A Complete Roundness Classification Procedure

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Abstract

We describe a *roundness classification procedure*, that is, a procedure to determine if the roundness of a planar object I is within some ϵ_0 from an ideal circle. The procedure consists of a probing strategy and an evaluation algorithm working in a feedback loop. This approach of combining probing with evaluation is new in computational metrology. For several definitions of roundness, our procedure uses $O(1/qual(I))$ probes and runs in time $O(1/qual(I)^2)$. Here, the quality $qual(I)$ of I measures how far the roundness of I is from the accept-reject criterion. Hence our algorithms are "quality sensitive".

1 Introduction

A basic task of metrology is to decide whether a given planar object is "round" within some specified bound. We call this the *roundness classification problem*. The literature on roundness classification is fairly large; recent algorithmic papers include [10, 7, 3, 13, 11, 2, 6]. The area of computational metrology addresses this and similar problems [5, 4, 12, 9, 14].

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In this paper, we not only raise the issue of how to define roundness but also give one of the first "complete" roundness classification procedures. It consists of a *probing strategy* that plans and makes the measurements, and a *computation and decision strategy* which uses the measurements to ultimately classify the object as good or bad. Traditionally, these two strategies are decoupled. In our case, the two strategies are coupled in a feedback loop, giving a holistic approach to the classification problem. This approach can be applied to other problems in computational metrology as well.

Let us briefly illustrate the "standard approach": suppose the roundness of I is defined to be the minimum width $w(I)$ of an annulus that contains the boundary of I. Assume that our classification problem is to accept I iff $w(I) \leq \epsilon_0$. There are 3 steps:

(i) Some probing strategy takes a predetermined number n of sample points on the boundary of I.

(ii) The set S of sample points is then submitted to an algorithm to compute the minimum width $w(S)$ of an annulus that contains S.

(iii) Finally, a decision policy decides to accept or reject I based on the computed value $w(S)$. Typically, we accept iff $w(S) \leq \epsilon_0$.

It is hardly clear that the policy (iii) gives a correct decision since we have substituted "$w(I)$" with "$w(S)$". Computational geometers have traditionally focused on step (ii) only. One of the few papers that discuss the decision policies in step (iii) is found in [15]. We say that the algorithm in this paper is "complete" in the sense that we integrate all three steps, and we prove that our decision policy is correct with respect to the original object I, not with respect to the sampled set S.

Let us fix some notations for the paper. An *object* I is defined to be any compact simply-connected subset of the plane, with boundary denoted by bd I.

For a point p, we use $R(p, I)$ and $r(p, I)$ to denote the maximal and minimal distance from p to a point in bd I respectively[1], i.e.,

$$R(p, I) = \max\{dist(p, q); q \in \text{bd}\, I\}$$

$$r(p, I) = \min\{dist(p, q); q \in \text{bd}\, I\}.$$

Our distance measure is the Euclidean distance. When I is understood, we write $R(p)$ instead of $R(p, I)$ and $r(p)$ instead of $r(p, I)$. Let $r_0 > 0$ and $\epsilon_0 > 0$ be parameters, which we consider fixed throughout the paper. An object I is called (r_0, ϵ_0)-*round* or *good*[2] if there is a point p such that $r_0(1 - \epsilon_0) \le r(p) \le R(p) \le r_0(1 + \epsilon_0)$. An object is called *bad* if it is not good.

In Figure 1, the shaded objects C_1 and C_2 are bad while C_3 and C_4 are good.

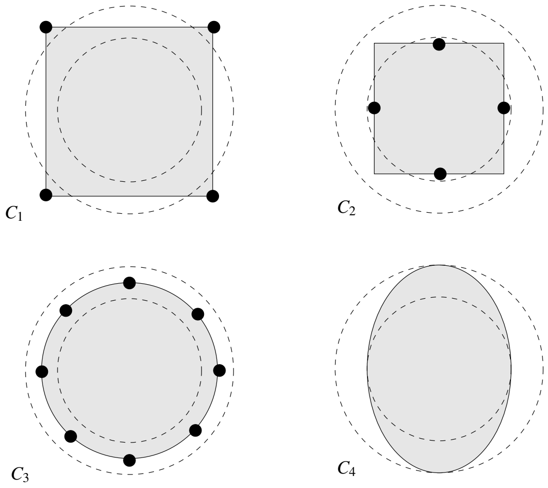

Figure 1: Objects C_1 and C_2 are "clearly" bad; C_3 is "clearly" good but the classification of C_4 as good is less evident. The four identical annuli indicated by dashed circles have radii $r_0(1 - \epsilon_0)$ and $r_0(1 + \epsilon_0)$.

We use the *finger probing* model of [1]. It postulates that the measurement device can identify a point in the interior of I and can probe along any ray, i.e., determine the first point on the ray in bd I. Note that this is a simple mathematical model for Coordinate Measurement Machines (CMM's). CMM's are considered the state of the art in this area [14]. In this probing model, n probes yield a set S_n of at most n points in bd I. A necessary condition for the classification procedure to be correct is that the set of all objects I' with $S_n \subseteq \text{bd}\, I'$ is either collectively good or bad. Without additional assumption on I this will not be the case for any finite n. In this paper, we use the following approximate roundness assumptions.

> **Minimum Quality Assumption (MQA and CMQA)**: I is a body and there is a point p and $\delta > 0$ such that $R(p) \le r_0(1 + \delta)$ and $r(p) \ge r_0(1 - \delta)$, where $\delta = 1/20$ and $\epsilon_0 \le \delta/2$. If we further assume that I is convex, then we denote the assumption by CMQA.

The constant $1/20$ in MQA is arbitrary and is easily replaced by larger values. But we feel that the MQA assumption is not critical for practical purposes: modern processes for manufacturing round bodies can easily satisfy our requirement. But the convexity assumption in CMQA is limiting. However, we believe that our techniques can be extended to replace convexity by a bound on the maximum possible negative curvature

2 Main Results

We explain the intuitive idea of our main results, using the objects $C_1, \ldots, C_4$ in Figure 1. We shall assume that C_i's are known a-priori to be convex. We say that C_1 is "clearly" bad because its violation of the outer circle of radius $r_0(1 + \epsilon_0)$ is easily witnessed by the four corners of C_1. (This depends on the convexity assumption.) So if our probe strategy yields a set of points that contains some approximation to each of these four corners, we can confidently reject C_1. Likewise, C_2 is bad as witnessed by the 4 indicated points on its boundary, and C_3 is witnessed to be good by the 8 indicated points on its boundary. Although C_4 is good, it cannot be witnessed by any finite set of points. Informally, the reason is that the roundness of C_4 is arbitrarily close to the boundary between "good" and "bad". We measure this closeness to the boundary by the "quality measure" (the quality of C_4 turns out to be 0).

For a point p, let

$$\begin{aligned} qual(p, I) &= \min\{(r(p) - (1 - \epsilon_0)r_0, \\ &\qquad (1 + \epsilon_0)r_0 - R(p)\}/r_0, \end{aligned}$$

and let

$$qual(I) = \max_p qual(p, I).$$

We call a point p with $qual(p, I) = qual(I)$ a *center* of I and use c_I to denote a center of I. Let A be the annulus with inner radius $(1 - \epsilon_0)r_0$ and outer radius $(1 + \epsilon_0)r_0$. For a good object, c_I defines a

[1] Since bd I is a closed bounded set the minimum and maximum exist.

[2] this definition of roundness is by far not the only conceivable one. It is called referenced rounding in [2]. Our results hold for several other definitions of roundness as we will show in section 5.

placement of I such that bd $I \subseteq A$ and the distance between bd I and bd A is maximized, and for a bad object, c_I defines a placement of I that minimizes the distance between bd $I \cap$ cpl A and bd A. Here, cpl A denotes the complement of A. The minimum quality assumption implies that $qual(I) \geq -(\delta - \epsilon_0)$ and that $(1-\delta)r_0 \leq r(c_I) \leq R(c_I) \leq (1+\delta)r_0$.

We now state our main results. Basically, we show classification procedures whose complexity is *quality sensitive*:

Theorem 1 *Under assumption CMQA there is a classification algorithm that classifies any body I in time $O(|1/qual(I)| \log |1/qual(I)|)$ with $O(|1/qual(I)|)$ probes.*

Theorem 1 is based on the following two theorems. The following theorem has independent interest:

Theorem 2 *Under assumption MQA, six probes and constant running time suffice to determine a point c_0 with $dist(c_0, c_I) \leq 2\delta r_0$.*

Let c_0 be as in the theorem above and let n be any positive integer. For any integer i, $0 \leq i < n$, send a ray along the direction $2\pi i/n$, and let S_n be the set of contact points. The quality of S_n is a good approximation of the quality of I.

Theorem 3 *Under assumption CMQA, $|qual(S_n) - qual(I)| = O(1/n)$ and $qual(S_n)$ can be computed in time $O(n \log n)$.*

The complete classification algorithm is given in Figure 2. By Theorem 3 there is a constant c such that $|qual(S_n) - qual(I)| \leq c/n$. Therefore, n cannot exceed $c \cdot qual(I)$ and Theorem 1 follows.

Note that this procedure runs forever if the quality is 0 (as when $I = C_4$ in Figure 1). In practice, we should stop the algorithm when the quality becomes sufficiently close to 0.

The structure of the paper is as follows. In section 3 we prove Theorem 2 and in section 4 we prove Theorem 3. In section 5 we extend our results to a larger class of quality measures and in section 6 we list open problems.

3 Initial Placement

We provide a simple strategy for finding a point c_0 close to the center c_I of I. This result has independent and practical interest. For instance, there are highly specialized roundness-measuring machines. Traditionally, the placement of an object in such machines are carried out manually. Our result means that one can easily automate this process with just 6 initial probes.

The strategy is based on two simple subroutines: For any point p, the pair of horizontal probes directed towards p, one from the left and one from the right, is called the *horizontal pair* for p. If p_ℓ and p_r are the two contact points returned by the horizontal pair for p, define $H(p) = H_I(p)$ to be $(p_\ell + p_r)/2$. Note that $y(p) = y(H(p))$. Also, $p_\ell = \infty$ iff $p_r = \infty$; in this case, $H(p)$ is defined to be ∞. Similarly, the two vertical probes directed towards p, one from above and one from below, is called the *vertical pair* for p. If the corresponding two contact points are p_a and p_b, we define $V(p) = V_I(p)$ to be $(p_a + p_b)/2$. For completeness, define $H(\infty) = V(\infty) = \infty$.

Theorem 4 *Let c_I be a center of I and let $\rho = R(c_I)/r(c_I) - 1$. Let p_0 be any point in the interior of I and let $c_0 = H(V(H(p_0)))$. If $\rho \leq 0.1$ then $dist(c_0, c_I) \leq \rho \cdot r(c_I)$.*

Proof: In the following analysis, assume without loss of generality that the center c_I is at the origin, and $r(c_I)$ is normalized to 1. Let

$$p_1 := H(p_0), \quad p_2 := V(p_1), \quad p_3 := H(p_2).$$

So $c_0 = p_3$. Without loss of generality, assume $x(p_1) \geq 0$ and $y(p_2) \geq 0$. First we show an upper bound on $x(p_1) = |x(p_1) - x(c_0)|$:

$$x(p_1) \leq \sqrt{2\rho + \rho^2}. \tag{1}$$

There are two cases to consider.

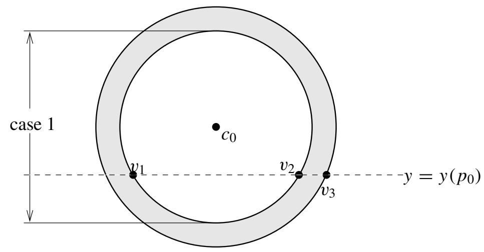

Figure 3: $-1 \leq y(p_1) \leq 1$

Case 1: $-1 \leq y(p_1) \leq 1$. Refer to Figure 3.

Let v_1, v_2 be the points where the horizontal line $y = y(p_0)$ intersects the unit circle centered at c_0. Let $x_i = x(v_i)$ $(i = 1, 2)$ and $x_1 < x_2$. Since the unit circle is contained in I, the left probe contacts I at the x-coordinate x_ℓ where $x_\ell \leq x_1$. Let x_3 be the positive x-coordinate where this line intersects the circle of radius $1 + \rho$ centered at

> Compute a point c_0 as in Theorem 2.
> $n = 2$;
> **Repeat**
> $n = 2 * n$;
> Determine S_n and compute $qual(S_n)$ and an interval $[l, u]$ containing $qual(I)$.
> **until** $0 \notin [l, u]$
> **If** $(l > 0)$ declare I good;
> **If** $(u < 0)$ declare I bad;

Figure 2: The complete classification algorithm.

c_0. Since I is contained in the circle of radius $1 + \rho$, the right probe contacts I at the x-coordinate x_r where $x_r \leq x_3$. So $x(p_1) = (x_\ell + x_r)/2 \leq (x_1 + x_3)/2$. Furthermore, since $x_1 = -x_2$, $x(p_1) \leq (x_3 - x_2)/2$. The quantity $x_3 - x_2$ is largest when $y(p_0) = 1$ or -1; in this case, the Pythagorean theorem shows $x_3 - x_2 = \sqrt{(1+\rho)^2 - 1^2} = \sqrt{2\rho + \rho^2}$. Thus $x(p_1) \leq \sqrt{2\rho + \rho^2}/2$.

Case 2: $y(p_1) < -1$ or $y(p_1) > 1$.

We may define x_3, x_ℓ and x_r as in case 1. Clearly both x_ℓ and x_r are no greater than x_3. Thus, $x(p_1) = (x_\ell + x_r)/2 \leq x_3$. The coordinate x_3 is largest when y equals 1 or -1, in which case $x_3 = \sqrt{2\rho + \rho^2}$. Thus $x(p_1) \leq \sqrt{2\rho + \rho^2}$.

In either case, we obtain the desired bound (1). Notice that p_1 need not lie inside I. Hence it is important to argue that $p_2 = V(p_1)$ is finite. But this follows from the fact that I is a connected set. But in fact, it is easy to show something stronger, namely p_i (for any $i \geq 2$) actually lies inside I. To see this for p_2, observe that

$$x(p_1) < 0.5, \tag{2}$$

which in turn follows from (1) and $\rho \leq 1/10$.

Next we show $|y(p_2) - y(c_0)| \leq 5\rho/8$, which, under our assumptions, amounts to:

$$y(p_2) \leq 5\rho/8. \tag{3}$$

Referring to Figure 4(i), let the line $x = x(p_1)$ intersect the unit circle centered at c_0 at the points v_1, v_2. Let $y_i := y(v_i)$ $(i = 1, 2)$ and $y_1 < y_2$. The line also intersects the circle of radius $1 + \rho$ at the positive y-coordinate y_3. As in the proof of (1), Case 1, we have $y(p_2) \leq (y_3 - y_2)/2$. Now the maximum value $y_3 - y_2$ obtains when $x(p_1)$ is maximized. By (1), $x(p_1) \leq \sqrt{2\rho + \rho^2}$. Let $g := \sqrt{2\rho + \rho^2}$ and e represent $y_3 - y_2$ when $x(p_1) = g$. So $y(p_2) \leq e/2$ and its suffices to show $e \leq 5\rho/4$. Define h as in Figure 4(ii). A simple calculation shows that

$$e + h = 1, \qquad h = \sqrt{1 - g^2}$$

and hence

$$e = 1 - \sqrt{1 - g^2}.$$

Using the Taylor expansion

$$\begin{aligned}(1-x)^{1/2} &= 1 - \left(\frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots\right) \\ &> 1 - \left(\frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{16(1-x)}\right),\end{aligned}$$

and on substituting $x = g^2$ in the expression for e,

$$e < \frac{g^2}{2} + \frac{g^4}{8} + \frac{g^6}{16(1-g^2)}.$$

Using $g^2 < 0.25$,

$$e < g^2\left(\frac{1}{2} + \frac{1}{32} + \frac{1}{128}\right) < \frac{9g^2}{16}.$$

From $g^2 = 2\rho + \rho^2 < 2.1\rho$, we conclude that $e < 5\rho/4$, which proves (3).

We next show $|x(p_3) - x(c_I)| \leq 13\rho/25$. Assuming without loss of generality that $x(p_3) \geq 0$, this amounts to

$$x(p_3) \leq 13\rho/25. \tag{4}$$

With reference to Figure 5(i), let the line $y = y(p_2)$ intersect the circles centered at c_0 with radii 1 and $1 + \rho$ at the positive x-coordinates x_2' and x_3', respectively. The above analysis shows $x(p_3) \leq (x_3' - x_2')/2$. Assuming without loss of generality that $y(p_3) \geq 0$, the value of $x_3' - x_2'$ is increasing with $y(p_2)$. Let e' be the value of $x_3' - x_2'$ when $y(p_2) = 5\rho/8$, as in Figure 5(ii). Since

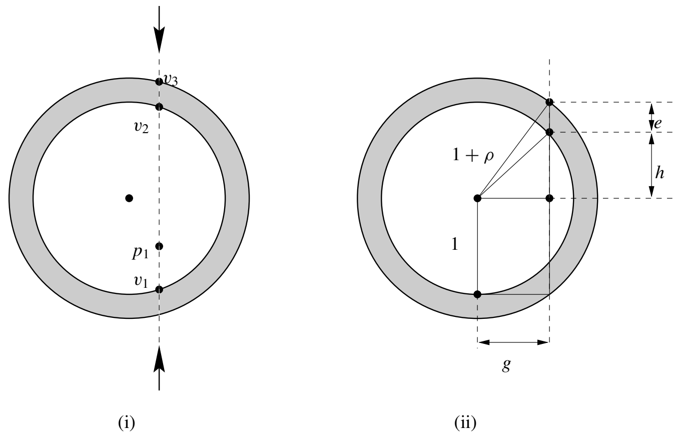

Figure 4: Second set of probes: bounding $y(p_2)$.

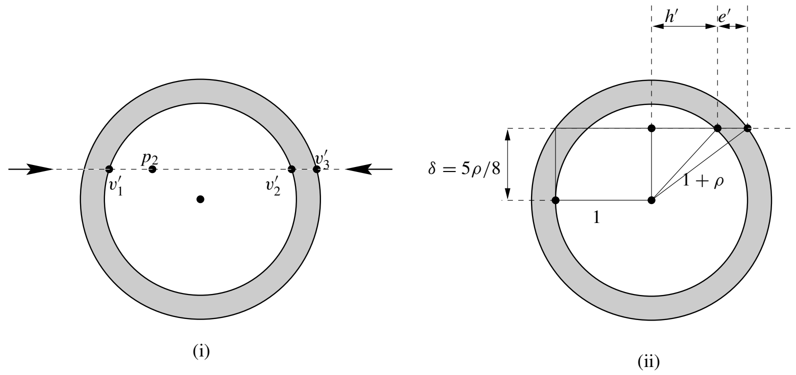

Figure 5: Third set of probes: bounding $x(p_3)$.

$y(p_2) \leq 5\rho/8$, this means $x(p_3) \leq e'/2$. It remains to bound e'. Putting $\delta = 5\rho/8$,

$$\begin{aligned} e' &= \sqrt{(1+\rho)^2 - \delta^2} - \sqrt{1-\delta^2} \\ &< (1+\rho) - 1 + \delta^2 \left(\frac{1}{2} + \frac{\delta^2}{8} \frac{1}{1-\delta^2} \right), \end{aligned}$$

using an abbreviated version of the previous Taylor expansion. Since $\delta^2 = 25\rho^2/64 < \rho/25$, we get $e' < 26\rho/25$, verifying (4).

From (3) and (4) we conclude that that $\| p_3 - c_I \| \leq \rho\sqrt{(5/8)^2 + (13/25)^2} < \rho$, and the theorem is proved. ∎

Theorem 2 is an easy consequence of Theorem 4 and MQA. MQA implies implies $\rho = R(c_I)/r(c_I) - 1 \leq (1+\delta)/(1-\delta) - 1 \leq 0.1$ and hence $dist(c_0, c_I) \leq \rho \cdot r(c_I) = R(c_I) - r(c_I) \leq 2\delta r_0$.

4 Uniform and Near-Uniform Probing

In this section we prove Theorem 3. Recall that we determine a set $S_n = \{v_0, \ldots, v_{n-1}\}$ of n points in bd I arranged in clockwise order by uniform probing about c_0, i.e., by probing along the directions $2\pi i/n$ for all i, $0 \leq i < n$. Let c_S be the center of S_n and define the core C of I by

$$C = \{p \in I;\ r(p, I) \geq (1 - 4\delta) r_0\}.$$

We show that c_I, c_0, and c_S lie in the core and that if p and q are points in the core and v and w are points in bd I then $\angle vpw \leq 6\angle vqw$. This implies that $\angle v_i c_S v_{i+1}$ is at most $12\pi i/n$ for all i, $0 \leq i < n$, i.e., the uniform probing about c_0 amounts to near-uniform probing about c_S with about the same sampling density. Finally, we show that the near-uniform probing about c_S implies that $|qual(I) - qual(S)| = O(1/n)$.

4.1 Uniform probing about c_0 amounts to near-uniform probing about c_S.

Lemma 5 *The points c_I, c_0, and c_S lie in C, $R(p, I) \le (1 + 6\delta)r_0$ for any point $p \in C$, and $dist(p, q) \le 10\delta r_0$ for any two points p and q from the core.*

Proof: The minimum quality assumption yields $r(c_I, I) \ge (1-\delta)r_0$ and hence $c_I \in C$ and Theorem 2 states that $dist(c_I, c_0) \le 2\delta r_0$. Thus $r(c_0, I) \ge (1 - 3\delta)r_0$ and hence $c_0 \in I$. For c_S we show $dist(c_S, c_I) \le 3\delta r_0$. Assume w.l.o.g. that c_I is at the origin. We show $x(c_S) \le 2\delta r_0$, from which the claimed bound on $dist(c_I, c_S)$ follows by symmetry and elementary calculation.

First note that $qual(I) \le qual S = qual(c_S, S)$ implies $R(c_S, S) \le (1 + \delta)r_0$. Next note that n is always a multiple of 4 and hence one of the probes is made along the negative x-axis. It hits I in a point whose x-coordinate is at most $-(1 - \delta)r_0$. Thus $R(c_S, S) \ge (1 - \delta)r_0 + x(c_S)$ and the bound on $x(c_S)$ follows.

I is contained in a disk D of radius $(1+\delta)r_0$ and hence no point in the core of I can have distance larger than $5\delta r_0$ from the center of D. This implies $R(p, I) \le (1 + 6\delta)r_0$ for any point $p \in C$, and $dist(p, q) \le 10\delta r_0$ for any two points p and q from the core. ∎.

Lemma 6 *Let p and q be points in the core of I and let v and w be points in* bd *I such that $2\alpha = \angle vqw$. Then $2\beta = \angle vpw \le 12\alpha$.*

Proof: If $\alpha \ge \pi/12$ there is nothing to show. Assume, otherwise. Then $\arcsin 2\tan\alpha \le 3\alpha$. For a point x we use r_x to denote $dist(p, x)$. We proceed in two steps.

First assume $r = r_v = r_w$. We show $\beta \le 3\alpha$. Let $L = 2\beta r/(2\pi)$ be the length of the circular arc from v to w. We bound L from above. We may assume w.l.o.g. that p is at the origin and that q lies on the positive x-axis, cf. Figure 6.

We claim that L is maximal if the axis of the wedge $W = \angle vqw$ aligns with the negative x-axis. Indeed consider any other orientation of W's axis and consider turning W's axis towards the negative x-axis. Then L will grow as long as the "forward leg" of W is longer than the "rear leg" of W. The worst case situation is therefore if W's axis aligns with the negative x-axis, cf. Figure 7. We have $r \ge (1 - 4\delta)r_0$, $z \le 10\delta r_0$, $\tan\alpha = y/(x + z)$, $\sin\beta = y/r$, and $x \le r$. Thus,

$$\sin\beta = y/r = \frac{(x+z)\cdot\tan\alpha}{r} \le 2\tan\alpha$$

and hence $\beta \le 3\alpha$.

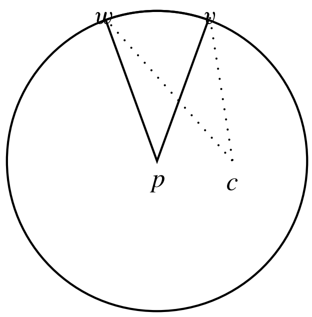

Figure 6: Situation in the proof of lemma 6

Now we come to the general case. Assume w.l.o.g, $r_v \le r_w$. Consider the circle with radius $r = r_v$ about v and let z be the intersection of this circle with the line l through q and w; z is unique since $r_w \ge r > r_q$. Then

$$2\beta = \angle vpw = \angle vpz + \angle zpw.$$

The angle $\angle vpz$ is bounded by 6α by the argument above. If z lies in the wedge $\angle vpw$ we also need to bound $\angle zpw$. Let a be the point on l closest to p, let $\gamma = \angle apz$ and $\eta = \angle zpw$, cf. Figure 8. Then $\cos\gamma = r_a/r_v$ and $\cos(\eta + \gamma) = r_a/r_w$.

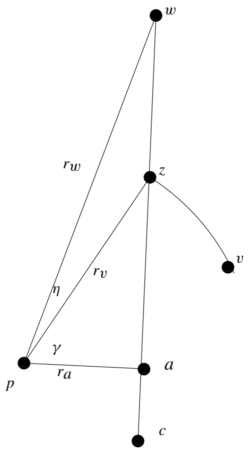

Figure 8: Bounding η

Since $\cos(\gamma + \eta) = \cos\gamma\cos\eta - \sin\gamma\sin\eta$ this implies

$$\begin{aligned}\sin\eta = & \ (\cos\gamma\cos\eta - \cos(\gamma+\eta))/\sin\gamma \\ \le & \ (\cos\gamma - \cos(\gamma+\eta))/\sin\gamma \\ = & \ ((r_w - r_v)r_a)/(r_v \cdot r_w)\cdot(1/\sin\gamma).\end{aligned}$$

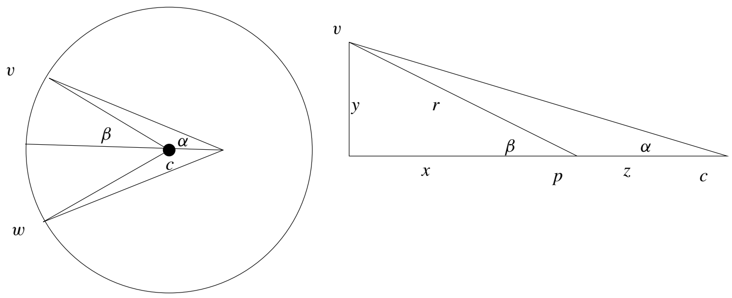

Figure 7: Worst Case

We have $\cos\gamma = r_a/r_v \le 6\delta/(1-4\delta) \le 1/10$ and hence $\sin\gamma = \sqrt{1-\cos^2\gamma} \ge \sqrt{99/100} \ge 9/10$. It remains to bound $(r_w - r_v)/r_w$. An obvious bound is $10\delta/(1-4\delta)$, which is, however, much too weak for our purposes.

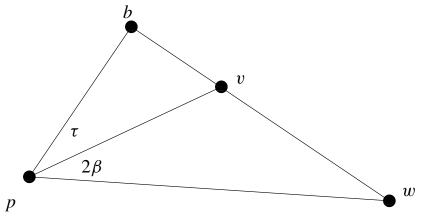

Figure 9: Bounding rw minus rv

Consider the line L through v and w and let b be the point closest to p on L; b may or may not lie between v and w. If b lies between v and w then

$$\begin{aligned}(r_w - r_v)/r_w &\le (r_w - r_b)/r_w\\ &\le 1-\cos 2\beta\\ &\le 2\sin\beta^2\\ &\le 2\beta.\end{aligned}$$

If b does not lie between v and w we have $r_b \ge (1-4\delta)r_0$ since otherwise a neighborhood of b is contained in I and hence $v \notin \operatorname{bd} I$ by the convexity of I. Let $\tau = \angle bpv$. We have $\cos\tau = r_b/r_v$ and $\cos(2\beta+\tau) = r_b/r_w$, cf. Figure 9, and hence

$$\begin{aligned}(r_w - r_v)/r_w &= 1-\cos(2\beta+\tau)/\cos\tau\\ &= 1-\cos 2\beta + (\sin\tau/\cos\tau)\sin 2\beta\\ &\le 2\sin^2\beta + (1/\cos\tau)\sin 2\beta\\ &\le 2\beta(1+1/\cos\tau).\end{aligned}$$

Since $\cos\tau = r_b/r_v \ge (1-4\delta)/(1+6\delta) \ge 1/2$ we conclude $(r_w - r_v)/r_w \le 6\beta$ and hence $\sin\eta \le 6\beta/9$. Substituting into $2\beta - 6\alpha \le \eta$ yields $\sin(2\beta - 6\alpha) \le 2\beta/3$. This implies $2\beta \le 12\alpha$. Note that the equation $\sin(2\beta-6\alpha) = 2\beta/3$ has a single solution with $2\beta \le \pi/4$ and that the iteration $2\beta_0 = 6\alpha$, $2\beta_{i+1} = 6\alpha + \arcsin 2\beta_i/3$ converges monotonically towards this solution. A simple induction shows that all iterates stay bounded by 12α. Indeed, $2\beta_0 \le 12\alpha$ and if $2\beta_i \le 12\alpha$ and hence $\arcsin 2\beta_i/3 \le \arcsin 4\alpha \le 6\alpha$ then $2\beta_{i+1} \le 12\alpha$. ∎

4.2 $qual(S_n)$ approximates $qual(I)$

We now conclude our proof of Theorem 3. Let $\theta = \pi/n$ and let $S = \{v_1 \dots v_n\} \subseteq \operatorname{bd} I$ be the points determined by probing along directions $2\theta i$, $0 \le i < n$, from c_0. Then $\angle v_i c_0 v_{i+1} = 2\theta$ and $\angle v_i c_S v_{i+1} \le 12\theta$, where c_S is the center of S by Lemma 6. Let $r_S = r(c_S, S)$ and $R_S = R(c_S, S)$.

Lemma 7

$$r_S(1 - 18\pi/n^2) \le r_S\cos 6\theta \le r(c_S, I) \le r_S. \quad (5)$$

Proof: The upper bound on $r(c_S, I)$ is immediate. For the lower bound, choose $q \in \operatorname{bd} I$ at distance $r(c_S, I)$ from c_S. The point q lies angularly between some two samples v_i and v_{i+1} around c_S. Let $2\sigma = \angle v_i c_S v_{i+1} \le 12\theta$. Since I is convex, the line segment $[v_i, v_{i+1}]$ must be contained in I. Convexity furthermore implies that the point q does not lie on the same side of this segment as c_S does. Thus, the distance $r(c_S, I)$ from c_S to q is at least the distance D from c_S to the segment. This distance D is minimized when the distances from c_S to v_i and from c_S to v_{i+1} are both r_S; in this case $D = r_S\cos\sigma \ge \cos 6\theta$, as desired. See Figure 10. The lower bound in terms of n follows from $\theta = \pi/n$ and $\cos\theta \ge 1 - \theta^2/2$. ∎

Lemma 8 *Let $z = R(c_S, I)/r(c_S, I)$. Then*

$$\begin{aligned}R_S \le R(c_S, I) &\le \frac{R_S}{\cos\theta - \sin\theta\sqrt{z^2-1}}\\ &\le R_S(1 + 18\pi/n).\end{aligned}$$

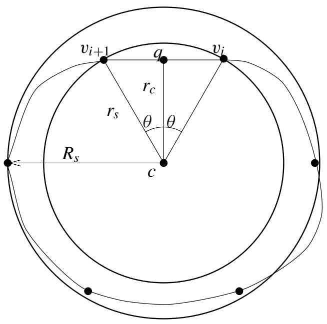

Figure 10: A lower bound on $r(c_S, I)$

Proof: The lower bound on $R(c_S, I)$ is immediate. For the upper bound, choose $q \in \operatorname{bd} I$ at distance $R(c_S, I)$ from c_S, lying angularly between two samples v_i and v_{i+1} when seen from c_S. Let $2\sigma = \angle v_i c_S v_{i+1} \le 12\theta$. The maximum value of $R(c_S, I)$ is obtained when v_i and v_{i+1} are both at distance R_S from c_S, q is on the bisector of $\angle v_i c_S v_{i+1}$, and the lines qv_i and qv_{i+1} are both tangent to the circle of radius $r(c_S, I)$ centered at c_S. This situation is illustrated in Figure 11.

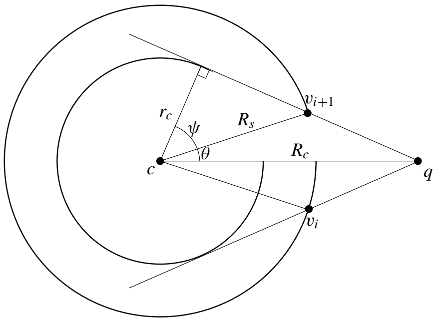

Figure 11: An upper bound on $R(c_S, I)$

Let ψ be the angle about c_S formed by v_{i+1} and the point where qv_{i+1} is tangent to the circle of radius $r(c_S, I)$. Then

$$\cos\psi = \frac{r(c_S, I)}{R_S}, \quad \cos(\psi+\sigma) = \frac{r(c_S, I)}{R(c_S, I)}.$$

Rearranging the latter equation,

$$\begin{aligned} R(c_S, I) &= \frac{r(c_S, I)}{\cos(\psi+\sigma)} \\ &= \frac{r(c_S, I)}{\cos\psi\cos\sigma - \sin\psi\sin\sigma} \\ &= \frac{R_S}{\cos\sigma - \sqrt{(R_S/r(c_S, I))^2 - 1}\,\sin\sigma}. \end{aligned}$$

For the upper bound in terms of n, observe that $\sin\sigma \le \sigma$ and $z = \le (1+\delta)/(1-\delta)$ implies

$$\begin{aligned} &\cos\sigma - \sin\sigma\sqrt{z^2-1} \\ &\quad = 1 - \sin^2\sigma - \sin\sigma\sqrt{z^2-1} \\ &\quad \ge 1 - \sigma(1+\sqrt{2}) \end{aligned}$$

and hence

$$R(c_S, I) \le R_S(1+3\sigma) \le R_S(1+18\pi/n).$$

∎

Lemma 9 *$qual(S_n) - 20\pi/n \le qual(I) \le qual(S)$ and $qual(S_n)$ can be computed in time $O(n \log n)$.*

Proof: $qual((I) \le qual(S_n)$ follows directly from $S \subseteq \operatorname{bd} I$. For the lower bound, let $r_S = r(c_S, S)$ and $R_S = R(c_S, S)$ and observe

$$\begin{aligned} qual(I) &= \max_p qual(p, I) \\ &\ge qual(c_S, I) \\ &= \min(r(c_S, I) - (1-\epsilon_0)r_0, \\ &\qquad (1+\epsilon_0)r_0 - R_S)/r_0 \\ &\ge \min(r_S(1 - 18\pi^2/n^2) - (1-\epsilon_0)r_0, \\ &\qquad (1+\epsilon_0)r_0 - R_S(1+18\pi/n))/r_0 \\ &\ge qual(S) - 18\pi\max(r_S/r_0, R_S/r_0)/n \\ &= qual(S) - 20\pi/n. \end{aligned}$$

The bound on the running time follows from [2].

∎

5 Other Quality Measures

There are many ways to quantify the roundness of an object besides the one used above. For example, we may require

- that the inner radius $r(p)$ lies in some interval $[r_0 - \epsilon_1, r_0 + \epsilon_2]$ and the outer radius $R(p)$ in some interval $[r_0 - \epsilon_3, r_0 + \epsilon_4]$, or
- that the width $R(p) - r(p)$ lies in some interval $[0, \epsilon_5]$ or
- that the relative width $(R(p) - r(p))/r(p)$ lies in some interval $[0, \epsilon_6]$ or
- that the average distance of a point in $\operatorname{bd} I$ from the center c lies in some interval $[r_0 - \epsilon_7, r_0 + \epsilon_7]$.

The approach described in this paper is readily extended to the first three constraints above and in fact to a large class of constraints that can be formulated in terms of an annulus containing the boundary of the body to be classified. Let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be a family of functions of two real arguments. We call f_i the i-th *measure of quality*. For a point p let

$$qual(p, I) = \min_{1 \le i \le k} f_i(r(p, I), R(p, I))$$

be the quality of the body I with respect to p and let

$$qual(I) = \max_p qual(p, I)$$

be the quality of I. We call a point p with $qual(I) = qual(p, I)$ an *$\mathcal{F}$-center* of I and use $c_{\mathcal{F},I}$ to denote an $\mathcal{F}$-center of I. The examples above lead to the quality functions $f_1(r, R) = r - (r_0 - \epsilon_1)$, $f_2(r, R) = r_0 + \epsilon_2 - r$, $f_3(r, R) = R - (r_0 - \epsilon_3)$, $f_4(r, R) = r_0 + \epsilon_4 - R$, $f_5(r, R) = \epsilon_5 - (R - r)$, and $f_6(r, R) = \epsilon_6 - (R - r)/r$. The specialization $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_0 r_0$ was considered in the preceding section.

Our approach hinges on the following two properties

- the quality of the sample $S = S_n$ can be computed efficiently and
- the quality of S_n is a good estimate for the quality of I.

With respect to the first property we observe that the overlay of the furthest and nearest site Voronoi diagram partitions the plane into cells (= regions, edges, and vertices) with the property that for all p in a fixed cell the radii $r(p, S)$ and $R(p, S)$ are determined by a small number of points in the sample S, namely two points for regions, three points for edges, and four points for vertices. For a cell f let S_f be the points in S that determine $r(p, S)$ and $R(p, S)$ for all p in f.

Assumption BM: Let f be a cell and let F be the affine hull of f; F is either the plane, or a line, or a vertex. We assume that the function $p \longrightarrow qual(p, S_f)$ for $p \in F$ has only a bounded number of local minima and that these minima are computable in constant time.

Assumption BM is certainly satisfied for any quality measure based on the functions $f_1, \ldots, f_6$ above. We proceed under assumption BM. For each f, the equations $f_i = f_j$, $1 \le i < j < k$, partition F into cells where in each cell we have a fixed ordering of the values of $f_1, \ldots, f_k$. Thus, determining the local minima within a cell amounts to a bounded size (since $|S_f| \le 4$) optimization problem.

We determine the local minima of $qual(p, S_f)$ in F and then test these minima for containment in f. The "surviving" minima are candidates for the $\mathcal{F}$-center of S. The total number of candidates is $O(n^2)$. We conclude:

Lemma 10 *Under assumption BM the quality of a sample S of size n can be computed in time $O(n^2)$.*

We next turn to the second property that $qual(S_n)$ is a good estimate of $qual(I)$.

Assumption CD: For all sufficiently large n, all $\mathcal{F}$-centers of I and all $\mathcal{F}$-centers of S_n lies in the core of I. The quality functions $f_1, \ldots, f_k$ are differentiable and have bounded derivative.

Lemma 11 *Under assumptions CMQA, BM, and CD we have $|qual(I) - qual(S_n)| = O(1/n)$.*

Proof: By the first part of assumption CD any $\mathcal{F}$-center $c_S = c_{\mathcal{F},S}$ of S lies in the core of I. Also c_0 lies in the core and hence $r(c_S, I) - O(1/n) \le r(c_S, I) \le r(c_S, S)$ and $R(c_S, S) \le R(c_S, I) \le R(c_S, S) + O(1/n)$ by Lemmas 7 and 8. Thus, $(|f_i(r(c_S, I), R(c_S, I)) - f_i((r(c_S, S), R(c_S, S))| = O(1/n)$ and hence

$$\begin{aligned} qual(I) \ge\ & qual(c_S, I) \\ \ge\ & qual(c_S, S) - O(1/n). \end{aligned}$$

The same argument with $c_{\mathcal{F},I}$ instead of c_S shows

$$\begin{aligned} qual(I) =\ & qual(c_I, I) \\ \le\ & qual(c_I, S) + O(1/n) \\ \le\ & qual(S) + O(1/n). \end{aligned}$$

■

Assumption CD is satisfied for all quality measures based on functions $f_1, \ldots, f_5$. It is not satisfied for the measure $qual(p, I) = f_6(r(p, I), R(p, I))$. Under this measure the center may lie at infinity and not within I.

Theorem 12 *Under assumptions CMQA, BM, and CD, there is a classification algorithm working in time $O(|1/qual(I)|^2)$ time with $O(|1/qual(I)|)$ probes.*

Proof: The algorithm was already given in section 1. Its correctness and performance follows immediately from Lemmas 10 and 11. ■

6 Discussion and Open Problems

Our work rises many open problems:

We have introduced several notions of roundness, and given a general procedure for their associated classification problems. It seems to us that the different notions of roundness are useful for different applications. It is unclear why the width measure $w(I)$ is dominant in practice.

What is the influence of the probing model? For example, what happens for a "line prober" that is able to determine the tangent to the object perpendicular to a specified direction. We conjecture that the error of approximation reduces from $O(1/n)$ to $O(1/n^2)$.

We claim only a running time of $O(n^2)$ for determining $qual(S_n)$. For two quality measures a time bound of $O(n \log n)$ can be obtained: For referenced roundness this is shown in [2] and for width this is shown in [6]. Can the time bound be improved for other quality measures.

Does the approach extend to more complicated classification tasks, e.g., determining roundness of a three-dimensional object?

In our approach we keep the probing center c_0 fixed although c_{S_n} is known to be a much better approximation of c_I as n gets large. Can this be exploited?

Is an approach that classifies "clearly good" and "clearly bad" objects quickly and takes longer on "borderline objects" of interest to actual metrology? Of course, in practice, one should modify our decision procedure to simply accept or reject once we have determined that the quality is smaller than some constant. This is the decision policy issue studied in [15].

Measurement devices make errors [8]. How can this be taken into account?

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