Elektronische Informationsverarbeitung und Kybernetik
EIK 12 (1976) 11/12, 523 – 524

An Improved Lower Bound on the Formula Complexity of Context-free Recognition

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In [1] G. Hotz proves an $O\left((n/\log n)^{3/2}\right)$ lower bound¹) on the formula complexity of context free recognition. In this note we improve this bound to $O(n^2/\log^3 n)$.

Let $L \subseteq \{0, 1\}^*$ be a context-free language, let $L_n = L \cap \{0, 1\}^n$ and let $C_L(n)$ be the size of the smallest formula (over the basis of all binary boolean functions) for the characteristic function of L_n. We call C_L the formula complexity of L.

Theorem. *There is a context-free language L such that*

$$C_L(n) \geqq O(n^2/\log^3 n) .$$

Proof. Neciporuk describes in [2] a method for proving lower bounds for the formula complexity of boolean functions. Applying this method Paul [3] gave a simple example of a boolean function having formula complexity $O(n^2/\log n)$. We give his example.

$$f_n\colon \{0, 1\}^{2n + \log n - \log\log n} \to \{0, 1\} .$$

The variables of f_n are divided in a group $A = a_1 \cdots a_{\log n - \log\log n}$ of $\log n - \log\log n$ variables and two groups X and $Y = y_1 \cdots y_n$ of n variables each. The n variables in X are divided into blocks $X_i = x_{i_1} \cdots x_{i\log n}$, $1 \leqq i \leqq n/\log n$ of $\log n$ variables each. We interpret A as a binary number between 1 and $n/\log n$: it indexes the blocks $X_1, \ldots, X_{n/\log n}$. We interpret a block X_i as a binary number between 1 and n: it indexes the table Y. Formally

$$f_n(A, X_1, \ldots, X_{n/\log n}, Y) = y_j$$

where A represents the binary number i and X_i represents j.

Fact [3]. $C(f_n) \geqq O(n^2/\log n)$.

We embed f_n into a context-free language and thus reduce the problem of evaluating f_n to the problem of context-free recognition. The lower bound in [1] was obtained by embedding matrix multiplication. Consider the following language $L \subseteq \{0, 1, \#\}^*$. For ease of description we define L over a 3 symbol alphabet.

$$\begin{aligned} L = \{ & a \#\#\# v_1 \# x_1 \#\# \cdots \#\# v_k \# x_k \#\#\# w_1 \# y_1 \#\# \cdots \\ & \cdots \#\# w_l \# y_l; \\ & a, v_i, x_i, w_j \in \{0, 1\}^*, y_j \in \{0, 1\} \text{ for all } i, j, 1 \leqq i \leqq k, 1 \leqq j \leqq l, \\ & \text{and } a = v_i^R \text{ and } x_i = w_j^R \text{ and } y_j = 1 \text{ for some } i, j\}. \end{aligned}$$

It is easy to see that L is context-free. We encode an input A, X, Y of f_n as a string z over $\{0, 1, \#\}$ of length approx. $n \log n$.

¹) For ease of writing we write $\log n$ instead of $\lceil \log_2 n \rceil$. $g(n) \geqq O(h(n))$ means that the growth rate of g is at least as large as the growth rate of h.

Let

$$a = a_1 \cdots a_{\log n - \log\log n}, \qquad x_i = x_{i1} \cdots x_{i \log n}, \qquad y_i = y_i$$

and let bin(j) be the binary representation of integer j (filled up with leading 0's to length $\log n$). Then

$$z = a \,\#\#\#\, \mathrm{bin}(1)^R \,\#\, x_1 \,\#\#\, \cdots \,\#\#\, \mathrm{bin}(n/\log n)^R \,\#\, x_n/\log n$$
$$\#\#\#\, \mathrm{bin}(1)^R \,\#\, y_1 \,\#\#\, \cdots \,\#\#\, \mathrm{bin}(n)^R \,\#\, y_n .$$

Fact. *$z \in L$ iff $f_n(A, X, Y) = 1$ and $|z| = O(n \log n)$.*

Let $m = |z|$ and consider any formula for the characteristic function of L_m. Such a formula is easily converted into a formula for f_n (by fixing the values of some of the inputs). Therefore

$$C_L(m) \geqq C(f_n) \geqq O(n^2/\log n)$$

and hence

$$C_L(n \cdot \log n) \geqq O(n^2/\log n)$$

which implies

$$C_L(n) \geqq O(n^2/\log^3 n);$$

q.e.d.[1])

References

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[3] Paul, W. J., A 2.5 N-Lower Bound on the Combinational Complexity of Boolean Functions. In: 7th ACM Symposium on Theory of Computing, 1975; pp. 27–36.
[4] Khrapchenko, On the Complexity of the Linear Function in the Class of Π-Circuits. Matem. Zametki **9** (1971) 1, 35–40 (Russian).

Abstract

For $L \subseteqq \{0, 1\}^*$ a language let $L_n = L \cap \{0, 1\}^n$ and let $C_L(n)$ be the size of the shortest boolean formula for the characteristic function of L_n. We exhibit a context-free language with $C_L(n) \geqq O(n^2/\log^3 n)$.

Kurzfassung

Für eine Sprache $L \subseteqq \{0, 1\}^*$ sei $L_n = L \cap \{0, 1\}^n$, und $C_L(n)$ sei die Größe des kleinsten Booleschen Ausdrucks für die charakteristische Funktion von L_n. Wir zeigen $C_L(n) \geqq$ $\geqq O(n^2/\log^3 n)$ für eine kontext-freie Sprache L_n.

Резюме

Предполагаем, что для языка $L \subseteqq \{0, 1\}^*$ имеет место $L_n = L \cap \{0, 1\}^n$ и что $C_L(n)$ является значением наименьшего булевого выражения для характеристической функции от L_n. Мы показываем, что для контекстно-свободного языка L_n имеет место соотношение $C_L(n) \geqq O(n^2/\log^3 n)$..

(Eingegangen am 28. 1. 1976)

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[1]) Zvi Galil and Mike Paterson improved upon the result of this paper. They show an $O(n^2/\log n)$ lower bound by exhibiting a better encoding of the functions f_n into a context-free language.