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THE 'ALMOST ALL' THEORY OF SUBRECURSIVE DEGREES IS DECIDABLE

Kurt Mehlhorn*
Department of Computer Science
Cornell University
Ithaca, New York 14850

Abstract:

We use constructive measure theory to show the decidability of the 'almost all' theory of subrecursive degrees. The formulas of this theory are built up using the constant 0 standing for the minimum degree, the functions $\cup$, $\cap$ standing for the join and meet of two degrees respectively, the relation $\leq$ standing for the reducibility the logical connectives &, $\neg$, and the quantifier (for almost all a). An efficient decision procedure is described.

1. Introduction

Subrecursive reducibility relations allow us to classify the set of all recursive functions into subrecursive degrees. Several such relations were studied by different investigators; e.g. 'primitive recursive in' [5], 'elementary recursive in' [9] and 'polynomial computable in' [3,4,8]. All research mentioned above investigates subrecursive degrees from a qualitative viewpoint. We take a different approach here (as we did in [6]). We are interested in quantitative statements about subrecursive degrees. Using constructive measure theory [1,2] we define: A property P is true for 'almost all' subrecursive degrees if and only if the set {f; f is recursive and P(deg(f))} has measure 1.

Many statements about subrecursive degrees can be formulated in the following language: The constant 0 stands for the minimum degree, the functions $\cap$, $\cup$ stand for the meet and join of two degrees respectively and the relation $\leq$ stands for the reducibility relation. Adding the logical connectives &, $\vee$, $\neg$ and the quantifier $\overset{\infty}{\forall}$ (for almost all) gives the 'almost all' theory of subrecursive degrees. This theory is decidable; an efficient decision procedure is described. Our proof follows closely Stillwell's proof of the corresponding result for Turing degrees [13].

As a byproduct we obtain a result about minimal pairs of degrees which generalizes a result of Ladner [4].

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2. A Constructive Measure on $\mathcal{R}$

In this section we define a constructive measure on the subsets of $\mathcal{R}$. We outline the construction (the full construction can be found in [7]).

(2.1) Definition: a) $\mathcal{R} = \{f;\ f$ is recursive and 0-1 valued$\}$

b) Q is the set of rational numbers

c) $\mathbb{R}$ is the set of constructive real numbers (Bishop). A sequence $\{x_n\}_{n=0}^{\infty}$ of rational numbers is a real if for all n and m $|x_n - x_m| \leq n^{-1} + m^{-1}$.

d) (a metric on $\mathcal{R}$) Let f, g $\in \mathcal{R}$.

$$d(f,g) = \begin{cases} 1/n+1 & \text{if } n = \mu x[f(x) \neq g(x)] \\ 0 & \text{otherwise} \end{cases}$$

e) A function $F: \mathcal{R} \to \mathbb{R}$ is continuous if there is a function $\omega: \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ $|F(f) - F(g)| < \varepsilon$ whenever $d(f,g) < \omega(\varepsilon)$

At a first glance, the definition of the metric d does not seem to be constructive; after all f=g is undecidable. But note that d maps $\mathcal{R} \times \mathcal{R}$ into the real numbers and not into the rational numbers. Thus d produces a sequence of rationals for every pair f,g of computable functions. d might proceed as follows. In order to produce the n-th element of the sequence, d checks if there is a $x < n$ such that $f(x) \neq g(x)$. If there is such an x, let x_0 be the least such x. Then d generates $1/(x_0+1)$ as the n-th element, otherwise it generates 1/n.

We denote the set of continuous functions from $\mathcal{R}$ to $\mathbb{R}$ by $C(\mathcal{R})$. The next theorem is an interesting characterization of $C(\mathcal{R})$.

(2.2) Definition: a) Let $g \in \mathcal{R}$ and $n \in \mathbb{N}$. g_n denotes the initial segment of g with domain $\{0,1,\ldots,n-1\}$.

$$g_n = g|\{0,1,\ldots,n-1\}$$

b) $T = \{t;\ t: \{0,\ldots,n-1\} \to \{0,1\}\}$ is the set of functions of finite support

(2.3) Theorem: $F: \mathcal{R} \to \mathbb{R}$ is continuous iff there is a pair $f_1: T \to Q$, $f_2: \mathbb{N} \to \mathbb{N}$ such that for all $g \in \mathcal{R}$

$$\{f_1(g_{f_2(n)})\}_{n=0}^{\infty} \text{ is a real number}$$

and

$$F(g) \equiv \{f_1(g_{f_2(n)})\}_{n=0}^{\infty}$$ □

Using this theorem we are able to define a positive measure on $C(\mathcal{R})$. Let $F \in C(\mathcal{R})$ be defined by $f_1: T \to Q$ and $f_2: \mathbb{N} \to \mathbb{N}$ in the sense of theorem (2.3). We set

$$\int F d\mu = \left\{ 2^{-f_2(n)} \sum_{\substack{t \in T \\ \ell h(t) = f_2(n)}} f_1(t) \right\}_{n=0}^{\infty}$$

<u>(2.5) Theorem</u>: The mapping $F \to \int F d\mu$ is a positive measure of $C(\mathcal{R})$ in the sense of [2].

<u>(2.6) Example</u>: Let $S(f,\frac{1}{n}) = \{g; d(f,g) < \frac{1}{n}\}$ be the open sphere of radius $\frac{1}{n}$ about $f \in \mathcal{R}$. Its characteristic function λ is continuous and $\int \lambda\ \mu = \mu S(f,\frac{1}{n}) = 2^{-n}$.

A subset of $\mathcal{R}$ is integrable if its characteristic function is integrable. Open spheres are integrable.

<u>(2.7) Definition</u>: The class $B(\mathcal{R})$ of Borel sets of $\mathcal{R}$ is the smallest class of sets which contains the open spheres and is closed under countable unions and intersections.

<u>(2.8) Theorem [2]</u>: All Borel sets are integrable.

3. <u>The Decision Procedure</u>: Let $\leq$ be any subrecursive reducibility relation, e.g. 'elementary recursive in'. For $f \in \mathcal{R}$ the degree of f is defined as

$$d(f) = \deg(f) = \{g;\ g \in \mathcal{R} \text{ and } g \leq f \text{ and } f \leq g\}.$$

Let $\{Op_i[\ ]\}$ be an enumeration of the elementary recursive operators, i.e. $f \leq g$ iff $f = Op_i[g]$ for some i. There is a minimum degree

$$0 = \{g;\ g \leq f \quad \text{for all } f \in \mathcal{R}\}$$

e.g. 0 is the class of elementary functions. The relation $\leq$ on $\mathcal{R}$ induces a partial order $\leq$ on the set of degrees.

$$\deg(f) \leq \deg(g) \iff f \leq g.$$

The set of subrecursive degrees forms an upper semi-lattice under this ordering; we denote the join of two degrees a and b by $a \cup b$. Ladner [4] showed that the meet of two degrees does not always exist; if it exists we denote it by $a \cap b$. We consider formulae which are built up from the constant 0, the functions $\cup$, $\cap$, the relation $\leq$, the logical connectives &, $\vee$ and $\neg$, and the quantifier $\overset{\infty}{\forall} a$ (for almost all a). Let P(a) be any formula. Then

$$\overset{\infty}{\forall} a\ P(a) \iff \mu\{f;\ P(\deg(f))\} = 1.$$

In the following we describe the decision procedure and give a correctness proof. Our proof resembles Stillwell's proof [13] for the decidability of the 'almost all' theory of Turing degrees. The proof consists of three parts. Firstly, we show a normal form theorem for terms: every term is "equivalent" to a term containing only the function symbol $\cup$. Secondly, we show that $t_1(a_1,\dots,a_n) \leq t_2(a_1,\dots a_n)$ is either true for almost all $a_1,\dots,a_n$ or false for almost all $a_1,\dots,a_n$. Thirdly, we prove by induction on the rank of a formula $F(a_1,\dots,a_n)$ that the set $\{(f_1,\dots,f_n); F(\deg(f_1),\dots,\deg(f_n))\}$ has either measure 0 or 1.

We need several lemmas. We have to know that the set of instances satisfying any formula in our theory is integrable.

<u>(3.1) Lemma</u>: Let F be a formula with the free variables $a_1,\dots,a_n$. Then the set $\{(f_1,\dots,f_n), F[d(f_1),\dots,d(f_n)]\}$ is integrable.

<u>Proof:</u> The proof is by induction on the rank of the formula F. The induction step is trivial (Fubini's theorem) and therefore left to reader. It remains to prove the assertion for atomic formula $t_1(a_1,\dots,a_n) \leq t_2(a_1,\dots,a_n)$ where $a_1,\dots,a_n$ are terms. We proceed by induction on the rank of the terms. Again the induction step is simple. The basic atomic formula are $a_1 \leq a_2$, $a_1 \leq a_2 \cup a_3$, $a_1 \cup a_2 \leq a_3$, $a_1 \cap a_2 \leq a_3$ and $a_1 \leq a_2 \cap a_3$. We show that the set of instances satisfying any of these formula is a Borel set. E.g. we consider $a_1 \leq a_2$. Let $t \in T$, and let U_t stand for the open sphere $\{f;\ t \subset f\}$. If $Op[t](w)$ is defined then $Op[f](w) = Op[t](w)$ for all $f \in U_t$. Therefore

$$\{(f,g);\ d(f) \leq d(g)\} = \{(f,g);\ f = Op_i[g] \text{ for some } i\}$$

$$= \bigcup_{i \in N} \{(f,g);\ f = Op_i[g]\}$$

$$= \bigcup_{i \in N} \bigcap_{s \in T} \left[(U_{Op_i[s]} \times U_s) \cup \bigcup_{\substack{s \not\subseteq t \\ t \not\subseteq s}} (\mathscr{R} \times U_t) \right]$$

The only other nontrivial case is $a_1 \cap a_2 \leq a_3$.

$$\{(f,g,h);\ d(f) \cap d(g) \leq d(h)\} =$$

$$= \{(f,g,h);\ \forall i \forall j \exists \ell [Op_i[f] = Op_j[g] \Rightarrow Op_i[f] = Op_\ell[h]]\}$$

$$= \bigcap_{i\in N} \bigcap_{j\in N} \bigcup_{\ell\in N} \{(f,g,h);\ Op_i[f] \neq Op_j[g] \vee Op_i[f]=Op_\ell[h]\}.$$

Furthermore

$$\{(f,g,h);\ Op_i[f] \neq Op_j[g] \vee Op_i[f] = Op_\ell[h]\}$$

$$= \{(f,g,h);\ Op_i[f] \neq Op_j[g]\} \cup \{(f,g,h);\ Op_i[f] = Op_\ell[h]\}$$

We show that the first set is Borel.

$$\{(f,g,h);\ Op_i[f] \neq Op_j[g]\} = \bigcap_{\substack{s,t,v \in T \\ Op_i[s] \not\subset Op_j[t] \vee \\ Op_j[t] \not\subset Op_i[s]}} U_s \times U_t \times U_v$$

□

<u>(3.2) Lemma</u>: Let F be a formula. Then

$$\mu\{f; F(d(f))\} = \mu\{(f,g);\ F(d(f) \cup d(g))\}$$

<u>Proof</u>: The set of instances d(f) satisfying F is a Borel set by lemma (5.1). It is therefore sufficient to show that the mapping G: $B(\mathscr{R}) \times B(\mathscr{R}) \to B(\mathscr{R})$ induced by $(f,g) \to f$ <u>join</u> g is measure preserving. This is obvious for open spheres; induction yields the assertion for all Borel sets. □

<u>(3.3) Lemma</u>: Given any $f,g \in \mathscr{R}$. If $f \nleq g$ then

$$\mu\{h;\ d(f) \leq d(g) \cup d(h)\} = 0.$$

<u>Proof</u>: Since $\mu\{h;\ d(f) \leq d(g) \cup d(h)\} =$

$$= \mu(\bigcup_{i \in N} \{h;\ f = Op_i[g \text{ join } h]\})$$

$$\leq \sum_{i \in N} \mu\{h;\ f = Op_i[g \text{ join } h]\}$$

we have only to prove $\mu\{h; f = Op_i[g \text{ join } h]\} = 0$ for all i. Assume $\mu\{h;\ f = Op_i[g \text{ join } h]\} > 0$. Since $\{h; f = Op_i[g \text{ join } h]\} = \bigcap_{n \in N} \{h;\ f_n \subset Op_i[g \text{ join } h]\}$ the sequence $\{\mu\{h;\ f_n \subset Op_i[g \text{ join } h]\}\}$ is a monotonically decreasing sequence of positive numbers. Therefore

$$\mu\{h;\ f = Op_i[f \text{ join } h]\}/\mu\{h;\ f_n \subset Op_i[g \text{ join } h]\}$$

is defined for all n and converges against 1. Hence we can find a $n_o \in N$ (by searching) such that

$$(*) \quad n \geq n_o \Rightarrow \mu\{h; f=Op_i[g \text{ join } h]\}/\mu\{h; f_n \subset Op_i[g \text{ join } h]\} > 3/4$$

Let S be the set of initial segments t such that

$$f_{n_o} \subseteq Op_i[g \text{ join } t] \quad \text{for } t \in S$$

but not

$$f_{n_o} \subseteq Op_i[g \text{ join } t'] \quad \text{for any } t' \subset t.$$

S is a finite set. We can rewrite (*) as:

$$(**) \qquad \frac{\displaystyle\sum_{t \in S} \mu(U_t) \frac{\mu\{h;\ f = Op_i[g \text{ join } h]\ \&\ t \subset h\}}{\mu(U_t)}}{\displaystyle\sum_{t \in S} \mu(U_t)} > 3/4$$

We can interpret (**) as a weighted average. Therefore there exists some $t_o \in S$ such that

$$3/4 \leq \mu\{h;\ f = Op_i[g \text{ join } h]\ \&\ t_o \subset h\} / \mu(U_{t_o})$$

We can now give a simple program for computing f from g. It is easy to show that there is an operator $Op_i'[\ \ ]$ such that $Op_i[g \text{ join } h](n)$ converges at $Op_i'[g](n)$ for all $h \in \mathcal{R}$; i.e. that the arguments of all calls of h are smaller than $Op_i'[g](n)$. (See [5]). The following program computes f.

```
if x < n_o then Op_i[g join t_o] (x)
            else
                 ℓ ← Op_i'[g](x) .
                 compute Op_i[g join t](x) for
                 all t such that
                 t_o ⊆ t and ℓh(t) = ℓ .
                 Output the value which is given
                 more than 3/4 of the times.
```

It is obvious from the remarks preceding the program that this program computes f. It is again an operator in the class $\{Op_i\}$ since all t's such that $\ell h(t) = \ell$ can be described by integers $\leq 2^{\ell}$ and exponentiation is already elementary recursive. □

<u>(3.4) Corollary</u>: Given any $f, g \in \mathcal{R}$. Then for almost all h $(d(f) \cup d(g)) \cap (d(f) \cup d(h))$ is defined and equal to $d(f)$.

Proof: The set of h such that

$$(d(f) \cup d(g) \cap (d(f) \cup d(h))$$

is either undefined or not equal to d(f) can be written as follows:

$$\bigcup_{d(i) \leq d(f) \cup d(g)} \{h;\ d(f) < d(i) \leq d(f) \cup d(h)\}$$

By lemma (3.3) all these sets have measure 0. □

Corollary (3.4) enables us to compute the meet of any two terms $t_1 = a_1 \cup \ldots \cup a_n \cup b_1 \cup \ldots \cup b_m$ and $t_2 = a_1 \cup \ldots \cup a_n \cup c_1 \cup \ldots \cup c_k$, namely $t_1 \cap t_2 = a_1 \cup \ldots \cup a_n$ for almost all instances of the variables. We tacitly used here that $b_1 \cup \ldots \cup b_m$ and $c_1 \cup \ldots \cup c_k$ are 'random degrees' (lemma 3.2). Repeated application of this process produces the normal form of any term.

(3.5) Theorem: For every term t there is a term $t' = a_1 \ldots a_n$ such that $t = t'$ for almost all instances of the variables. □

We turn now to the relation $\leq$. Let t_1, t_2 be terms. Then $t_1 \leq t_2$ if and only if $t_1 \cap t_2$ exists and $t_1 \cap t_2 = t_1$. We can assume w. l. o. g. that t_1, t_2 are in normal form, i.e. $t_1 = a_1 \cup \ldots \cup a_n \cup b_1 \cup \ldots \cup b_k$ and $t_2 = a_1 \cup \ldots \cup a_n \cup c_1 \cup \ldots \cup c_m$. Then $t_1 \cap t_2 = a_1 \cup \ldots \cup a_n$ for almost all instances of the variables. Therefore $t_1 \leq t_2$ if and only if $k = 0$. Hence either $t_1 \leq t_2$ for almost all instances of the variables or $t_1 \nleq t_2$ for almost all instances of the variables. So every atomic formula is satisfied by a set of instances whose measure is either 0 or 1. We say that atomic formulas are 0-1 valued.

We show this fact for all formulas by induction.

If $\phi = \phi_1 \ \& \ \phi_2$ with ϕ_1 and ϕ_2 0-1 valued then the 0-1 valuedness of ϕ follows from the following property of measures:

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

The same property yields the assertion for $\phi = \neg \phi_1$.

Assume now: $\phi = (\overset{\infty}{\forall} a_1) \phi_1(a_1, a_2, \ldots, a_n)$ with 0-1 valued ϕ_1. If $n = 1$ then ϕ is either true or false and hence 0-1 valued, if $n > 1$ then the assertion follows from Fubini's theorem.

(3.6) Theorem: Every formula is 0-1 valued.

(3.7) Corollary: The 'almost all' theory of subrecursive degrees is decidable.

We give now two examples of theorems which can be proved using the decision procedure.

$$\underbrace{\overset{\infty}{\forall} a \; \underbrace{\overset{\infty}{\forall} b \; \underbrace{(\underbrace{\neg\underbrace{(a \leq b)}_{\text{0-valued}}}_{\text{1-valued}} \;\&\; \underbrace{\neg\underbrace{(b \leq a)}_{\text{0-valued}}}_{\text{1-valued}})}_{\text{1-valued}}}_{\text{1-valued}}}_{\text{true}}$$

This theorem states that almost all pairs of degrees are incomparable. The next theorem strengthens this result. It states that almost all pairs of degrees are incomparable in a very strong sense.

$$\overset{\infty}{\forall} a \; \overset{\infty}{\forall} b \quad (a \cap b = 0)$$

i.e. their meet is the least degree.

Lemma (3.2) states that $a \cup b$ is a 'random degree'. Therefore we can infer from the second example.

$$\overset{\infty}{\forall} c \; \exists a \; \exists b \quad (c = a \cup b \;\&\; a \neq 0 \;\&\; b \neq 0 \;\&\; a \cap b = 0)$$

From (3.4) we infer an interesting result about minimal pairs of degrees.

<u>(3.8) Theorem</u>: Let $0 < a$. Then there is a degree b such that $0 < b$ and $c \leq a, b$ implies $c = 0$.

<u>Proof</u>: Let $0 = \deg(f)$ and $a = \deg(g)$. Then for almost all h $0 = a \cap \deg(h)$. Hence there is a h such that $0 < b = \deg(h)$ and $0 = a \cap b$.

Finally we estimate the complexity of the decision procedure. A formula of length n contains at most n different variables. Thus we may describe a term by a binary vector of length n. The basic operations on terms correspond to boolean operations on these binary vectors. The cost of one such operationiis bounded by n, the number of such operations is also bounded by n. Hence the decision procedure can be implemented in time n^2.

(3.9) Theorem: The decision procedure has time complexity n^2. □

4. Conclusion: The 'global' structure of subrecursive degrees is simple.

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