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A PROBABILISTIC ALGORITHM FOR VERTEX CONNECTIVITY OF GRAPHS

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A probabilistic algorithm is presented which computes the vertex connectivity of an undirected graph $G=(V, E)$ in expected time $O((-\log\varepsilon)|V|^{3/2}|E|)$ with error probability at most ε provided that $|E|\leqslant\frac{1}{2}d|V|^2$ for some universal constant $d<1$.

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Let $G=(V, E)$ be an undirected graph. Let $a\nmid b$ denote that there is no edge between a and b. If $a\nmid b$, then $S\subseteq V-\{a, b\}$ is an (a, b) vertex separator if every path from a to b passes through a vertex of S. $N(a, b)$ is the least cardinality of an (a, b) vertex separator, i.e.,

$$N(a, b)=\min\{|S|;\ S\subseteq V-\{a, b\}\text{ is an (a, b) vertex separator}\}.$$

$N(a, b)$ can be determined in time $O(|V|^{1/2}|E|)$ by network flow methods [1]. The vertex connectivity c of graph G is defined by

$$c=\begin{cases} n-1 & \text{if G is a complete graph,}\\ \min\limits_{a\nmid b} N(a, b) & \text{if G is not complete.}\end{cases}$$

Even and Tarjan [1] described an $O(c\,|V|^{3/2}\,|E|)=O(|V|^{1/2}|E|^2)$ algorithm for computing c. In this note we describe a probabilistic variant of their method, which computes c with error probability ε in expected time $O((-\log\varepsilon)|V|^{3/2}|E|)$ provided that $c\leqslant d\,|V|$ for some constant $d<1$. Since $c\leqslant 2|E|/|V|$, this will always be the case for sparse graphs.

In the following, n is often used for $|V|$.

Lemma 1. *Let* $G=(V, E)$ *be an undirected graph with vertex connectivity* $c<|V|-1$.

(a) *Let* $S\subseteq V$ *be a vertex separator for some nodes* v, w *with* $|S|=c$. *Then*

$$c=\min_{a\nmid b} N(a, b)$$

for all $a\in V-S$.

(b) *Let* $\varepsilon>0$,

$$k=-\log\varepsilon/(\log n-\log c),$$

let $a_1,\dots,a_k$ *be* k *random nodes of* G *and let*

$$\mu=\min_{1\leqslant i\leqslant k}\ \min_{a_i\nmid b} N(a_i, b).$$

Then $\Pr(\mu>c)\leqslant\varepsilon$.

Proof. (a) Graph $G-S$ consists of at least two components. Let b be a node which does not belong to the same component as a. Then S separates a from b and hence $N(a, b)\leqslant|S|=c$. Thus $c=N(a, b)$ by definition of c.

(b) Let S be a vertex separator with $S=c$. If $\mu>c$, then $a_1,\dots,a_k$ all belong to S. Hence $\Pr(\mu>c)\leqslant(c/n)^k\leqslant\varepsilon$.

Lemma 1 suggests a probabilistic algorithm for computing c.

Algorithm.

Step 1. COUNT $\leftarrow 0$; $\mu\leftarrow|V|-2$;
Step 2. **while** COUNT $<-\log\varepsilon/(\log n-\log\mu)$
Step 3. **do** COUNT $\leftarrow$ COUNT $+1$;

Step 4. select $a \in V$ at random;
Step 5. $\mu \leftarrow \min\{\mu, \min_{a \nmid b} N(a, b)\}$
Step 6. **od**
Step 7. output (μ)

Theorem 2. *Let* $G = (V, E)$ *be an undirected graph with vertex connectivity* $0 < c < |V| - 1$ *and let* $\varepsilon > 0$, $\varepsilon \leq 1/n$. *Then the algorithm above terminates in expected time*

$$O\big(|V|^{3/2}|E|(-\log \varepsilon)/(\log n - \log c)\big)$$

and computes c *with error probability* $\leq \varepsilon$.

Proof. Note first that for fixed a, b one can compute N(a, b) in time $O(|V|^{1/2}|E|)$ by network flow methods [1]. Hence one execution of Step 5 takes time $O(|V|^{3/2}|E|)$. Let N be the number of loop iterations. Since $\mu \geq c$, we certainly have

$$N \geq k := -\log \varepsilon/(\log n - \log c),$$

and therefore the error probability $\leq \varepsilon$ by Lemma 1(b). It remains to compute the expected number of iterations of the loop. Let p_m be the probability that $\geq m$ iterations are performed, $m \geq k$. Then

$$p_k = 1$$

and

$$p_m \leq (c/n)^{m-1} \quad \text{for } m > k,$$

because the fact that $\geq m$ iterations are performed implies that $\mu > c$ after $m - 1$ iterations, the probability of which is bounded by $(c/n)^{m-1}$. Hence the expected number of iterations is bounded by

$$\begin{aligned} k + \sum_{m \geq k} p_m &\leq \\ &\leq k + (c/n)^k \sum_{\ell \geq 0} (c/n)^\ell \\ &\leq k + (c/n)^k \sum_{\ell \geq 0} (c/n)^\ell \\ &\leq k + (c/n)^k/(1 - c/n) \\ &\leq k + \varepsilon/(1 - (n-1)/n) \\ &\leq k + \varepsilon n \leq k + 1 = O(k). \end{aligned}$$

Thus the expected running time of our algorithm is $O(k|V|^{3/2}|E|)$.

It c is sufficiently smaller than |V|, say $c \leq 0.9\ |V|$, then our algorithm will run in expected time $O((-\log \varepsilon)\ |V|^{3/2}|E|)$. Since

$$c \leq \max\{\text{degree}(v);\ v \in V\} \leq 2|E|/|V|,$$

this will certainly be the case if $|E| \leq 0.9|V|^2/2$, i.e., if the graph is sparse.

Corollary 3. *Let* $d < 1$. *The algorithm above determines the vertex connectivity of an undirected graph* $G = (V, E)$ *in expected time*

$$O\big((-\log \varepsilon)|V|^{3/2}|E|\big)$$

with error $\leq \varepsilon$ *provided that* $|E| \leq \frac{1}{2}d|V|^2$.

Proof. Immediate from Theorem 2 and the discussion above.

For $\varepsilon = 2^{-100}$ the algorithm will run in expected time $O(|V|^{3/2}|E|)$ – a significant improvement with respect to Even and Tarjan's result.

Reference

[1] S. Even and R.E. Tarjan, Network flow and testing graph connectivity, SIAM J. and Comput. 4 (1975) 507–518.