_p and g_p' be the edges preceding and following e in a counterclockwise traversal of bd f_p and let g_q and g_q' be the edges preceding and following e in a clockwise traversal of bd f_q. The four edges are called the *neighbouring edges* of e and $G_R(e) = \{g_p, g_p', g_q, g_q'\}$ is used to denote the set of neighbouring edges. Let sites r_p and r_p' be such that edges g_p and g_p' separate f_p from a face of $VR^*(r_p, R)$ and $VR^*(r_p', R)$, respectively. Similarly, let sites r_q and r_q' be such that edges g_q and g_q' separate f_q from a face of $VR^*(r_q, R)$ and $VR^*(r_q', R)$, respectively.

Definition 7. Let e be a bounded edge of $V^*(R)$ and let $p, q, r_p, r_p', r_q, r_q'$ be as explained above. Then $D_R(e) \stackrel{def}{=} \{(p, q, r_q, r_p), (q, p, r_p', r_q')\}$ is called the description of e w.r.t. R.
We also define $\mathcal{F}(R) = \{D \mid D$ is a description over R and $V^*(set(D))$ contains a bounded edge with description $D\}$.

Remarks:
1. Note that one of the endpoints of e is a (p, q, r_p)- and (p, q, r_q)-vertex and the other one is a (q, p, r_p')- and (q, p, r_q')-vertex.
2. $|\{r_p, r_p', r_q, r_q'\}|$ lies between 1 and 4, $|G_R(e)|$ varies between 2 and 4. For example, if $|R| = 3$ and $0 \in R$ the minimal values are attained for each edge in $V^*(R)$.

Our next aim is to establish basic properties of the mapping between the bounded edges and their descriptions. The following two lemmas show that distinct edges have distinct descriptions and that an edge retains its description if sites are removed from the Voronoi diagram which are not in the description of the edge.

Lemma 11 *Let e be a bounded edge of $V^*(R)$ and let R' be such that $set(D_R(e)) \subseteq R' \subseteq R$. Then:*

a. *e exists in $V^*(R')$.*

b. *The description of e in $V^*(R')$ is the same as in $V^*(R)$, i.e., $D_{R'}(e) = D_R(e)$.*

Proof: We have $VR^*(s, R) \subseteq VR^*(s, R')$ for every site $s \in R'$. The condition $set(D_R(e)) \subseteq R'$ ensures that the Voronoi regions involved in forming e also appear in $V^*(R')$. Thus e exists in $V^*(R')$. Also, for each edge $g \in G_R(e)$ separating the Voronoi regions $VR^*(r_1, R)$ and $VR^*(r_2, R)$ of two sites $r_1, r_2 \in set(D_R(e))$ there is an edge $g' \in G_{R'}(e)$ separating $VR^*(r_1, R')$ and $VR^*(r_2, R')$ with $g \subseteq g'$ and $g \cap U = g' \cap U$ for all sufficiently small neighbourhoods U of e. Thus $D_{R'}(e) = D_R(e)$. $\square$

Lemma 12 *Let e and e' be distinct bounded edges of $V^*(R)$. Then $D_R(e) \neq D_R(e')$.*

Proof: We will show that $D_R(e) = D_R(e')$ implies $e = e'$. Let $D_R(e) = D_R(e') = \{(p, q, r_q, r_p), (q, p, r'_p, r'_q)\}$. Then both edges have a (p, q, r_p)-vertex and a (q, p, r'_p)-vertex as an endpoint. Thus e and e' have the same endpoints according to Lemma 5 and hence are identical (again by Lemma 5). □

Next, we turn to the definition of a conflict. We give two definitions, a topological and a combinatorial definition, and show their equivalence. The combinatorial definition gives the conflict relation in the sense of the incremental paradigm, the topological definition links the concept with the intuition that a site $t \in S \setminus R$ conflicts with an edge e in $V^*(R)$ if the edge e no longer exists in $V^*(R \cup \{t\})$; more precisely, if the insertion of t affects e or one of the neighbouring edges at the endpoint shared with e.

Definition 8.

a. topological definition of conflict:
   Let e be a bounded edge of $V^*(R)$ and let $t \in S \setminus R$. Then t conflicts with e in $V^*(R)$ if and only if

$$U \cap (e \cup \bigcup_{g \in G_R(e)} g) \cap VR^*_o(t, R \cup \{t\}) \neq \emptyset$$

   for every neighbourhood U of e.

b. combinatorial definition of conflict:
   Let $D \in \mathcal{F}(S)$ and let $t \in S \setminus set(D)$. t conflicts with D if and only if there is no bounded edge in $V^*(set(D) \cup \{t\})$ with description D.
   $\mathcal{F}_o(R)$ denotes the set of conflict-free descriptions in $\mathcal{F}(R)$, i.e., $\mathcal{F}_o(R) = \{D \in \mathcal{F}(R) \mid D \text{ does not conflict with any } t \in R \setminus set(D)\}$.

Remark: Recall that edges are relatively open sets, i.e., the endpoints of an edge do not belong to the edge. Thus it is possible that an endpoint of e belongs to $VR^*_o(t, R \cup \{t\})$ but t does not conflict with e (in the topological sense).

We next show the equivalence of the two notions of conflict.

Lemma 13 *Let $0 \in R$ and $t \in S \setminus R$. Then t conflicts with e in $V^*(R)$ if and only if t conflicts with $D_R(e)$.*

Proof: Recall that the condition $0 \in R$ ensures that all edges in $V^*(R)$ are bounded, and hence $D_R(e)$ is defined for each edge e in $V^*(R)$. We will prove the lemma by showing the contrapositions. Let $D = D_R(e)$.

($\Rightarrow$) **Claim:** If t does not conflict with $D_R(e)$ then t does not conflict with e in $V^*(R)$.

By Lemma 11, the edge e is also an edge in $V^*(set(D))$ and moreover has the same description D. Since t does not conflict with $D_R(e)$, there is an edge e' in $V^*(set(D) \cup \{t\})$ with description D. The edge e' exists also in $V^*(set(D))$ according to Lemma 11 and hence $e = e'$ (by Lemma 12), i.e., e is an edge of $V^*(set(D) \cup \{t\})$ and $D_{set(D)\cup\{t\}}(e) = D$. The last observations now ensure that for each sufficiently small neighbourhood U of e the following holds:

$$U \cap (e \cup \bigcup_{g \in G_{set(D)}(e)} g) \cap VR_o^*(t, set(D) \cup \{t\}) = \emptyset.$$

Since $\bigcup_{g \in G_R(e)} g \subseteq \bigcup_{g \in G_{set(D)}(e)} g$ and $VR^*(t, R \cup \{t\}) \subseteq VR^*(t, set(D) \cup \{t\})$ it follows that

$$U \cap (e \cup \bigcup_{g \in G_R(e)} g) \cap VR_o^*(t, R \cup \{t\}) = \emptyset$$

and hence t does not conflict with e.

($\Leftarrow$) **Claim:** If t does not conflict with e in $V^*(R)$ then there is no conflict between t and $D_R(e)$.

When t does not conflict with e, e is also an edge of $V^*(R \cup \{t\})$ and moreover has the same description D. By Lemma 11, the edge e is also an edge in $V^*(set(D) \cup \{t\})$ and moreover has the same description D. $\Box$

Theorem 1 *If $0 \in R$ then the mapping $e \mapsto D_R(e)$ is a bijection between the edge set of $V^*(R)$ and $\mathcal{F}_o(R)$.*

Proof: We first show that the function really maps only into $\mathcal{F}_o(R)$. Let $D = D_R(e)$ and s be an element of $R \setminus set(D_R(e))$, if any. Then e is also an edge of $V^*(set(D))$ and $V^*(set(D) \cup \{s\})$ and $D = D_{set(D_R(e))}(e) = D_{set(D_R(e))\cup\{s\}}(e)$ according to Lemma 11. Thus no site $s \in R \setminus set(D_R(e))$ conflicts with e and hence $D_R(e) \in \mathcal{F}_o(R)$.

The injectivity of the mapping was shown in Lemma 12.

It remains to show surjectivity. Let $D \in \mathcal{F}_o(R)$ be arbitrary and assume that $D \neq D_R(e)$ for all edges e of $V^*(R)$. $D \in \mathcal{F}_o(R)$ implies that there is an edge in $V^*(set(D))$ with description D. Thus there must be a set R' with $set(D) \subseteq R' \subseteq R$ and a site $s \in R \setminus R'$ such that $V^*(R')$ contains an edge e with description D but $V^*(R \cup \{s\})$ does not. Thus s conflicts with $D_{R'}(e)$ by Lemma 13. Also, $D_{R'}(e) = D$ according to Lemma 11 and hence $D \notin \mathcal{F}_o(R)$. $\Box$

Remark: What have we achieved? Theorem 1 links a topological concept, namely the edges of $V^*(R)$, with a combinatorial concept, namely the descriptions in $\mathcal{F}_o(R)$. We use this bijection as follows: Lemma 4 gives us a bound on the number of edges of a furthest site diagram. Theorem 1 translates this into a bound on the size of $\mathcal{F}_o(R)$. The general theory of randomized incremental constructions (RICs) then gives a bound on the number of combinatorial objects constructed in a RIC which, by Theorem 1, translates into a bound on the number of topological objects constructed.

Edges (as point sets) could also be characterized by four sites, namely by the two sites separated by the edge and one additional site incident to each endpoint of the edge. But then an edge incident to a high degree vertex has many descriptions and there would be no bijection between combinatorial and topological objects. This would make it impossible to apply the general results about RICs.

The equivalence between the combinatorial and the topological definition of conflict is also important. Our algorithm detects certain topological conflicts. The general theory of RICs gives a bound on the number of combinatorial conflicts encountered which, by the equivalence, translates into a bound on the topological conflicts and hence into a bound for the running time.

4. The Algorithm

This section gives the algorithm for constructing the furthest site abstract Voronoi diagram. In section 4.1 we introduce the basic operation underlying our algorithm, in Section 4.2 we outline the algorithm, and in the remaining sections we give the details.

4.1. The Basic Operation

We first characterize the intersection of an edge with the region of a new site (Lemma 15) and show that this intersection can be computed by considering the diagram of five sites (Lemma 14). We then define our basic operation.

Lemma 14 *Let e be an edge of $V^*(R)$ with $D_R(e) = \{(p,q,r_q,r_p),(q,p,r'_p,r'_q)\}$, let $r \in \{r_p,r_q\}$ and $r' \in \{r'_p,r'_q\}$, let $R' = \{p,q,r,r'\}$, and let $t \in S \setminus R$. Then $e \cap VR^*_o(t,R\cup\{t\}) = e \cap VR^*_o(t,R'\cup\{t\})$.*

Proof: Since $VR^*(s,R) \subseteq VR^*(s,R')$ for all $s \in R'$ the point set e is also an edge of $V^*(R')$ separating the Voronoi regions $VR^*(p,R')$ and $VR^*(q,R')$.

($\subseteq$) Since $VR^*(t,R\cup\{t\}) \subseteq VR^*(t,R'\cup\{t\})$ we have $e\cap VR^*_o(t,R\cup\{t\}) \subseteq e\cap VR^*_o(t,R'\cup\{t\})$.

($\supseteq$) To show the converse, we assume for the sake of a contradiction that a point $x \in e\cap VR^*_o(t,R'\cup\{t\})\setminus e\cap VR^*_o(t,R\cup\{t\}) = e\cap(VR^*_o(t,R'\cup\{t\})\setminus VR^*_o(t,R\cup\{t\}))$ exists. If x is in $VR^*_o(t,R'\cup\{t\})$, but not in $VR^*_o(t,R\cup\{t\})$, then there must be a site $s \in R\setminus R'$ such that $x \in D^*(s,t)$. From $x \in VR^*_o(t,R'\cup\{t\})$ we conclude that in each neighbourhood U_x of x there must be a point y which lies in $\operatorname{int} VR^*(t,R'\cup\{t\})$. y can be chosen such that either $y \in \operatorname{int} VR^*(p,R)$ or $y \in \operatorname{int} VR^*(q,R)$ holds. Assuming w.l.o.g. that $y \in \operatorname{int} VR^*(p,R)$, we obtain $y \in D^*(p,s)$. Moreover, we have $y \in D^*(t,p)$ because of $y \in \operatorname{int} VR^*(t,R'\cup\{t\})$. Combining the last two observations we obtain $y \in D^*(t,p)\cap D^*(p,s) \subseteq D^*(t,s)$. On the other hand x is an element of the open set $D^*(s,t)$. This implies that all sufficiently small neighbourhoods U_x of x also belong to $D^*(s,t)$. Consequently, $y \in D^*(s,t)$. But y cannot be an element of $D^*(s,t)$ and $D^*(t,s)$ simultaneously. $\square$

Lemma 15 *Let e be an edge of $V^*(R)$, let $t \in S\setminus R$, and let I denote the intersection of e and $VR^*_o(t,R\cup\{t\})$, i.e., $I = e\cap VR^*_o(t,R\cup\{t\})$.*

a. *If I contains a connected component I' which is not incident to either endpoint of e then $I = I'$ and $VR^*(t,R\cup\{t\})$ consists of a single unrooted face.*

b. *I consists of at most two connected components.*

c. *If I has two components then both are incident to an endpoint of e.*

Proof: Part a). Let I' be a connected component of I which is not incident to an endpoint of e. Since the closures of distinct faces of $VR^*(t, R \cup \{t\})$ are disjoint (by Lemma 3) there is a unique face f of $VR^*(t, R \cup \{t\})$ with $I' \subseteq f$. We now distinguish cases.
Assume first that e is an edge located on Γ. By the tree property, $\Gamma \cap \mathrm{cl}\, f$ is a connected set. Since I' is not incident to an endpoint of e, we obtain $I' = \Gamma \cap \mathrm{cl}\, f = e \cap \mathrm{cl}\, f$. In particular, $\mathrm{cl}\, f$ contains no Voronoi vertex of $V^*(R)$ located on Γ. Consequently, f is an unrooted face of $VR^*(t, R \cup \{t\})$ (by Lemma 8) and hence f is the only face of $VR^*(t, R \cup \{t\})$ (by Lemma 9).

Assume next that e is an edge in $V^*(R) \setminus \Gamma$. By Lemma 10, we obtain $I' = e \cap \mathrm{cl}\, f$. Since I' is not incident to an endpoint of e we conclude from Lemma 7 that $I' = e \cap \mathrm{cl}\, f = (V^*(R) \setminus \Gamma) \cap \mathrm{cl}\, f = T_f$. Therefore T_f^+ is not connected and hence f is an unrooted face.

Finally, in both cases we observe $I = e \cap VR_o^*(t, R \cup \{t\}) = e \cap \mathrm{cl}\, f = I'$.

Parts b) and c) follow immediately from part a). □

Remark: If I has two components, $VR^*(t, R \cup \{t\})$ consists of rooted faces. The two components usually belong to different faces of $VR^*(t, R \cup \{t\})$. An exception may occur when e is located on Γ.

We can now define our basic operation. The procedure is designed to decide whether a site $t \in S \setminus R$ intersects a given edge e of $V^*(R)$. When an intersection is detected, it determines the type of intersection. Input as well as output are of combinatorial type and have constant size. We will charge one time unit for each call of the basic operation.

The basic operation is the only part of our algorithm which depends on the particular kind of abstract Voronoi diagram. This allows us to adapt our algorithm to a specific situation simply by exchanging this procedure.

Basic Operation

input: a description $D_R(e) = \{(p, q, r_q, r_p), (q, p, r'_p, r'_q)\} \in \mathcal{F}(S)$ and a site t with $t \notin set(D_R(e))$.

output: Let $r \in \{r_p, r_q\}$, let $r' \in \{r'_p, r'_q\}$, and let $R' = \{p, q, r, r'\}$. A symbol is reported to describe the combinatorial type of $I = e \cap VR_o^*(t, R' \cup \{t\})$:

EMPTY:	the intersection is empty ($I = \emptyset$)
ENTIRE_EDGE:	the intersection is equal to e ($I = e$)
SEGMENT_1:	the intersection consists of a segment having the (p, q, r)-vertex as one endpoint ($I \subset e$)
SEGMENT_2:	the intersection consists of a segment having the (q, p, r')-vertex as one endpoint ($I \subset e$)
INNER_SEGMENT:	the intersection is a segment of e incident neither to the (p, q, r)-vertex nor to the (q, p, r')-vertex ($I \subset e$)
TWO_SEGMENTS:	the intersection consists of two disjoint segments each of which is incident to an endpoint of e ($I \subset e$)

We will use basic_op(t,D) to denote the output of the basic operation applied to site t and description D. An implementation of the basic operation can be based on the construction of the furthest site diagrams of four, namely R', and five sites, namely $R' \cup \{t\}$, and the comparison of the two diagrams.

The correctness of the procedure follows from the preceding discussion: We have $I = e \cap VR_o^*(t, R' \cup \{t\}) = e \cap VR_o^*(t, R \cup \{t\})$ according to Lemma 14, and Lemma 15 ensures that the list of symbols used to describe I exhausts all possible cases.

Next, we link the basic operation to the notion of conflict.

Definition 9. Let e be an edge in $V^*(R)$, let v be an endpoint of e, and let $t \in S \setminus R$.

a. t *intersects* the edge e if and only if $e \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset$.

b. t *clips* e at v if and only if $U_v \cap e \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset$ for all neighbourhoods U_v of v.

c. t *intersects* $D_R(e)$ if and only if basic_op(t, $D_R(e)$) $\neq$ EMPTY.

d. t *clips* $D_R(e)$ at v if and only if basic_op(t, $D_R(e)$) $\in$ { ENTIRE_EDGE, TWO_SEGMENTS, SEGMENT_i }, where v is the vertex referred to in the definition of case SEGMENT_i.

The intention behind these definitions is as follows:

Lemma 16 *Let e be an edge in $V^*(R)$ and let $t \in S \setminus R$.*

a. *t intersects e if and only if t intersects $D_R(e)$.*

b. *t clips e at its endpoint v if and only if t clips $D_R(e)$ at v.*

c. *t conflicts with e if and only if t intersects e or t clips an edge $g \in G_R(e)$ at the common endpoint of e and g.*

d. *t conflicts with $D_R(e)$ if and only if t intersects $D_R(e)$ or t clips a description $D_R(g)$ for some $g \in G_R(e)$ at the common endpoint of e and g.*

Proof: Parts a) and b) follow directly from the definition of the symbols used as output of the basic operation.

Part c). Let v be the common endpoint of e and some edge $g \in G_R(e)$. Then t clips g at v if and only if $U_v \cap g \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset$ for all neighbourhoods U_v of v. Since $e \cap g = \emptyset$ and v is the common endpoint of e and g, this is equivalent to $U \cap g \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset$ for all neighbourhoods U of e. The claim now follows since

$$\begin{array}{lll} & & t \text{ conflicts with } e \\ \Leftrightarrow & \forall U: & U \cap (e \cup \bigcup_{g \in G_R(e)} g) \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset \\ \Leftrightarrow & \forall U: & U \cap e \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset \text{ or } U \cap \bigcup_{g \in G_R(e)} g \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset \\ \Leftrightarrow & \forall U: & e \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset \text{ or } \exists g \in G_R(e) (U \cap g \cap VR_o^*(t, R \cup \{t\}) \neq \emptyset) \\ \Leftrightarrow & \multicolumn{2}{l}{t \text{ intersects } e \text{ or } t \text{ clips an edge } g \in G_R(e) \text{ at the common endpoint of } e \text{ and } g.} \end{array}$$

Part d) is an immediate consequence of parts a), b), c) and Lemma 13. $\square$

4.2. A Global View of the algorithm

In this section we give a global view of the algorithm and define essential data structures.

The algorithm chooses a random order $\{t_1, \ldots, t_{n-1}\}$ of the sites $\{1, \ldots, n-1\}$. Let R_{i+1} denote $\{0, t_1, \ldots, t_i\}$. Initially, it computes $V^*(R_3)$ and then it successively adds t_i to obtain $V^*(R_{i+1})$ from $V^*(R_i)$. The following data structures are maintained for the current set $R = R_i$ of sites:

1. The furthest site Voronoi diagram $V^*(R)$ of the set R of sites already inserted is stored as a planar map:
   (a) For a vertex $v \in V^*(R)$ we store the cyclic list of edges incident to v in clockwise order. This data structure is denoted by $list_R(v)$.
   (b) An edge e in $V^*(R)$ is connected with its two endpoints. e also knows the two sites whose Voronoi regions share edge e.
2. The history graph $H(R)$ provides information about conflicts, cf. Ref.[2]. In contrast to the terms vertex and edge used to describe the Voronoi diagram we use the terms node and arc for $H(R)$. $H(R)$ is a directed acyclic graph with a single source. The node set is given by $\{source\} \cup \bigcup_{3 \le j \le i} \{D_{R_j}(e) \mid e \text{ is an edge of } V^*(R_j)\}$. The following history graph invariants hold:
   (a) Every edge e of $V^*(R)$ is linked with its description $D_R(e)$ in $H(R)$.
   (b) Each node of $H(R)$ has outdegree at most 5 and the nodes corresponding to edges in $V^*(R)$ have outdegree 0.
   (c) For every site $t \in S \setminus R$ and every edge e of $V^*(R)$, such that t intersects e, there is a path from *source* to $D_R(e)$ that visits only descriptions intersected by t.

The general outline of the algorithm is as follows:

algorithm
begin
  choose a random permutation $\{t_1, \ldots, t_{n-1}\}$ of $\{1, \ldots, n-1\}$;
  $R = R_3$ /* $R = \{0, t_1, t_2\}$ */ ;
  compute $V^*(R)$ and $H(R)$;
  for $i = 3, \ldots, n-1$ **do**
    $t = t_i$;
    compute $E_t = \{\, e \mid e \text{ is an edge of } V^*(R) \text{ and conflicts with } t \,\}$;
    compute $V^*(R \cup \{t\})$ from E_t and $V^*(R)$;
    compute $H(R \cup \{t\})$ using $H(R)$ and $V^*(R \cup \{t\})$;
    $R = R \cup \{t\}$;
  end
end

In the following we will show in detail how the iteration treating t works. We also show that the insertion of t takes $O(c)$ time, where c denotes the number of nodes in $H(R)$ in conflict with t.

4.3. Collecting the Edges of E_t

We proceed in two steps: In a first step we identify the edges in $V^*(R)$ which are intersected by t. Starting at node *source* a simple variant of breadth first search in $H(R)$ extracts all these edges. Each intersection test requires a call to the basic operation. Only if the basic operation indicates a nonempty intersection we search the successors of the node. The fact that no edge is missed follows from the third history graph invariant. Since the outdegree of a node is bounded by 5, the search in $H(R)$ takes time proportional to the number of descriptions in $H(R)$ intersected by t.

In a second step we determine all edges which conflict with t. According to Lemma 16 this is tantamount to checking all neighbours of intersected edges.

Altogether, the computation of E_t can be accomplished in time proportional to $O(c)$. We summarize in:

Lemma 17 *The set E_t can be computed in time $O(c)$.*

4.4. Construction of $V^*(R \cup \{t\})$

As above, let $T = (V^*(R) \setminus \Gamma) \cap VR_o^*(t, R \cup \{t\})$ and let $T_f = (V^*(R) \setminus \Gamma) \cap \operatorname{cl} f$ and $T_f^+ = V^*(R) \cap \operatorname{cl} f$ for a specified face $f \subseteq VR^*(t, R \cup \{t\})$.

We know from Lemma 6 that $T = \emptyset$ iff $VR_o^*(t, R \cup \{t\}) = \emptyset$. The case $T = \emptyset$ can be checked by the predicate $E_t = \emptyset$. If so, $V^*(R) = V^*(R \cup \{t\})$ and we are done. Otherwise we have $T \neq \emptyset$ and $E_t \neq \emptyset$. We start by classifying the vertices in $V^*(R)$ and $V^*(R \cup \{t\})$:

UNCHANGED	$= \{v \mid$	v is a vertex of $V^*(R)$ and no edge incident to v in $V^*(R)$ is clipped at $v\}$
CHANGED	$= \{v \mid$	v is a vertex of $V^*(R)$ and some but not all edges incident to v are clipped at v by $t\}$
DELETED	$= \{v \mid$	v is a vertex of $V^*(R)$ and all edges incident to v in $V^*(R)$ are clipped at $v\}$
NEW	$= \{v \mid$	v is an endpoint of a segment of $e \cap VR^*(t, R \cup \{t\})$ which is not an endpoint of $e\}$

Intuitively, UNCHANGED collects all vertices of $V^*(R)$ which are not affected by the insertion of t, CHANGED collects all vertices of $V^*(R)$ which are also vertices of $V^*(R \cup \{t\})$ but with a modified edge list, DELETED collects all vertices of $V^*(R)$ which are not vertices of $V^*(R \cup \{t\})$, and NEW collects all vertices of $V^*(R \cup \{t\})$ which were not already a vertex of $V^*(R)$.

Next, we will describe this intuition more precisely and also characterize the cyclic egde lists of the vertices of $V^*(R \cup \{t\})$:

Consider the set UNCHANGED first. We claim that the elements of UNCHANGED lie outside $VR_o^*(t, R \cup \{t\})$ and are also vertices of $V^*(R \cup \{t\})$.

Let $v \in$ UNCHANGED and assume for the sake of a contradiction that $v \in \mathrm{cl}\, f$ for some face $f \subseteq VR^*(t, R \cup \{t\})$. Then v either lies on Γ or belongs to T_f which is a connected set and not just a single point by Lemma 7. In the former case the only edge in $V^*(R) \setminus \Gamma$ incident to v is clipped by t at v according to Lemma 8. The latter case implies that one of the Voronoi edges incident to v is clipped by t at v by Lemma 7. In either case we have $v \notin$ UNCHANGED. Thus $v \notin VR_o^*(t, R \cup \{t\})$ and v is a vertex of $V^*(R \cup \{t\})$, too. By the same argument we have $list_{R\cup\{t\}}(v) = list_R(v)$.

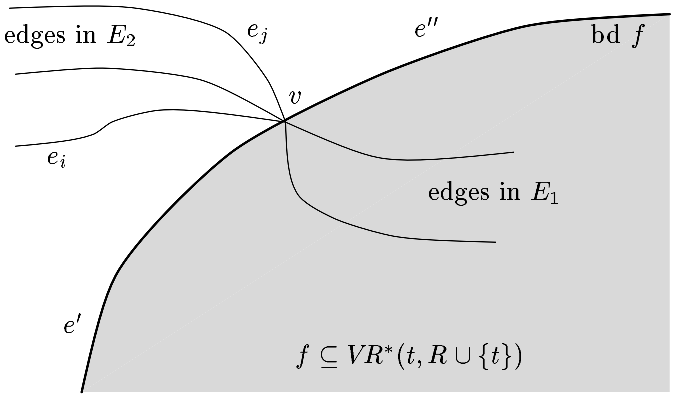


Fig. 7. $v \in$ CHANGED

Now consider the set CHANGED. We claim that the elements of CHANGED belong to $\mathrm{bd}\, VR_o^*(t, R \cup \{t\})$ and are vertices of $V^*(R \cup \{t\})$.

Let $v \in$ CHANGED and let $list_R(v) = (e_1, \ldots, e_k)$. Additionally, let $p_1, \ldots, p_k$ be sites such that e_i $(1 \leq i \leq k)$ separates the Voronoi regions of sites p_i and $p_{i \bmod k+1}$. Some of the edges $(e_1, \ldots, e_k)$ are clipped by t at v and some are not. Consequently, v is a vertex on $\mathrm{bd}\, VR_o^*(t, R \cup \{t\})$. Lemma 3 ensures that there is only one face $f \subseteq VR^*(t, R \cup \{t\})$ with $v \in \mathrm{bd}\, f$. The boundary of f splits the edges $(e_1, \ldots, e_k)$ into two nonempty and uninterrupted subsequences. One of them, say E_1, contains the edges clipped by t and the other the unclipped edges. Suppose that $E_2 = (e_i, \ldots, e_j)$ is the latter subsequence. In $V^*(R \cup \{t\})$ vertex v is shared by the Voronoi regions of the sites $p_i, \ldots, p_j, p_{j \bmod k+1}$ and t. Suppose that e', resp. e'', is the Voronoi edge in $V^*(R \cup \{t\})$ separating t-region from p_i-region, resp. $p_{j \bmod k+1}$-region. To update $list_R(v)$ we have to replace the subsequence E_1 by the two edges e' and e'', i.e. $list_{R\cup\{t\}}(v) = (e', e_i, \ldots, e_j, e'')$. See also Figure 7.

Next, we turn to the set NEW. We claim that the elements of NEW are located on $\mathrm{bd}\, VR_o^*(t, R \cup \{t\})$ and are vertices of $V^*(R \cup \{t\})$, but not of $V^*(R)$.

If $v \in$ NEW then there is an edge e of $V^*(R)$ such that v is an endpoint of a segment of $e \cap VR_o^*(t, R \cup \{t\})$ which is not an endpoint of e. Thus v is not a vertex of $V^*(R)$ and v lies on $\mathrm{bd}\, VR_o^*(t, R \cup \{t\})$. If e has separated p-region and q-region in $V^*(R)$ then v lies also on $\mathrm{bd}\, VR_o^*(p, R \cup \{t\})$ and $\mathrm{bd}\, VR_o^*(q, R \cup \{t\})$.

Thus v is a vertex of $V^*(R \cup \{t\})$ and the cyclic edge list $list_{R\cup\{t\}}(v)$ contains precisely three edges, one for each pair of the three Voronoi regions meeting at v. The cyclic order is readily inferred from the basic operation applied to t and $D_R(e)$. See also Figure 8.

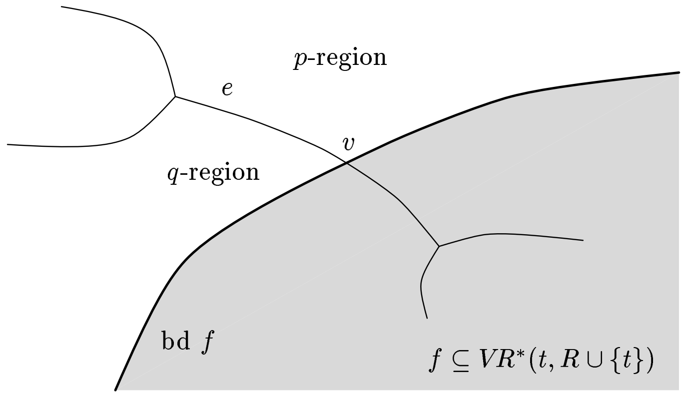


Fig. 8. $v \in$ NEW

Finally, we regard the set DELETED. We claim that the elements of DELETED do not appear in the vertex set of $V^*(R \cup \{t\})$.

When all edges incident to a vertex $v \in$ DELETED are clipped by t then either v lies in $\operatorname{int} VR^*(t, R \cup \{t\})$ or v lies on the boundary of exactly two Voronoi regions, namely the t-region and a Voronoi region which had v on its boundary before t was inserted. In either case v is no longer incident to three Voronoi regions and vanishes from the vertex set.

We summarize these observations in the following lemma:

Lemma 18 *The set of vertices of* $V^*(R)$ *equals* UNCHANGED$\cup$CHANGED$\cup$DELETED, *the set of vertices of* $V^*(R \cup \{t\})$ *equals* UNCHANGED $\cup$ CHANGED $\cup$ NEW.

Proof: The distinction made in the definition of UNCHANGED, CHANGED and DELETED is exhaustive. This proves the first part. A vertex in $V^*(R \cup \{t\})$ is either a vertex of $V^*(R)$ or it is not. In the former case UNCHANGED $\cup$ CHANGED includes the vertex, in the latter case it is contained in NEW. $\square$

Vertices in UNCHANGED have no importance for updating the Voronoi diagram. Their edge lists stay unchanged and they do not require any treatment. The vertices contained in CHANGED, NEW and DELETED can be identified when E_t is calculated.

At this point we have achieved the following: We have shown how to compute the vertex set of $V^*(R \cup \{t\})$ and the cyclic edge list of every vertex.

In order to complete the planar map for $V^*(R \cup \{t\})$ we still need to do the following: Each new Voronoi edge has two endpoints. So each such edge appears exactly twice in the cyclic lists of the vertices. It remains to explain how to link the two occurrences of each new edge.

There are two kinds of new Voronoi edges in $V^*(R \cup \{t\})$:

type 1: edges which are on $\operatorname{bd} VR^*_o(t, R \cup \{t\})$.

type 2: edges which are proper subsets of edges in $V^*(R)$.

The task is easy for edges of type 2. They can be determined during the computation of E_t. An edge of this type is detected whenever the basic operation does not return EMPTY or ENTIRE_EDGE. Note that type 2 edges have at least one endpoint in NEW.

The computation of the type 1 edges is much more involved. We distinguish cases according to whether $VR^*(t, R \cup \{t\})$ has unrooted faces or not. A criterion to decide this question is given in the next Lemma.

Lemma 19 *$VR^*(t, R \cup \{t\})$ has an unrooted face if and only if* basic_op$(t, D_R(e))$=INNER_SEGMENT *for some edge e of $V^*(R)$.*

Proof: ($\Rightarrow$) Let f be the unrooted face of $VR^*(t, R \cup \{t\})$. By the tree property, there must be an edge e on Γ with $e \cap \mathrm{cl}\, f \neq \emptyset$. By Lemma 8, $\mathrm{cl}\, f$ cannot contain the endpoints of e and hence basic_op$(t, D_R(e))$=INNER_SEGMENT. According to the third history graph invariant e is found when the set E_t is computed.

($\Leftarrow$) The converse follows from Lemma 15. □

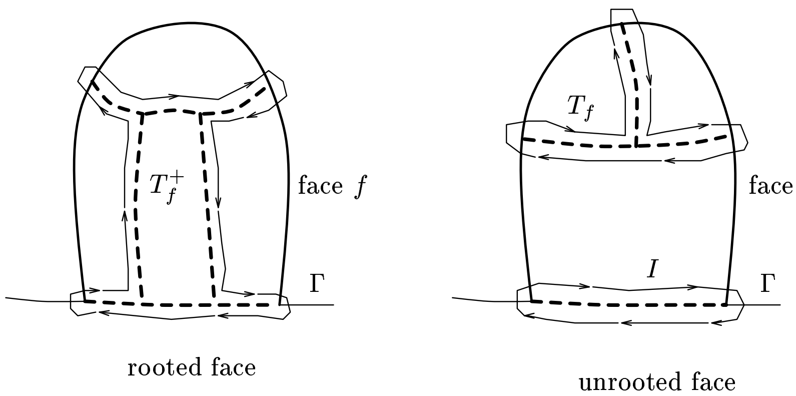


Fig. 9. Touring around T_f^+

The procedure completing the update of the Voronoi diagram works as follows:

1. Assume first that $VR^*(t, R \cup \{t\})$ has only rooted faces. Then $T_f^+ = V^*(R) \cap \mathrm{cl}\, f$ is connected for each face $f \subseteq VR^*(t, R \cup \{t\})$ and $T_f^+ \cap T_{f'}^+ = \emptyset$ for distinct faces f and f' of $VR^*(t, R \cup \{t\})$ by Lemma 3. We conclude that the faces of $VR^*(t, R \cup \{t\})$ are in one-to-one correspondence to the connected components of $V^*(R) \cap VR^*(t, R \cup \{t\})$. Let T_f^+ be one such connected component for a particular face $f \subseteq VR^*(t, R \cup \{t\})$. $V^*(R)$ provides a planar embedding of T_f^+ in the plane. T_f^+ induces exactly one outer domain and a possibly empty set of domains surrounded by T_f^+. A traversal of the boundary of the outer domain meets all endpoints of the new Voronoi edges on $\mathrm{bd}\, f$ and also the two occurrences of each new edge, cf. Figure 9. This allows the two occurrences to be linked.

2. Assume next that $VR^*(t, R \cup \{t\})$ has an unrooted face f. Then f is the only face of $VR^*(t, R \cup \{t\})$ (by Lemma 9), $T_f = (V^*(R) \setminus \Gamma) \cap \mathrm{cl}\, f$ is connected

(Lemma 7), and $I = \Gamma \cap \mathrm{cl}\, f$ is a subsegment of some edge e on Γ which is not incident to an endpoint of e (Lemma 8). I and T_f are disjoint, cf. Figure 9.

There are two kinds of vertices on $\mathrm{bd}\, f$. Two vertices are located on Γ and all other vertices belong to T_f. The cyclic order of the latter kind can again be determined by a traversal of the boundary of the outer domain of T_f. The only problem unresolved is where to insert the two vertices on Γ into this cyclic order.

Overcoming this difficulty requires a more detailed inspection: The unrooted face f splits a face f_p of some Voronoi region $VR^*(p, R)$ into smaller faces belonging to $VR^*(p, R \cup \{t\})$. The border between f and these two faces is formed by two edges which must lie on $J(p, t)$. In $V^*(R \cup \{t\})$ there are two new Voronoi vertices on Γ: a $(t, p, 0)$-vertex called v and a $(p, t, 0)$-vertex called w. See also Figure 10.

Among the other vertices on $\mathrm{bd}\, f$ we single out those vertices which lie also on $\mathrm{bd}\, VR^*_o(p, R \cup \{t\})$. Let $(v_1, w_1, v_2, w_2, \ldots, v_k, w_k)$ with $k \geq 1$ be the cyclic clockwise sequence of those vertices with v_i being a (p, t, a_i)-vertex and w_i being a (t, p, b_i)-vertex for some sites a_i and b_i ($1 \leq i \leq k$). We need to find out which vertex v_j has to be connected with vertex v by a new Voronoi edge and which vertex w_j has to be linked with w. If $k = 1$ the problem is trivial. If $k > 1$ then the next two lemmas show how the basic operation can be used to determine v_j. We first show that v, w, and v_i are vertices of $V^*(\{p, t, 0, a_i\})$ and that there is an edge e_i connecting v and v_i in $V^*(\{p, t, 0, a_i\})$ and we then show that $j = i$ if and only if e_i exists in $V^*(\{p, t, 0, a_i, b_i\})$.

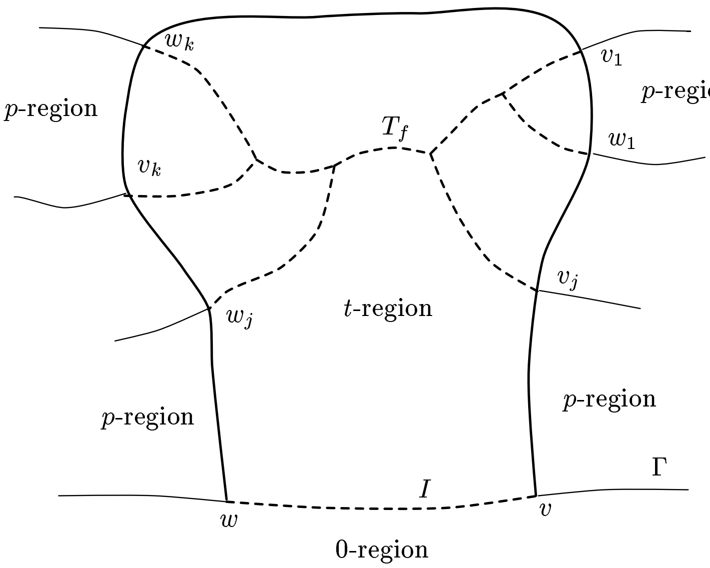


Fig. 10. $VR^*(t, R \cup \{t\})$ consists of an unrooted face

Lemma 20 *Let* $1 \leq i \leq k$ *and let* $D_i = \{(p, t, a_i, a_i), (t, p, 0, 0)\}$. $V^*(\{p, t, 0, a_i\})$ *contains an edge* e_i *separating* $VR^*(p, \{p, t, 0, a_i\})$ *and* $VR^*(t, \{p, t, 0, a_i\})$ *and connecting* v *and* v_i. *Moreover,* $D_{\{p,t,0,a_i\}}(e_i) = D_i$.

Proof: Since $VR^*(s, R) \subseteq VR^*(s, R')$ for each $\emptyset \subset R' \subseteq R$ and $s \in R'$ the vertices v, w and v_i also occur in $V^*(\{p,t,0,a_i\})$ as $(t,p,0)$-vertex, $(p,t,0)$-vertex and (p,t,a_i)-vertex, respectively. Recall that a 3-tuple of sites uniquely determines a Voronoi vertex according to Lemma 5. In $V^*(\{p,t,0,a_i\})$ there is the p-region in the neighbourhood of v and w. On the other hand the existence of the a_i-region prevents that v and w can be connected by a path inside $\operatorname{int} VR^*(p,\{p,t,0,a_i\})$. Thus $VR^*(p,\{p,t,0,a_i\})$ consists of two faces. From Lemma 4 we conclude that $VR^*(t,\{p,t,0,a_i\})$ and $VR^*(a_i,\{p,t,0,a_i\})$ can only have one face. By the tree property, the $(t,p,0)$-vertex v and the (p,t,a_i)-vertex v_i are endpoints of the same edge.

D_i is the description of e since v_i cannot be located on $\operatorname{bd} VR^*(0,\{p,t,0,a_i\})$ and v cannot be on $\operatorname{bd} VR^*(a_i,\{p,t,0,a_i\})$. Otherwise, there would be a vertex of degree 4 on Γ. $\square$

Lemma 20 shows that $D_i \in \mathcal{F}(S)$. Thus it is possible to use D_i as input for a call to the basic operation.

Lemma 21 *v directly follows v_i in the cyclic clockwise ordering of the vertices of* $\operatorname{bd} f$ *if and only if* basic_operation(b_i, D_i) $=$ EMPTY.

Proof: $(\Rightarrow)$ If v follows v_i in the cyclic clockwise ordering then e_i as defined in Lemma 20 is identical to the edge connecting v_i and v in $V^*(R \cup \{t\})$. Thus the basic operation returns EMPTY.

$(\Leftarrow)$ Conversely, if the basic operation returns EMPTY, then w_i does not lie on e_i. This is only true if v follows v_i. $\square$

We summarize in:

Lemma 22 *Given E_t, $V^*(R \cup \{t\})$ can be computed from $V^*(R)$ in time $O(c)$.*

Proof: The vertices in CHANGED $\cup$ DELETED $\cup$ NEW can be calculated as a by-product when computing E_t. Also, the update of the cyclic edge lists does not take more then $O(c)$ time.

Next, we show that the construction of the new edges also consumes no more than $O(c)$ time. $VR^*(t, R \cup \{t\})$ can have rooted faces or an unrooted face. If $VR^*(t, R \cup \{t\})$ consists of rooted faces the construction of the new edges requires a walk around T_f^+ for each face $f \subseteq VR^*(t, R \cup \{t\})$. In case of an unrooted face all edges but the edges connecting v and v_j, resp. w and w_j, can be found by walking around $T_f^+ = I \cup T_f$. The construction of the latter two edges again requires a walk around T_f to find v_j and w_j. Each traversal on T_f^+, resp. T_f, takes time proportional to the number of edges of $V^*(R)$ contributing to T_f^+, resp. T_f. Summing over all faces of $VR^*(t, R \cup \{t\})$ this number coincides with the number of edges in $V^*(R)$ intersected by t. Hence $O(c)$ time suffices to compute $V^*(R \cup \{t\})$. $\square$

We close this section with two definitions which will be needed in the next section. An edge e on the boundary of $\operatorname{int} VR^*(t, R \cup \{t\})$ is called *critical* if $VR^*(t, R \cup \{t\})$ has an unrooted face and exactly one endpoint of e lies on Γ. Otherwise edge e is called *noncritical.*

For an edge e on the boundary of $\operatorname{int} VR^*(t, R \cup \{t\})$ we associate a certain point set $T(e)$:

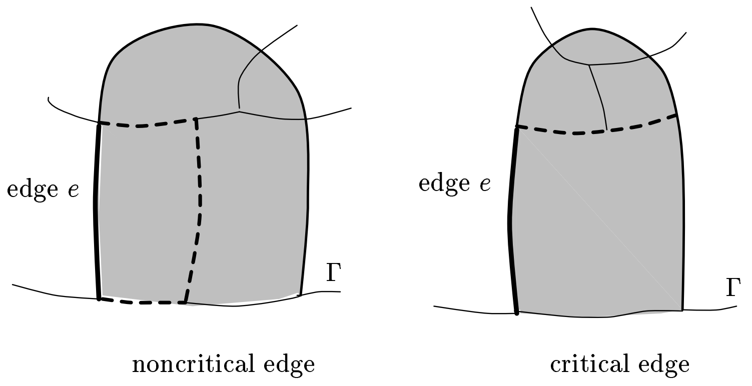


Fig. 11. $T(e)$ is shown by a dashed line

$$T(e) = \begin{cases} e & \text{if } e \subseteq \Gamma \\ \text{the part of } T_f^+ \text{ traversed to construct } e \text{ except its endpoints} & \text{if } e \text{ is a noncritical edge and } e \not\subseteq \Gamma \\ \text{the part of } T_f \text{ leading from } v_j \text{ to } w_j \text{ (as defined above) except its endpoints} & \text{if } e \text{ is a critical edge} \end{cases}$$

The definition of $T(e)$ is illustrated by Figure 11.

4.5. Computation of $H(R \cup \{t\})$

In this section we show how to update the history graph. We first characterize the nodes which are added to it, then define the set of arcs to be added, and finally argue that the history graph invariants are maintained. Throughout this section we use B to denote the boundary of $\operatorname{int} VR^*(t, R \cup \{t\})$.

An edge e is called

new	if $e \subseteq B$
affected	if e was already an edge in $V^*(R)$ and at least one edge $g \in G_R(e)$ was clipped at an endpoint of e, but e is not a subset of B
shortened	if e does not belong to B and there is an edge $\overline{e}$ in $V^*(R)$ such that $e \subset \overline{e}$

If e is an affected or shortened edge of $V^*(R \cup \{t\})$ we use *super*(e) to denote the edge of $V^*(R)$ containing e. Thus we have $e = super(e)$ for affected edges and $e \subset super(e)$ for shortened edges; see Figure 12 for an example.

Lemma 23 *Let N_t be the set of nodes of $H(R \cup \{t\})$ which are not already nodes of $H(R)$. Then $N_t = \{D \mid D = D_{R\cup\{t\}}(e)$ for some new, affected or shortened edge e of $V^*(R \cup \{t\})\}$.*

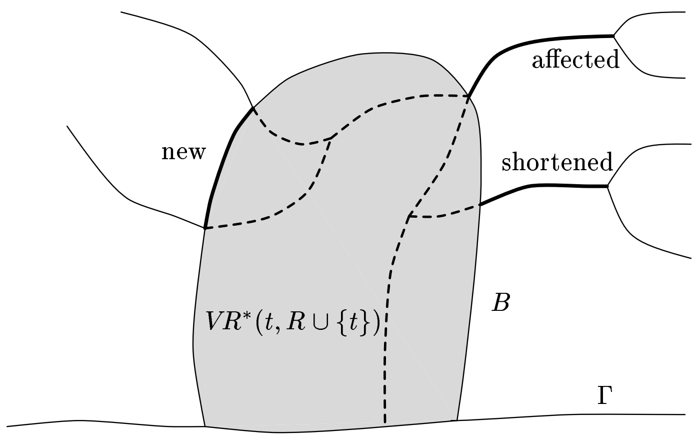


Fig. 12. new, affected, and shortened edges

Proof: (⇒) Let $D \in N_t$. Then $D = D_{R\cup\{t\}}(e)$ for some edge of $V^*(R \cup \{t\})$. Assume that e is neither new, shortened, nor affected. Then e was already an edge in $V^*(R)$ and no edge of $G_R(e)$ was clipped at an endpoint of e. So the descriptions of e in $V^*(R \cup \{t\})$ and $V^*(R)$ are equal, i.e., $D_{R\cup\{t\}}(e) = D_R(e)$. Thus $D \notin N_t$, a contradiction.

(⇐) It is enough to show that $t \in set(D)$. Assume first that e is a new edge. Then $e \subseteq B$ and hence $t \in set(D)$. Assume next that e is a shortened edge. Then at least one endpoint of e must be in NEW. Consequently, at least one edge in $G_{R\cup\{t\}}(e)$ belongs to B and hence $t \in set(D)$. Finally, if e is an affected edge, then at least one edge in $G_R(e)$ is clipped by t. Each such edge is replaced by an edge lying in B. Hence $t \in set(D)$. □

Next we define the new arcs of the history graph. Each new arc goes from a node of $H(R)$ to a node in N_t. There are four types of arcs:

- **type 1:** For each affected or shortened edge e of $V^*(R \cup \{t\})$ there is an arc $D_R(super(e)) \rightarrow D_{R\cup\{t\}}(e)$.
- **type 2:** For each affected edge e and each new edge $g \in G_{R\cup\{t\}}(e)$ there is an arc $D_R(super(e)) \rightarrow D_{R\cup\{t\}}(g)$.
- **type 3:** For each new and critical edge e and each edge $\overline{e}$ in $V^*(R) \cap \Gamma$ which contains an endpoint of e there is an arc $D_R(\overline{e}) \rightarrow D_{R\cup\{t\}}(e)$.
- **type 4:** For each new edge e and each edge $\overline{e}$ of $V^*(R)$ such that $\overline{e} \cap T(e)$ is nonempty and more than just a point there is an arc $D_R(\overline{e}) \rightarrow D_{R\cup\{t\}}(e)$.

It remains to verify the history graph invariants and to estimate the time needed to construct $H(R \cup \{t\})$. The first history graph invariant is clearly maintained.

Lemma 24 *The second history graph invariant is maintained:*
1. No node in $H(R \cup \{t\})$ has more than five children and
2. precisely the nodes corresponding to edges in $V^*(R \cup \{t\})$ have outdegree 0.

Proof: Observe first that in $H(R)$ precisely the nodes corresponding to edges in $V^*(R)$ have outdegree 0, that all arcs added go from nodes conflicting with t to nodes in N_t, and that for each node conflicting with t at least one outgoing arc is added. This proves the second claim.

For the first claim, let $\overline{e}$ be an arbitrary edge of $V^*(R)$ in conflict with t and let $\overline{D} = D_R(\overline{e})$. We distinguish the following cases:
Case 1: Assume first that there is an affected edge e in $V^*(R\cup\{t\})$ with $super(e) = \overline{e}$. Then there is at most one type 1 arc out of $\overline{D}$ and there are at most four type 2 arcs out of $\overline{D}$ (at most two for each endpoint of $\overline{e}$), for a total of five arcs.
Case 2: Otherwise there is no affected edge in $V^*(R\cup\{t\})$ with $super(e) = \overline{e}$. Then $\overline{I} = \overline{e} \cap VR^*_o(t, R\cup\{t\})$ is nonempty. Again we distinguish several cases:
Case 2.1: We first assume that $VR^*(t, R\cup\{t\})$ consists of rooted faces.
The edge $\overline{e}$ is either contained in Γ or it is not. In either case $\overline{e}\setminus\overline{I}$ is a single connected component (by Lemma 19) and hence there is at most one type 1 arc to a shortened edge e with $super(e) = \overline{e}$. There are no type 2 and type 3 arcs and there at most four type 4 arcs as there can be at most four edges e where $\overline{e}\cap T(e)$ is a non-trivial subsegment of $\overline{e}$. (This also covers the case when $\overline{e}$ is an edge on Γ, $VR^*(t, R\cup\{t\})$ has only one rooted face, and $\overline{I}$ consists of two segments.)
Case 2.2: If $VR^*(t, R\cup\{t\})$ consists of a single unrooted face then $\overline{I}$ is a single component by Lemma 10. There are two cases which have to be considered.
Case 2.2.1: Suppose now that $\overline{e}\subseteq\Gamma$.
Then there are two type 1 arcs to shortened edges, no type 2 arc, two type 3 arcs to the critical edges having an endpoint on $\overline{e}$, and one type 4 arc to the boundary edge of $VR^*(t, R\cup\{t\})$ on Γ.
Case 2.2.2: Finally, assume that $\overline{e}\subseteq V^*(R)\setminus\Gamma$.
Then there can be at most two type 1 arcs to shortened edges out of $\overline{D}$, no type 2 and 3 arcs, and at most four type 4 arcs to the new edges incident to the endpoints of $\overline{I}$. Moreover, if there are two type 1 arcs out of $\overline{D}$ then there must be a new edge connecting the two endpoints of $\overline{I}$ (by Lemma 7). Thus there are at most three type 4 arcs in this case. □

Finally, we turn to the third history graph invariant.

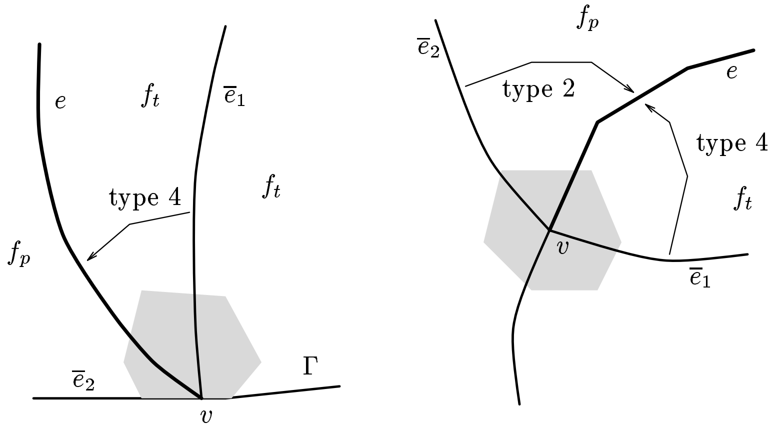
Lemma 25 *The third history graph invariant is maintained.*

Proof: It suffices to show that for all $D\in N_t$ and all $u\in S\setminus R\cup\{t\}$ which intersect D there is a node $\overline{D}\in H(R)$ such that u intersects $\overline{D}$ and $\overline{D}\to D$ is an arc in $H(R\cup\{t\})$.

Let e be the edge of $V^*(R\cup\{t\})$ with $D_{R\cup\{t\}}(e) = D$. By Theorem 1 e is unique. We distinguish several cases depending on whether e is new, shortened, or affected.
Case 1: Let e be either a shortened or an affected edge. Then let $\overline{e} = super(e)$ and $\overline{D} = D_R(\overline{e})$. Now observe that $e\subseteq\overline{e}$ and hence $e\cap VR^*_o(u, R\cup\{t,u\}) \subseteq \overline{e}\cap VR^*_o(u, R\cup\{u\})$. Thus u intersects $\overline{e}$ in $V^*(R)$ if u intersects e in $V^*(R\cup\{t\})$. Consequently, u intersects $\overline{D}$ according to Lemma 16. Now the type 1 arc $\overline{D}\to D$ supplies the desired connection.
Case 2: The case where e is new is more complicated. We distinguish several cases according to whether $e\subseteq\Gamma$ or not.
Case 2.1: Let $e\subseteq\Gamma$. Then we have type 4 arcs $D_R(\overline{e})\to D$ for all edges $\overline{e}\subseteq V^*(R)\cap\Gamma$ with $e\cap\overline{e}\neq\emptyset$. Thus if u intersects e in $V^*(R\cup\{t\})$ then u intersects

some edge $\overline{e}$ in $V^*(R)$ with $D_R(\overline{e}) \rightarrow D$.
Case 2.2: From now on assume that $e \subseteq V^*(R \cup \{t\}) \setminus \Gamma$. We need some additional notation. Let f_u be a face of $VR^*(u, R \cup \{t, u\})$ with $e \cap \mathrm{cl}\, f_u \neq \emptyset$, let f'_u be the face of $VR^*(u, R \cup \{u\})$ with $f_u \subseteq f'_u$, let $p \in R$ be such that e separates face f_p of $VR^*(p, R \cup \{t\})$ and face f_t of $VR^*(t, R \cup \{t\})$, and let f'_p be the face of $VR^*(p, R)$ with $f_p \subseteq f'_p$.

Assume first that some endpoint v of e lies in $\mathrm{cl}\, f'_u$. v is either a vertex of $V^*(R)$ or lies on an edge $\overline{e}$ of $V^*(R)$. In the latter case we have $\overline{e} \cap \mathrm{cl}\, f'_u \neq \emptyset$, i.e., u intersects $\overline{e}$ in $V^*(R)$. By Lemma 16, u must also intersect $D_R(\overline{e})$. But $D_R(\overline{e}) \rightarrow D$ was added as an arc of type 3 or type 4. In the former case, v is a vertex on the boundary of f'_p. Let $\overline{e}_1$ and $\overline{e}_2$ be the two edges of $\mathrm{bd}\, f'_p$ incident to v. Assume first that v is located on Γ and that $\overline{e}_1$ is the edge on $\mathrm{bd}\, f'_p$ which does not lie on Γ. Then $U_v \cap \overline{e}_1 \cap \mathrm{cl}\, f'_u \neq \emptyset$ for all neighbourhoods U_v of v by Lemma 8. Now observe that $D_R(\overline{e}_1) \rightarrow D$ is an arc of type 4. See also Figure 13a. Assume next that v is a Voronoi vertex in $V^*(R) \setminus \Gamma$. From Lemma 7 we get that $T_{f'_u} = (V^*(R) \setminus \Gamma) \cap \mathrm{cl}\, f'_u$ is a connected set and more than just a point. Moreover, we have $v \in T_{f'_u}$. On the other hand $e \cap \mathrm{cl}\, f'_u \neq \emptyset$ ensures that $\mathrm{cl}\, f'_u$ and $\mathrm{cl}\, f'_p$ have a nonempty intersection. We conclude that at least one edge out of $\{\overline{e}_1, \overline{e}_2\}$ is among the edges of $V^*(R)$ clipped by u at v. Thus $\mathrm{cl}\, f'_u$ intersects $\overline{e}_1$ or $\overline{e}_2$, i.e., u intersects $\overline{e}_1$ or $\overline{e}_2$ and hence $D_R(\overline{e}_1)$ or $D_R(\overline{e}_2)$. But $D_R(\overline{e}_1) \rightarrow D$ and $D_R(\overline{e}_2) \rightarrow D$ are arcs of type 2 and 4, respectively. See also Figure 13b.



a: $v \in \Gamma$ b: $v \notin \Gamma$

Fig. 13. $v \in \mathrm{cl}\, f'_u$ (The shaded region belongs to $\mathrm{cl}\, f'_u$)

So assume from now on that no endpoint of e lies in $\mathrm{cl}\, f'_u$. Since $e \cap \mathrm{cl}\, f_u \subseteq e \cap \mathrm{cl}\, f'_u$,the set $e \cap \mathrm{cl}\, f_u$ must be an "INNER_SEGMENT" of e and hence f_u is an unrooted face. Thus $(V^*(R) \setminus \Gamma) \cap \mathrm{cl}\, f_u = e \cap \mathrm{cl}\, f_u$ is an inner segment of e. Since e separates f_p and f_t it follows that $\mathrm{bd}\, f_u \cap \Gamma$ is either an inner segment of $\mathrm{bd}\, f_p \cap \Gamma$ or an inner segment of $\mathrm{bd}\, f_t \cap \Gamma$. Again we distinguish several cases:

Assume first that e is critical and hence f_t is unrooted. Let $\overline{e}$ be the edge of

$V^*(R)$ with $\overline{e} = \mathrm{bd}\, f'_p \cap \Gamma$. Then $\overline{e} = (\mathrm{bd}\, f_p \cup \mathrm{bd}\, f_t) \cap \Gamma$ and hence

$$\begin{array}{rlcl} \emptyset & \neq \mathrm{bd}\, f_u \cap \Gamma & = & \mathrm{bd}\, f_u \cap (\mathrm{bd}\, f_u \cap \Gamma) \\ & \subseteq \mathrm{bd}\, f_u \cap ((\mathrm{bd}\, f_p \cup \mathrm{bd}\, f_t) \cap \Gamma) & = & \mathrm{bd}\, f_u \cap \overline{e} \\ & \subseteq \mathrm{cl}\, f_u \cap \overline{e} & \subseteq & \mathrm{cl}\, f'_u \cap \overline{e}. \end{array}$$

Thus u intersects $\overline{e}$, resp. $D_R(\overline{e})$. The type 3 arc $D_R(\overline{e}) \to D$ supplies the desired connection.

Assume finally, that e is noncritical. Our goal is to show that $T(e) \cap \mathrm{cl}\, f'_u$ is nonempty and more than just a point. Then we can infer that u intersects an edge $\overline{e}$ of $V^*(R)$ with $\overline{e} \cap T(e) \neq \emptyset$ and observe that $D_R(\overline{e}) \to D$ is an arc of type 4.

Claim: $T(e) \cap \mathrm{cl}\, f'_u$ is nonempty and more than just a single point.

Proof: We have either $\mathrm{bd}\, f_u \cap \Gamma \subseteq \mathrm{bd}\, f_t \cap \Gamma$ or $\mathrm{bd}\, f_u \cap \Gamma \subseteq \mathrm{bd}\, f_p \cap \Gamma$. In the former case the claim obviously holds because each curve connecting e and $\mathrm{bd}\, f_u \cap \Gamma$ and running inside $f_t \cap f_u$ must intersect $T(e)$.

So assume that $\mathrm{bd}\, f_u \cap \Gamma \subseteq \mathrm{bd}\, f_p \cap \Gamma$. (For an illustration see Figure 14.) We will argue next that $f_u \cap f_p = f'_u \cap f_p$. Since $f_u \subseteq f'_u$, the relation $f_u \cap f_p \subseteq f'_u \cap f_p$ is certainly true. To show the converse, recall that $f'_u \subseteq D^*(u,p)$ and that $f_p \subseteq D^*(p,t)$. Thus, by Fact 1,

$$f'_u \cap f_p = (f'_u \cap D^*(u,p)) \cap (f_p \cap D^*(p,t)) \subseteq f'_u \cap f_p \cap D^*(u,t) \subseteq f_u \cap f_p$$

Also recall that $\mathrm{cl}\, f_u$ does not intersect $\mathrm{bd}\, f_p \setminus (e \cup \Gamma)$. Consequently, $\mathrm{cl}\, f'_u$ cannot intersect $\mathrm{bd}\, f_p \setminus (e \cup \Gamma)$. Since $f_p \subseteq f'_p$ we conclude that f'_u is an unrooted face in $V^*(R \cup \{u\})$. Lemma 7 ensures that $T_{f'_u} = (V^*(R) \setminus \Gamma) \cap \mathrm{cl}\, f'_u$ is nonempty and more than just a single point. Now $f_p \subseteq f'_p$ and $\mathrm{cl}\, f'_u \cap (\mathrm{bd}\, f_p \setminus (e \cap \Gamma)) = \emptyset$ imply that $\mathrm{cl}\, f'_u$ must intersect $T(e)$ in more than one point. $\square$

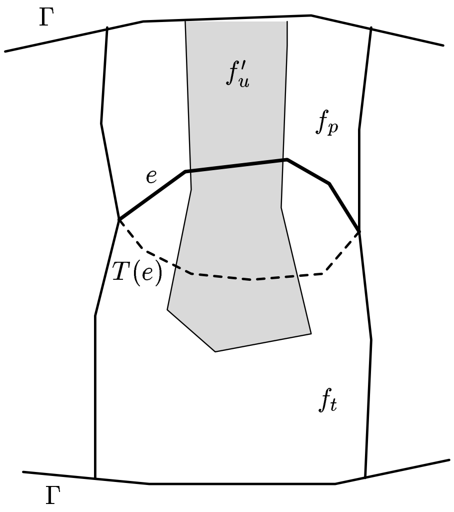


Fig. 14. $\mathrm{bd}\, f_u \cap \Gamma \subseteq \mathrm{bd}\, f_p \cap \Gamma$

Let x be a point in $T(e) \cap \mathrm{cl}\, f'_u$. x can be chosen to lie on an edge $\overline{e}$ of $V^*(R)$. Since $x \in \mathrm{cl}\, f'_u$, u intersects $\overline{e}$ and hence $D_R(\overline{e})$, according to Lemma 16. Thus the type 4 arc $D_R(\overline{e}) \rightarrow D$ supplies the desired connection. This completes the proof of Lemma 25. □

It remains to estimate the time consumed to update the history graph.

Lemma 26 *$H(R \cup \{t\})$ can be constructed from E_t and $V^*(R \cup \{t\})$ in time $O(|E_t|) = O(c)$.*

Proof: The descriptions in N_t can be inferred from $V^*(R \cup \{t\})$ in constant time per description. Also, $|N_t| \leq 5|E_t|$ according to Lemma 24. Computing the arcs of types 1,2 and 4 only requires another traversal around T_f^+. This takes time $O(|E_t|)$. For arcs of type 3 note that there can be at most two arcs of this type which can be found in constant time. □

4.6. Complexity Analysis

We summarize our result in:

Theorem 2 *The furthest site abstract Voronoi diagram of a set of n sites can be computed by a randomized algorithm in expected time $O(n \log n)$ and expected space $O(n)$. The expected time to insert the n-th site is $O(\log n)$.*

Proof: We apply the analysis of Ref.[4] for randomized incremental constructions. Initializing the data structures requires $O(1)$ time. In the i-th iteration we have to compute E_t, $V^*(R \cup \{t\})$, $H(R \cup \{t\})$ for $t = t_i$ $(3 \leq i < n)$. Lemmas 17, 22, 26 ensure that $O(c)$ time suffices to perform these steps, where c is the number of nodes of $H(R_i)$ in conflict with t.

Thus the assumptions made in Ref.[4] are met. By Theorems 3 and 4 of Ref.[4] and Lemma 4 the expected size of c is $O(\log i)$ and the expected size of $H(R_i)$ is $O(i)$. This implies the stated time and space bounds. □

5. Concluding remarks

We have presented an algorithm computing the furthest site abstract Voronoi diagram. Its most important features are its generality, as it applies to all abstract Voronoi diagrams, its modularity, as only the basic operation depends on the particular kind of diagram, and its simplicity. We admit, however, that the correctness proof is complicated.

It would be desirable to extend the algorithm such that it can compute abstract Voronoi diagrams of arbitrary order.

We want to add a few remarks on the applicability of the abstract Voronoi diagram approach. We admitted already that the topological and combinatorial concepts presented in the paper require a detailed and careful discussion in several cases. However, the application of the framework is usally easy as we argue next.

The key point is here, that one only needs to go through a few steps to get a solution for a new situation. So let us assume that the framework has been established and the purely combinatorial part of the algorithm has been implemented.

We outline the procedure of how to adapt the framework to a new type of Voronoi diagram. There are two tasks to perform.

preparation step: In the preparation step one has to prove that the Voronoi diagram being under consideration belongs to the class of abstract Voronoi diagrams. This can be done by proving the conditions established in section 2.1.

implementation step: The second step consists in the implementation of the basic operation declared in section 4.1.

With respect to the first step we observed that for the most common types of diagrams very similiar arguments work. So we found it sometimes more convenient to establish stronger properties than the definition of abstract Voronoi diagrams requires. We summarize this experience in the following methodical hints.

- Usually, the sites are convex, closed, and pairwise disjoint subsets of $\mathbb{R}^2$. Thus the first two conditions are obtained by observing $p \cap q = \emptyset$ and $p \subseteq D(p,q)$ for $p, q \in S$. The boundary condition $\operatorname{bd} D(p,q) = \operatorname{bd} D(q,p)$ usually follows from the continuity of the distance function.
- Conditions 2 and 3 are concerned with the behaviour of bisectors. The investigation of the bisectors seems to be the most time consuming step. We recommend a careful inspection of the bisectors, especially with regard to the implementation step. Therefore bisectors should be described in its general form.

  Bisectors are very often composed of a finite set of straigt line segments, parabolic arcs, and hyperbolic arcs. For example, parabolas occur in regions of the plane, where the bisector of two sites $p, q \in S$ is formed by a corner of p and a boundary line segment of q or vice versa. Hyperbolas occur, if a part of the bisector is defined by two circle arcs or in the case, when additional additive weights come into play.

  Modelling real scenarios may result in sites with complex shapes. Generally, it is preferable to consider complex sites as composed objects formed by elementary geometric objects like line segments or circle segments since it results in simple bisectors and less efforts in the implementation of the basic operation.
- Next, the shape of the dominance regions has to be investigated. If possible, it seems to be more convenient to characterize the dominance regions as *star shaped point sets* with respect to the sites. For non-trivial sites, star shapeness should be defined as follows: A set $M \subseteq \mathbb{R}^2$ is called *star shaped* w.r.t. a site $p \in S$ if and only if for each $x \in M \setminus p$ the straight line segment connecting x with its nearest point within p is contained in M. Observe, that the property of star shapeness is maintained, if finitely many star shaped sets are intersected. Since star shaped sets are path connected sets, condition 4A very often can be based on the star shapeness of the dominance regions in conjunction with the property that the sites are disjoint.

- In order to show the final condition 4B one may utilize fact 1. According to fact 1, it suffices to show the relation $R_<(p,q) \cap R_<(q,r) \subseteq R_<(p,r)$ for any three sites $p,q,r \in S$. We recommend to choose an arbitrary point $x \in \mathbb{R}^2$ and to compare the distances of x to the three sites p,q,r.

The types of Voronoi diagrams known to be in the class of abstract Voronoi diagrams cover disjoint line segment, disjoint circle sites under the Euclidean metric, point sites under the L_p-norms $(1 < p < \infty)$ as well as power diagrams for disjoint circles.

If one succeeds in showing that a particular kind of Voronoi diagram belongs to the class of abstract Voronoi diagrams it remains to implement the basic operation for this kind of diagram. This operation involves up to five sites in the computation. However, the interface of our basic operation does not require the computation of a Voronoi diagram. It only requires to report a symbol describing the combinatorial structure of the intersection of an Voronoi edge e with the region of a new site t. This observation simplifies the efforts in implementing the basic operation since one may reduce this task to the comparision of the relative positions taken by the sites.

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