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A PARTIAL ANALYSIS OF HEIGHT-BALANCED TREES UNDER RANDOM INSERTIONS AND DELETIONS*

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Abstract. We describe a fringe analysis of AVL-trees (and 2-3-trees and HB-trees) under random insertions and deletions. Previously, only the case of random insertions was dealt with.

Key words. AVL-trees, random insertions and deletions, fringe analysis.

1. Introduction. Balanced tree schemes such as 2-3-trees [1], B-trees [2], AVL-trees [12] and BB[α]-trees [17] are very popular and useful data structures for manipulation of ordered lists of keys. Their worst case behavior for a single search, insertion or deletion is well understood (Knuth [12], Mehlhorn [14]). Recently, there has been progress in the study of the worst case behavior of sequences of such operations (Blum and Mehlhorn [3], Brown and Tarjan [6], Mehlhorn [16], Huddleston and Mehlhorn [9]); in these papers bounds on the total cost of rebalancing operations required to process a sequence of searches, insertions and deletions are derived. In contrast, very little is known about the average case behavior of balanced tree schemes, i.e., about their behavior for random inputs. To analyse the storage utilization in 2-3-trees and B-trees under random insertions, Yao [19] introduced the concept of fringe analysis. Subsequently, Brown [4] applied this type of analysis to AVL-trees under random insertions.

In this paper, we will carry out a fringe analysis of AVL-trees and 2-3-trees under random insertions and deletions. Our model of randomness is the following: At any instant of time, we are equally likely to perform an insertion or deletion, and each external node (leaf) is equally likely to be split in the case of an insertion or removed in the case of a deletion.

AVL-trees can be used to store ordered sets S of keys. In terms of the key space, our randomness assumption may be formulated as follows: At any instant of time, we are equally likely to perform an insertion or deletion. In the case of an insertion, the new key goes into each of the gaps between the keys present which equal probability. In the case of a deletion, each key present is chosen for deletion with equal probability. This assumption corresponds to assumption (I_0, D_r) in Knuth [13], and was used before in Flajolet et al. [8]. As shown by Knott's phenomenon [11], our assumption is not equivalent to the condition that keys are drawn independently from a uniform distribution. The latter assumption, called (I_r, D_r) in [13], leads to an extremely involved analysis even for very simple algorithms (Jonassen and Knuth [10]).

The fringe of a tree is obtained by deleting all nodes which are not close to the leaves, e.g., by deleting all nodes of height $\geqq k$ or by deleting all nodes which have at least k leaves below them. Note that only a small percentage of the total number of nodes will be deleted in this way, and hence fringe analysis can give valuable insights. The fringe is a system of trees of small height; in particular, it is a system of trees drawn from a finite collection C of trees. The composition of the fringe can be described by counting, for each tree of that finite collection, the number of times it occurs in the fringe. We can now study the effect of random insertions and deletions on the composition of the fringe.

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If our finite collection C of trees is "closed", i.e., if the effect of an insertion or deletion on a tree $T \in C$ is determined by T alone and transforms T into other (maybe several) trees of C, then the effect of an insertion or deletion on the composition of the fringe can be studied without reference to the entire tree. Hence the study of random trees reduces to the study of a relatively simple Markov chain.

If *insertion is the only operation* under consideration, then closed classes are known for some balanced tree schemes. The subtrees of height k form a closed class for 2-3-trees and B-trees (Yao [19], Brown [5]). The subtrees with three or fewer leaves form a closed class for AVL-trees (Brown [4]). No other closed classes are known for AVL-trees. Even worse, if we use B-trees with the overflow mechanism (Bayer and McCreight [2]), i.e., if a node is only split if the brother pages are also full, then no closed classes exist.

If insertions and deletions are the operations under consideration, then the situation is completely hopeless. Closed classes do not exist for any balanced tree scheme.

In this paper we show how to do fringe analysis without the burden of finding a closed class of subtrees, and (partially) analyse AVL-trees under random insertions and deletions. The same approach can be applied to obtain a higher order analysis of AVL-trees under insertions only (Mehlhorn [15]), and to B-trees with the overflow mechanism (Eisenbart and Mehlhorn [7]).

The idea behind our approach is quite simple. Suppose that the fringe consists of two types of trees, type I and type II, and suppose further that the effect of a deletion from a type I tree depends on the environment of that tree in the entire tree, say whether the brother is type II or not. Since the information about the type of the brother is lost when we pass to the fringe, we introduce an (unknown) probability for the event that the brother of a type I tree is a type II tree. A key observation is the fact that this probability, though unknown, cannot assume arbitrary values between 0 and 1 (Lemma 6 below). Recurrence relations describing the effect of a random insertion or deletion on the composition of the fringe are then derived in the standard way, i.e., by recurrence relations for the behavior of a Markov chain with unknown transition probabilities (§ 2). These recurrence relations can be solved very easily under the (unjustified) assumption that the probability mentioned above does not depend on the size of the trees. Fortunately, one can show that these solutions are also valid without that assumption (§ 3). In § 4 we use our results to obtain bounds on the number of balanced nodes in random AVL-trees, and on the total number of nodes in random 2-3-trees and HB-trees.

2. A partial analysis of AVL-trees. Let T be an AVL-tree. The *fringe* of T is obtained by deleting all nodes which have more than three leaves below them. (See Fig. 1 for an example.) This definition of fringe is due to Brown [4]. The fringe of an AVL-tree consists of two types of trees, subtrees with 3 leaves form the first type and

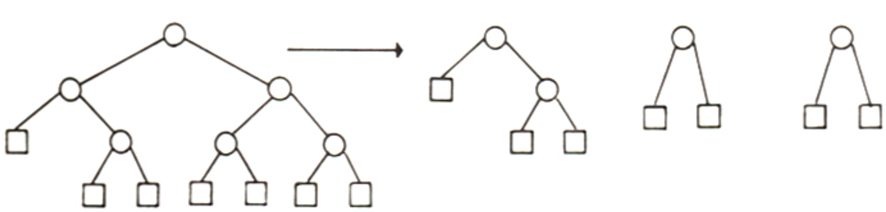

FIG. 1. *An AVL-tree and its fringe.*

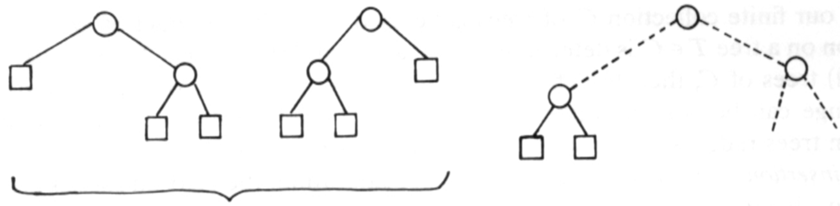

FIG. 2. *The subtrees in the fringe.*

subtrees with 2 leaves form the second type (see Fig. 2). Note that the brother of the root of a type II subtree is an internal node and therefore has two sons. Brown calls type I and type II subtrees M- and N-subtrees respectively.

DEFINITION 1. For an AVL-tree T let $a_1(T)$ ($a_2(T)$) be the number of type I (type II) subtrees in the fringe of T.

LEMMA 1. *Let T be an* AVL-*tree with n leaves. Then* $3a_1(T)+2a_2(T)=n$.

Proof. A type I subtree has 3 leaves and a type II subtree has 2 leaves, and every leaf is in a type I subtree or a type II subtree. □

Next we need to study the effect of an insertion into (or a deletion from) an AVL-tree T on the number of type I and type II subtrees. We need some notation first. The height of a node is the length (= number of edges) of the longest path to a leaf. The height of a tree is the height of its root. The balance factor $b(v)$ of an internal node v is the height of the left son of v minus the height of the right son of v. Finally, we assume that the insertion and deletion algorithms are as described in Knuth [12].

LEMMA 2. *Let T be an* AVL-*tree. Suppose that a new leaf is inserted into T and tree T' is obtained after rebalancing.*

a) *If the insertion is into a type* I *subtree, then* $a_1(T')=a_1(T)-1$ *and* $a_2(T')=a_2(T)+2$.

b) *If the insertion is into a type* II *subtree, then* $a_1(T')=a_1(T)+1$ *and* $a_2(T')=a_2(T)-1$.

Proof. This result is proved in Brown [4]. □

Deletion is somewhat harder to deal with. We first prove a general lemma on the effect of a rotation or double rotation on the composition of the fringe.

LEMMA 3. *Let T_1 and T_2 be* AVL-*trees of height $h+2$ and h respectively. Let u be a new node and consider the tree T consisting of root u, left subtree T_1 and right subtree T_2. Either a rotation or double rotation about u transforms T into an* AVL-*tree T'.*

If $h \geqq 1$ then $a_1(T')=a_1(T)$ and $a_2(T')=a_2(T)$.

Proof. Let T_{11} and T_{12} be the left and right subtrees of T_1. At least one of them has height $h+1$; the other one has height $h+1$ or h.

Case 1. T_{11} has height $h+1$. Then a rotation (see Fig. 3) rebalances the tree. Since T_{11}, T_{12} and T_2 have height at least 1 and hence at least two leaves each, the composition of the fringe is not changed.

Case 2. T_{11} has height h and T_{12} has height $h+1$. Let T_{121} and T_{122} be the left and right subtrees of T_{12}. One of them has height h, the other one has height h or $h-1$. A double rotation rebalances the tree (cf. Fig. 4). If $h \geqq 2$ or both T_{121} and T_{122} have height h, then all four trees have at least two leaves each, and hence the composition of the fringe is not changed. Otherwise, we have $h=1$ and either T_{121}

FIG. 3. *A rotation.*

FIG. 4. *A double rotation* (DR).

or T_{122} is a single leaf. Figure 5 shows that the composition of the fringe is not changed in this case. □

Upon the deletion of a leaf, the father of that leaf is replaced by the other subtree, and then the tree is rebalanced along the path back to the root (cf. Knuth [12] for more detail).

LEMMA 4. *Let T be an* AVL-*tree with at least three leaves. Suppose that a leaf is deleted from T and tree T' is obtained after rebalancing.*

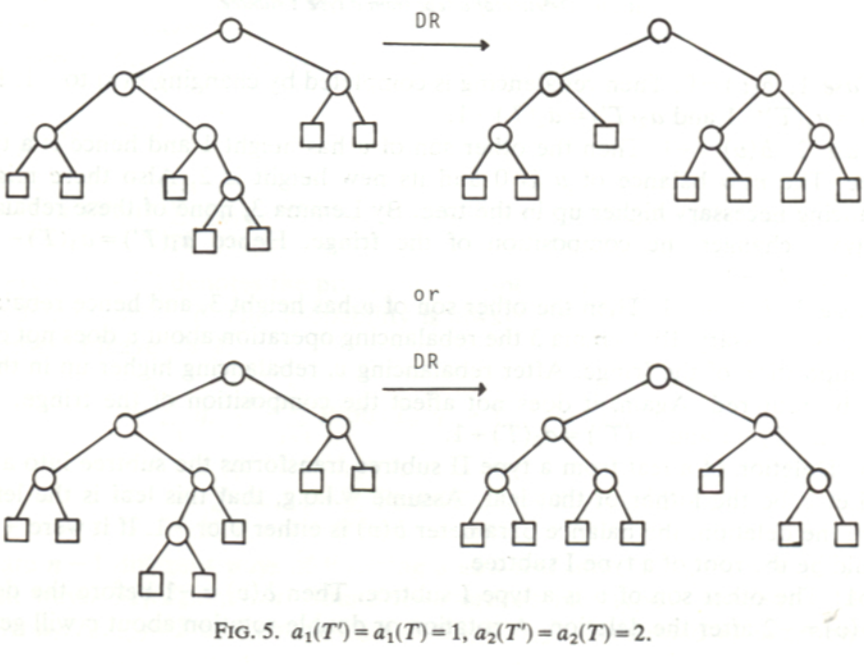

FIG. 5. $a_1(T') = a_1(T) = 1$, $a_2(T') = a_2(T) = 2$.

a) *If the deletion is from a type* I *subtree, then* $a_1(T') = a_1(T) - 1$ *and* $a_2(T') = a_2(T) + 1$.

b) *If the deletion is from a type* II *subtree, then*

b1) *If the brother of that type* II *subtree is a type* I *subtree, then* $a_1(T') = a_1(T) - 1$ *and* $a_2(T') = a_2(T) + 1$.

b2) *If the brother of that type* II *subtree is not a type* I *subtree, then* $a_1(T') = a_1(T) + 1$ *and* $a_2(T') = a_2(T) - 2$.

Proof. a) Deletion of a leaf from a type I subtree transforms that tree into a subtree with two leaves; call its root u (cf. Fig. 6). Let v be the father of u. We distinguish cases according to the *old* balance $b(v)$ of node v. We assume w.l.o.g. that u is the left son of v.

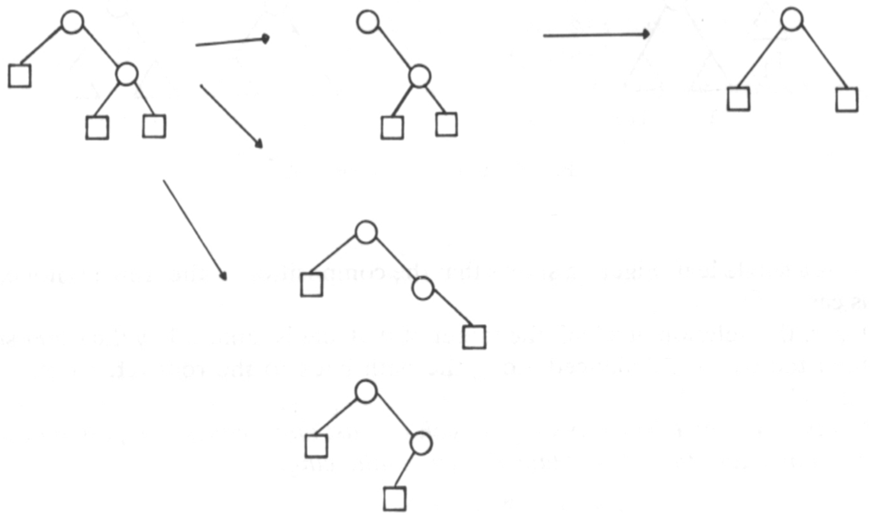

FIG. 6. *Deletion of a leaf from a type* I *subtree.*

Case 1. $b(v) = 0$. Then rebalancing is completed by changing $b(v)$ to -1. Hence $a_1(T') = a_1(T) - 1$ and $a_2(T') = a_2(T) + 1$.

Case 2. $b(v) = +1$. Then the other son of v has height 1 and hence is a type II subtree. The new balance of u is 0 and its new height is 2. Also there might be rebalancing necessary higher up in the tree. By Lemma 3, none of these rebalancing operations changes the composition of the fringe. Hence $a_1(T') = a_1(T) - 1$ and $a_2(T') = a_2(T) + 1$.

Case 3. $b(v) = -1$. Then the other son of v has height 3, and hence rebalancing about v is necessary. By Lemma 3 the rebalancing operation about v does not change the composition of the fringe. After rebalancing v, rebalancing higher up in the tree might be required. Again, it does not affect the composition of the fringe. Hence $a_1(T') = a_1(T) - 1$ and $a_2(T') = a_2(T) + 1$.

b) Deletion of a leaf from a type II subtree transforms the subtree into a single leaf. Let v be the father of that leaf. Assume w.l.o.g. that this leaf is the left son. Before the deletion, the balance parameter $b(v)$ is either 0 or -1. If it were $+1$ then v would be the root of a type I subtree.

b1) The other son of v is a type I subtree. Then $b(v) = -1$ before the deletion and $b(v) = -2$ after the deletion. A rotation or double rotation about v will generate two type II subtrees. Subsequent rebalancing operations higher up in the tree will not change the composition of the fringe. Hence $a_1(T')=a_1(T)-1$ and $a_2(T')=a_2(T)+1$.

b2) The other son of v is not a type I subtree.

Case 1. $b(v)=0$. Then the other son of v is a type II subtree. After the deletion, v is the root of a type I subtree. No rebalancing is required. Hence $a_1(T')=a_1(T)+1$ and $a_2(T')=a_2(T)-2$.

Case 2. $b(v)=-1$. Then the other son of v has height 2, but it is not the root of a type I subtree. Hence its two sons are type II subtrees. A rotation about v will be performed. Hence $a_1(T')=a_1(T)+1$ and $a_2(T')=a_2(T)-2$. □

In the remainder of this section, we will set up recurrence relations in order to study the effect of random insertions and deletions on the composition of the fringe of AVL-trees. We review our randomness assumption.

1) For $n \geqq 3$, insertion and deletion are equally likely. Only insertions occur for $n=2$.
2) In the case of insertion, each one of the n leaves is equally likely to be split.
3) In the case of deletion, each leaf is deleted with equal probability.

Let k_n be the number of AVL-trees with n leaves. Let $T_{n,j}$ represent the jth AVL-tree with n leaves in some arbitrary ordering of the n-leaf AVL-trees. The above randomness assumptions define a Markov chain with states $T_{n,j}$. There is a transition from $T_{n,j}$ to $T_{n+1,k}$ if insertion into $T_{n,j}$ and subsequent rebalancing can generate $T_{n+1,k}$, and there is a transition to $T_{n-1,i}$ if deletion from $T_{n,j}$ and subsequent rebalancing yields $T_{n-1,i}$. For each leaf of $T_{n,j}$ we have two transitions, one corresponding to splitting that leaf (insertion) and one corresponding to deleting that leaf. Each transition has probability $1/(2n)$. Let $q_{n,j}$ be the stationary (conditional) probability of being in tree $T_{n,j}$, under the assumption of being in a tree with n leaves.

DEFINITION 2.

$$a_1(n) \equiv \sum_{j=1}^{k_n} q_{n,j} a_1(T_{n,j}), \qquad a_2(n) \equiv \sum_{j=1}^{k_n} q_{n,j} a_2(T_{n,j}),$$

where $a_1(n)$ ($a_2(n)$) is the average number of type I (type II) subtrees in a random AVL-tree with n leaves.

We proceed to derive recurrence relations for $a_2(n)$.

$$a_2(n) = \sum_{j=1}^{k_n} q_{n,j} \cdot a_2(T_{n,j})$$

$$= \sum_{j=1}^{k_n} a_2(T_{n,j}) \left[\sum_{i=1}^{k_{n-1}} q_{n-1,i} \cdot \text{Prob}\,(T_{n-1,i} \to T_{n,j}) + \sum_{h=1}^{k_{n+1}} q_{n+1,h} \cdot \text{Prob}\,(T_{n+1,h} \to T_{n,j}) \right],$$

where Prob $(T \to T')$ denotes the probability of obtaining T' after a random insertion into or deletion from T. This probability is always a multiple of $1/(2n)$, where n is the number of leaves of T.

$$= \sum_{i=1}^{k_{n-1}} q_{n-1,i} \cdot \sum_{j=1}^{k_n} \text{Prob}\,(T_{n-1,i} \to T_{n,j}) \cdot a_2(T_{n,j})$$

$$+ \sum_{h=1}^{k_{n+1}} q_{n+1,h} \cdot \sum_{j=1}^{k_n} \text{Prob}\,(T_{n+1,h} \to T_{n,j}) \cdot a_2(T_{n,j}).$$

There are $n-1$ different ways of inserting a new leaf into $T_{n-1,i}$. Out of these $n-1$ different ways, $3a_1(T_{n-1,i})$ correspond to insertions into type I subtrees (and then $a_2(T_{n,j}) = a_2(T_{n-1,i})+2$ by Lemma 2), and $2a_1(T_{n-1,i})$ correspond to insertions into type II subtrees (and then $a_2(T_{n,j}) = a_2(T_{n-1,i}) - 1$). Hence

$$\sum_{i=1}^{k_{n-1}} q_{n-1,i} \cdot \sum_{j=1}^{k_n} \text{Prob}\,(T_{n-1,i} \to T_{n,j}) \cdot a_2(T_{n,j})$$
$$= \sum_{i=1}^{k_{n-1}} q_{n-1,i} \cdot \frac{1}{2}\left[\frac{3a_1(T_{n-1,i})}{n-1}(a_2(T_{n-1,i})+2) + \frac{2a_2(T_{n-1,i})}{n-1}(a_2(T_{n-1,i})-1)\right]$$
$$= \frac{1}{2}\left[a_2(n-1) + \frac{6a_1(n-1)}{n-1} - \frac{2a_2(n-1)}{n-1}\right].$$

There are $n+1$ different ways of deleting a leaf from $T_{n+1,h}$. Exactly $3a_1(T_{n+1,h})$ of these deletions are deletions from type I subtrees, and then $a_2(T_{n,j}) = a_2(T_{n+1,h}) + 1$. The other $2a_2(T_{n+1,h})$ deletions are deletions from type II subtrees.

DEFINITION 3. Let $p(T_{n,j})$ be the probability that the brother of a type II subtree is a type I subtree in $T_{n,j}$.

LEMMA 5.

$$0 \leqq p(T_{n,j}) \leqq \min\,(1, a_1(T_{n,j})/a_2(T_{n,j})).$$

Proof. Since $p(T_{n,j})$ is a probability, we have $0 \leqq p(T_{n,j}) \leqq 1$. Furthermore, $p(T_{n,j})$ is the fraction of type II subtrees of $T_{n,j}$ whose brothers are type I subtrees of $T_{n,j}$. Hence $a_2(T_{n,j}) \cdot p(T_{n,j}) \leqq a_1(T_{n,j})$. □

Out of the $2a_2(T_{n+1,h})$ deletions from type II subtrees, exactly $2a_2(T_{n+1,h})p(T_{n+1,h})$ are deletions from type II subtrees whose brothers are type I subtrees. In this case we have $a_2(T_{n,j}) = a_2(T_{n+1}, h) + 1$. The remaining $2a_2(T_{n+1,h}) \cdot (1 - p(T_{n+1,h}))$ deletions give $a_2(T_{n,j}) = a_2(T_{n+1,h}) - 2$. Hence

$$\sum_{h=1}^{k_{n+1}} q_{n+1,h} \cdot \sum_{j=1}^{k_n} \text{Prob}\,(T_{n+1,h} \to T_{n,j}) \cdot a_2(T_{n,j})$$
$$= \sum_{h=1}^{k_{n+1}} q_{n+1,h} \cdot \frac{1}{2}\left[\frac{3a_1(T_{n+1,h})}{n+1}(a_2(T_{n+1,h})+1) + \frac{2a_2(T_{n+1,h})}{n+1}(p(T_{n+1,h})(a_2(T_{n+1,h})+1) + (1-p(T_{n+1,h}))(a_2(T_{n+1,h})-2))\right]$$
$$= \frac{1}{2}\left[a_2(n+1) + \frac{3a_1(n+1)}{n+1} + \frac{2a_2(n+1)}{n+1}(3p(n+1)-2)\right],$$

where

$$p(n+1) \equiv \left(\sum_{h=1}^{k_{n+1}} q_{n+1,h} \cdot a_2(T_{n+1,h})p(T_{n+1,h})\right) \Big/ a_2(n+1)$$

is the probability that the brother of a type II subtree is a type I subtree in a random AVL-tree with $n+1$ leaves.

LEMMA 6.

$$0 \leqq p(n) \leqq \min\,(1, a_1(n)/a_2(n)).$$

Proof. $0 \leqq p(n) \leqq 1$ is obvious. Also

$$p(n) = \frac{1}{a_2(n)} \cdot \sum_{j=1}^{k_n} a_2(T_{n,j}) p(T_{n,j}) \cdot q_{n,j}$$

$$\leqq \frac{1}{a_2(n)} \sum_{j=1}^{k_n} q_{n,j} a_1(T_{n,j}) \qquad \text{(by Lemma 5)}$$

$$= a_1(n)/a_2(n). \qquad \square$$

Putting the derivations together, we arrive at our main equation:

$$\text{(M)} \qquad \begin{aligned} 2a_2(n) = a_2(n-1) + \frac{6a_1(n-1)}{n-1} - \frac{2a_2(n-1)}{n-1} \\ + a_2(n+1) + \frac{3a_1(n+1)}{n+1} + \frac{2a_2(n+1)}{n+1}(3p(n+1)-2), \end{aligned}$$

where $0 \leqq p(n+1) \leqq \min(1, a_1(n+1)/a_2(n+1))$.

With the use of Lemma 1, we can eliminate $a_1(n-1)$, $a_1(n+1)$ from equation (M) and obtain equations (M′) and (M″):

$$\text{(M}'\text{)} \qquad 2a_2(n) = a_2(n-1) - \frac{6a_2(n-1)}{n-1} + a_2(n+1) + \frac{6a_2(n+1)}{n+1}[p(n+1)-1] + 3,$$

or

$$\text{(M}''\text{)} \qquad a_2(n+1)\left[1 + \frac{6p(n+1)-6}{n+1}\right] = 2a_2(n) - a_2(n-1)\left[1 - \frac{6}{n-1}\right] - 3,$$

where

$$0 \leqq p(n+1) \leqq \min\left(1, \frac{n+1-2a_2(n+1)}{3a_2(n+1)}\right).$$

In the next section we will derive bounds on the solutions of equation (M″).

3. Solving the equation. In this section we will solve equation (M″); more precisely, we will show $c_1 \cdot n \leqq a_2(n) \leqq c_2 \cdot n$ for suitable constants c_1 and c_2. We proceed in two steps. In the first step, we solve the equation under the simplifying, however unjustified, assumption $p(n) = p$ and $a_2(n) = a_2 \cdot n$. In the second step, we try to show that the solutions obtained in the first step suffice to describe the system, even without the additional assumption.

Suppose $p(n) = p$ and $a_2(n) = a_2 \cdot n$ for all n. We want to stress again that this assumption is completely unjustified. Then equation (M″) transforms into

$$a_2 \cdot (n+1) \cdot \left[\frac{n+6p-5}{n+1}\right] = 2a_2 \cdot n - a_2 \cdot (n-1) \cdot \left[\frac{n-7}{n-1}\right] - 3$$

or

$$a_2(6p-5) = 7a_2 - 3 \quad \text{or} \quad a_2 = \frac{3}{12-6p}.$$

The extremal values of p are 0 and $\min(1, a_1(n)/a_2(n))$. Using Lemma 1 and $a_2(n) = a_2 \cdot n$, we obtain $0 \leqq p \leqq \min(1, (1-2a_2)/3a_2)$. For $p = 0$ we have $a_2 = \frac{1}{4}$ and for $p = (1-2a_2)/3a_2$ we have $a_2 = \frac{5}{16}$. Hence our simplifying assumptions imply $n/4 \leqq a_2(n) \leqq 5n/16$.

In our main theorem, we show that the above inequality is essentially true even without the simplifying assumption:

THEOREM 1. a) *For* $n \geqq 7$, $a_2(n) \geqq n/4$;

b) $\lim_{n\to\infty} \sup a_2(n)/n \leqq \frac{5}{16}$.

Proof. Up to symmetry, there is only one AVL-tree with 7 leaves (Fig. 7). Hence $a_2(7)=2$.

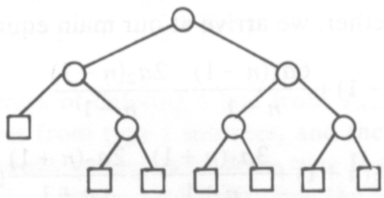

FIG. 7. *The only (up to symmetry) AVL-tree with 7 leaves.*

Define $a_2(n)=b_2(n)\cdot n$. Then equation (M″) transforms into (M‴):

(M‴) $$b_2(n+1)[n+6p(n+1)-5]=2b_2(n)\cdot n-b_2(n-1)[n-7]-3.$$

We will show $b_2(n)\geqq\frac{1}{4}$ for $n\geqq 7$, and $\lim\sup b_2(n)\leqq\frac{5}{16}$.

LEMMA 7. $0\leqq b_2(n)\leqq\frac{1}{2}$ *for all* n.

Proof. $a_1(n)\geqq 0$ and $a_2(n)\geqq 0$ are obvious. An application of Lemma 1 finishes the proof. □

We will first show $b_2(n)\geqq\frac{1}{4}$ for $n\geqq 7$. This was already shown for $n=7$. Lemma 8 below shows that once $b_2(n)$ goes below $\frac{1}{4}$, it will be monotonically decreasing. Hence by Lemma 7 the sequence $b_2(n)$ converges to some number between 0 and $\frac{1}{4}$ (exclusive). A contradiction is then derived in Lemma 9.

LEMMA 8. *For* $n\geqq 7$, *if* $b_2(n)<b_2(n-1)$ *and* $b_2(n)<\frac{1}{4}$, *then* $b_2(n+1)\leqq b_2(n)+(12b_2(n)-3)/(n-5)$.

Proof. Since $p(n+1)\geqq 0$, equation (M‴) implies

$$(n-5)b_2(n+1)\leqq 2nb_2(n)-(n-7)b_2(n-1)-3$$

or

$$(n-5)(b_2(n+1)-b_2(n))\leqq(n-7)(b_2(n)-b_2(n-1))+(12b_2(n)-3).$$

Thus

$$b_2(n+1)\leqq b_2(n)+(12b_2(n)-3)/(n-5).$$ —

LEMMA 9. $b_2(n)\geqq\frac{1}{4}$ *for all* $n\geqq 7$.

Proof. Assume otherwise. Since $b_2(7)\geqq\frac{1}{4}$, there is at least $n_0>7$ with $b_2(n_0)<\frac{1}{4}\leqq b_2(n_0-1)$. Let $\varepsilon=3-12b_2(n_0)<0$. A simple induction argument based on Lemma 8 shows $b_2(n+1)<b_2(n)-\varepsilon/(n-5)$ for all $n\geqq n_0$. Hence $b_2(n+1)\leqq b_2(n_0)-\varepsilon\cdot\sum_{i=n_0}^{n}1/(i-5)$ for $n\geqq n_0$. Thus $b_2(n)$ must become negative, a contradiction to Lemma 7. □

It remains to prove $\lim\sup b_2(n)\leqq\frac{5}{16}$; $\lim\sup b_2(n)$ exists, since $b_2(n)$ is bounded. We will first prove the analogue of Lemma 8.

LEMMA 10. *For* $n\geqq 9$, *if* $b_2(n)>b_2(n-1)$ *and* $b_2(n)>\frac{5}{16}$, *then*

$$b_2(n+1)\geqq b_2(n)+(16b_2(n)-5)/(n-9).$$

Proof. Since $p(n+1) \leqq ((n+1)-2a_2(n+1))/3a_2(n+1) = (1-2b_2(n+1))/3b_2(n+1)$, equation (M‴) implies

$$(n-5+6\cdot(1-2b_2(n+1))/3b_2(n+1))\cdot b_2(n+1) \geqq 2nb_2(n)-(n-7)b_2(n-1)-3$$

or

$$(n-9)(b_2(n+1)-b_2(n)) \geqq (n-7)(b_2(n)-b_2(n-1))+(16b_2(n)-5),$$

and hence $b_2(n+1) \geqq b_2(n)+(16b_2(n)-5)/(n-9)$. □

LEMMA 11.

$$\limsup b_2(n) \leqq \tfrac{5}{16}.$$

Proof. Assume otherwise; say $b = \limsup b_2(n) > \frac{5}{16}$. Then there is either an $n_0 \geqq 9$ such that $b_2(n_0) > b_2(n_0-1)$ and $b_2(n_0) > \frac{5}{16}$, or $b_2(n) > \frac{5}{16}$ and $b_2(n) \leqq b_2(n-1)$ for all $n \geqq 9$. In either case we will derive a contradiction.

Case 1. *There is an $n_0 > 9$ such that $b_2(n_0) > b_2(n_0-1)$ and $b_2(n_0) > \frac{5}{16}$.* Let $\varepsilon = 16b_2(n_0)-5$. A simple induction argument based on Lemma 10 shows that $b_2(n+1) > b_2(n)+\varepsilon/(n-9)$ for all $n \geqq n_0$. Hence $b_2(n)$ is unbounded, a contradiction to Lemma 7.

Case 2. *$b_2(n) > \frac{5}{16}$ and $b_2(n) < b_2(n-1)$ for all $n \geqq 9$.* Then $b_2(n)$ is nonincreasing, and hence $b = \lim b_2(n)$. Let $\varepsilon = b - \frac{5}{16} > 0$. Then there is n_0 such that for $n \geqq n_0$ we have $b_2(n)-b_2(n+1) \leqq \varepsilon$. Hence the next to last inequality in Lemma 10 implies for $n > n_0$

$$\begin{aligned}(n-9)&(b_2(n+1)-b_2(n))\\ &\geqq (n-9)(b_2(n)-b_2(n-1))+2(b_2(n)-b_2(n-1))+(16b_2(n)-5)\\ &\geqq (n-9)(b_2(n)-b_2(n-1))-2\varepsilon+16\varepsilon,\end{aligned}$$

and thus

$$b_2(n+1)-b_2(n) \geqq (b_2(n)-b_2(n-1))+(14\varepsilon)/(n-9).$$

This shows that the difference $b_2(n+1)-b_2(n)$ must become unbounded, a contradiction to Lemma 7.

In either case we have derived a contradiction. Hence $\limsup b_2(n) \leqq \frac{1}{16}$. □

This finishes the proof of Theorem 1. □

4. Applications. In this section we apply our results to derive bounds on the number of balanced nodes in random AVL-trees, and we indicate how to extend the results to HB-trees (Ottman and Six [18], Mehlhorn [14]) and 2-3-trees (Aho, Hopcroft and Ullman [1]).

THEOREM 2. *Let $\bar{B}(n)$ denote the average number of balanced nodes (= nodes of balance 0) in a random (in the sense of § 1) AVL-tree with n leaves. Then*

$$4n/9 \leqq \bar{B}(n) \leqq 7n/8 + o(n).$$

Proof. Let T be an AVL-tree with a_1 type I subtrees and a_2 type II subtrees, and n leaves. Then $3a_1+2a_2 = n$. T contains a_1+a_2 balanced nodes in the fringe and a_1+a_2-1 nodes which are not in the fringe.

Out of these a_1+a_2-1 nodes, all may be balanced. Hence T contains at most $2a_1+2a_2-1 = 2n/3+2a_2/3-1$ balanced nodes. Passing to averages yields $\bar{B}(n) \leqq \frac{7}{8}n + o(n)$ by Theorem 1.

Next we want to derive a lower bound on the number of balanced nodes. The brother of a type II subtree is either a type I subtree or a type II subtree or a tree consisting of two type II subtrees. Since the former situation arises at most a_1 times, the latter two situations must occur at least $a_2 - a_1$ times. Every three occurrences lead to at least one balanced node. This yields another $(a_2 - a_1)/3$ balanced nodes. Passing to averages yields

$$\bar{B}(n) \geqq a_1(n) + a_2(n) + (a_2(n) - a_1(n))/3$$
$$\geqq 2n/9 + 8a_2(a)/9 \geqq 4n/9. \quad \square$$

Theorem 2 shows that $0.44n \leqq \bar{B}(n) \leqq 0.875n + o(n)$. Previously AVL-trees were analyzed under random insertions only. In that case, Brown [4] showed $0.47n \leqq \bar{B}(n) \leqq 0.85n$ and Mehlhorn [15] showed $0.51n \leqq \bar{B}(n) \leqq 0.81n$.

In the remainder of this section, we indicate how to extend our results to 2-3-trees and HB-trees. In a 2-3-tree, all leaves are on the same level, and every interior node has degree either two or three (Aho, Hopcroft and Ullman [1]). We obtain the fringe by deleting all nodes of height at least two, i.e., we do a first-order analysis in the sense of Yao [19]. The fringe consists of two types of trees: nodes with three sons (type I) and nodes with two sons (type II) (Fig. 8).

FIG. 8. *The subtrees in the fringe of a 2-3-tree.*

The crucial observation is that Lemma 2 remains true if we replace "AVL-tree" by "2-3-tree" and that Lemma 4 remains true if we replace "AVL-tree" by "2-3-tree" and "if the brother" by "if a brother" in b1) and "if the brother" by "if the brothers" in b2). The simple proofs are left to the readers.

Next we need to define $p(T)$ as the probability that one of the brothers of a type II subtree is a type I subtree. Then Lemma 5 transforms into $0 \leqq p(T) \leqq \min(1, 2 \cdot a_1(T)/a_2(T))$, since one type I tree can serve as the brother of two type II trees. Nevertheless, we obtain the same recurrence relation for $a_2(n)$ as before; however, we have the weaker condition $0 \leqq p(n) \leqq \min(1, 2a_1(n)/a_2(n))$ on $p(n)$.

Proceeding as above, we obtain $a_2(n) \geqq n/4$ and $a_2(n) \leqq 7n/20 + o(n)$. The proofs are almost literally the same; in the proofs of Lemmas 10 and 11, one has to use the weaker restriction on $p(n)$. These bounds on $a_2(n)$ can be used to derive bounds on the expected number of nodes in a random 2-3-tree and hence on the storage utilization. (cf. Yao [19]). If the fringe of a 2-3-tree T consists of $a_1(a_2)$ nodes with 3(2) sons, then the number of internal nodes of T lies between $(3(a_1 + a_2) - 1)/2$ and $2(a_1 + a_2) - 1$. Also $3a_1 + 2a_2$ is equal to the number of leaves of T. Hence the expected number of nodes in a random 2-3-tree with n leaves is at least $(3(a_1(n) + a_2(n)) - 1)/2 = (n + a_2(n) - 1)/2 \geqq (5n - 4)/8 \approx 0.625n$, and at most $2(a_1(n) + a_2(n)) - 1 \leqq 2(n + a_2(n))/3 - 1 \leqq 9/10n + o(n) \approx 0.9n$. The obvious bounds are $(n-1)/2$ and $n - 1$. We summarize the discussion in

THEOREM 3. *Let $\bar{N}(n)$ be the expected number of nodes in a random (in the sense of § 1) 2-3-tree with n leaves. Then*

$$(5n - 4)/8 \leqq \bar{N}(n) \leqq (9/10) \cdot n + o(n).$$

For HB-trees (Ottman and Six [18]), one obtains the fringe by deleting all nodes with at least four leaves below them. Figure 9 shows the trees which appear in the

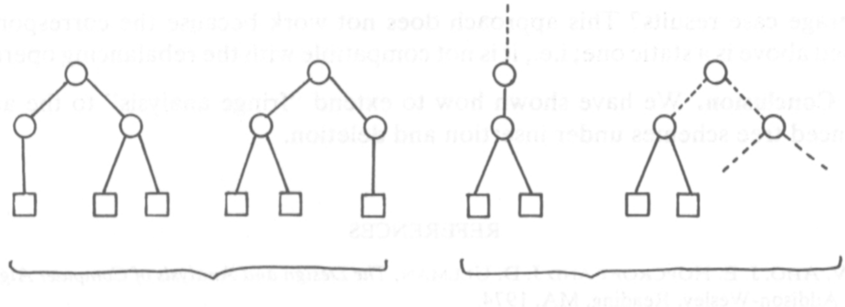

FIG. 9. *The subtrees in the fringe of an* HB-*tree.*

fringe. Lemmas 1–6 stay true with "AVL-tree" replaced by "HB-tree" throughout. Hence Theorem 1 gives the number of type I and type II subtrees in a random HB-tree. This leads to

THEOREM 4. *Let* $\bar{N}(n)$ *be the expected number of nodes in a random (in the sense of* § 1) HB-*tree with n leaves. Then*

$$(9/8)\cdot n-o(n)\leqq \bar{N}(n)\leqq 14n/9.$$

Proof. Let T be an HB-tree with a_1 type I subtrees and a_2 type II subtrees and n leaves. Since T has n leaves, there are exactly $n-1$ nodes with two sons. It remains to derive bounds on the number of nodes with one son. Since every type I subtree contains one node with only one son, there are at least a_1 nodes with only one son. Passing to averages gives $N(n)\geqq n-1+a_1(n)\geqq(9/8)n-o(n)$. For the upper bound on the number of nodes with one son, we use the following correspondence between AVL-trees and HB-trees. If one deletes all nodes with only one son from an HB-tree by combining these nodes with their fathers, then one obtains an AVL-tree T' (cf. Fig. 10). Furthermore, this correspondence preserves the composition of the fringe; i.e., type I(II) subtrees in the HB-tree sense are mapped onto type I(II) subtrees in the AVL-tree sense, and the number of nodes with one son in T is equal to the number of unbalanced nodes in T'. So T' is an AVL-tree with n leaves, a_1 type I subtrees and a_2 type II subtrees. In the proof of Theorem 2 we have shown that T' contains at least $a_1+a_2+(a_2-a_1)/3$ balanced nodes. Thus T contains at most $n-1-a_1-a_2-(a_2-a_1)/3$ nodes with only one son. Passing to averages give $\bar{N}(n)\leqq 2n-2-a_1(n)-a_2(n)-(a_2(n)-a_1(n))/3\leqq 14n/9$. □

The proof of Theorem 4 may seem unnecessarily complicated to some readers. Why not use the correspondence between AVL-trees and HB-trees directly to lift

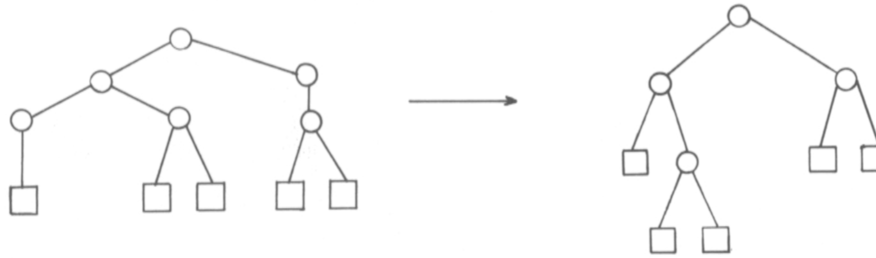

FIG. 10. *An* HB-*tree and the corresponding* AVL-*tree.*

the average case results? This approach does not work because the correspondence described above is a static one; i.e., it is not compatible with the rebalancing operations.

5. Conclusion. We have shown how to extend "fringe analysis" to the analysis of balanced tree schemes under insertion and deletion.

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