The Boolean Column and Column–Row Matrix Decompositions

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16 September 2008

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This Talk

1. Background.
2. Propose new decompositions combining two previusly-proposed ideas.
3. Study the computational complexity of the problems.
    - Relate the results to other, known ones.
4. Propose simple algorithms for the problems.
5. Some experimental evaluation.

Outline

Background: Column and Column–Row Decompositions

Given a matrix $\mathbf{A}$, represent it using

- linear combinations of a subset of its columns, i.e. $\mathbf{A} \approx \mathbf{CX}$ (CX decomposition)
  - Finding columns of $\mathbf{C}$ is known as the Column Subset Selection problem.
  - Resembles feature selection.
- combinations of a subset of its columns and a subset of its rows, i.e. $\mathbf{A} \approx \mathbf{CUR}$ (CUR decomposition)

Lot-studied in math, recently gained interest in CS

- 1 + 1 papers in KDD'08, 2 papers in SODA'09 . . .

Background: Boolean Matrix Decompositions

Given a binary matrix A, represent it as $\mathbf{A} \approx \mathbf{X} \circ \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are binary.

- Matrix multiplication is done over the Boolean semiring.
  - i.e. addition defined as $1 + 1 = 1$
- Can yield increased interpretability and decreased reconstruction error.
- Combinatorial problem, results from numerical linear algebra do not apply.
- Studied in combinatorics (Boolean or Schein rank), and in data mining
  - discrete basis problem (PKDD'06), role mining problem (ICDE'08), KDD'08 workshop on data mining using matrices and tensors, ...

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Outline

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Boolean CX and CUR Decompositions

Problem (Boolean CX Decomposition, BCX)

Given a matrix $A \in \{0, 1\}^{m \times n}$ *and an integer* k*, find an* $m \times k$ *binary matrix* C *of* k *columns of* A *and a matrix* $X \in \{0, 1\}^{k \times n}$ *minimizing* $d_1(\mathbf{A}, \mathbf{C} \circ \mathbf{X}) = \sum_{i,j} \left|(\mathbf{A})_{ij} - (\mathbf{C} \circ \mathbf{X})_{ij}\right|$.

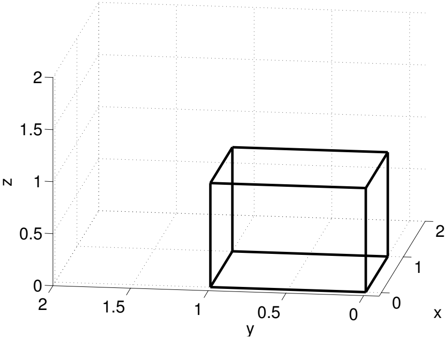
Problem (Boolean CUR Decomposition, BCUR)

Given a matrix $A \in \{0, 1\}^{m \times n}$ *and integers* k *and* r*, find an* $m \times k$ *binary matrix* C *of* k *columns of* A*, an* $r \times n$ *binary matrix* R *of* r *rows of* A*, and a matrix* $U \in \{0, 1\}^{k \times r}$ *minimizing* $d_1(\mathbf{A}, \mathbf{C} \circ \mathbf{U} \circ \mathbf{R}) = \sum_{i,j} \left|(\mathbf{A})_{ij} - (\mathbf{C} \circ \mathbf{U} \circ \mathbf{R})_{ij}\right|$.

BCX Visualized

- Columns of $\boldsymbol{A}$ represent corners in Boolean hypercube

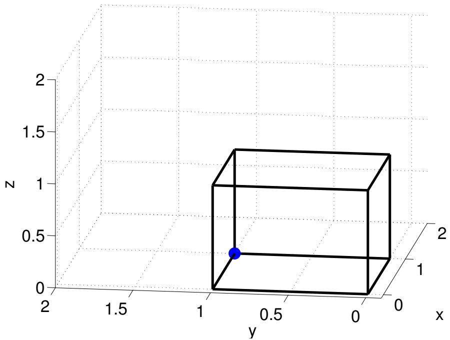
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$



BCX Visualized

- Columns of A represent corners in Boolean hypercube

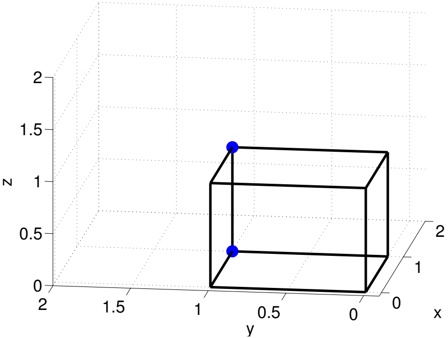
$$\begin{pmatrix} \color{blue}{1} & 1 & 0 \\ \color{blue}{1} & 1 & 1 \\ \color{blue}{0} & 1 & 1 \end{pmatrix}$$



BCX Visualized

- Columns of A represent corners in Boolean hypercube

$$\begin{pmatrix} \color{blue}{1} & \color{blue}{1} & 0 \\ \color{blue}{1} & \color{blue}{1} & 1 \\ \color{blue}{0} & \color{blue}{1} & 1 \end{pmatrix}$$



BCX Visualized

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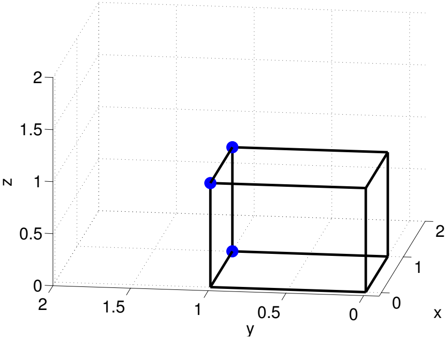
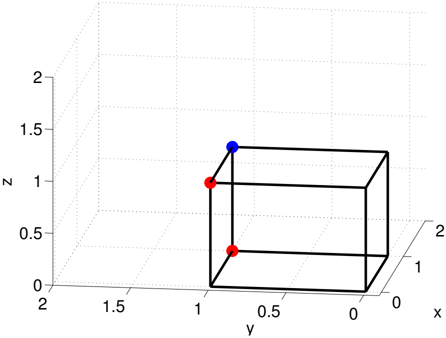


BCX Visualized

- $\mathbf{C}$ selects some of the corners

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$



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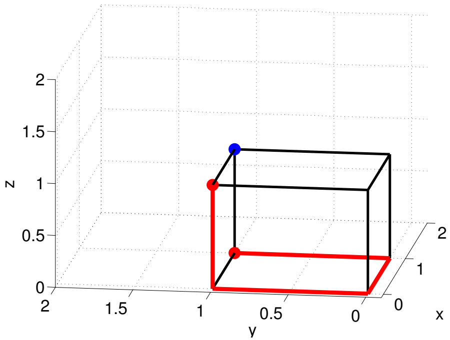
BCX Visualized

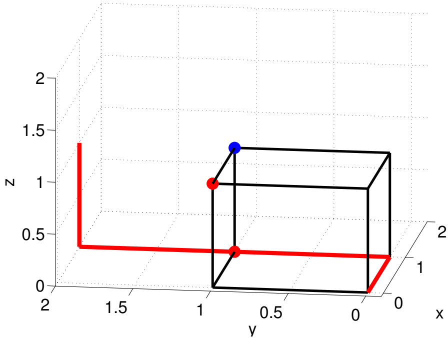
- $\mathbf{C}$ selects some of the corners

$$\begin{pmatrix} \color{red}{1} & \color{blue}{1} & \color{red}{0} \\ \color{red}{1} & \color{blue}{1} & \color{red}{1} \\ \color{red}{0} & \color{blue}{1} & \color{red}{1} \end{pmatrix}$$



BCX Visualized

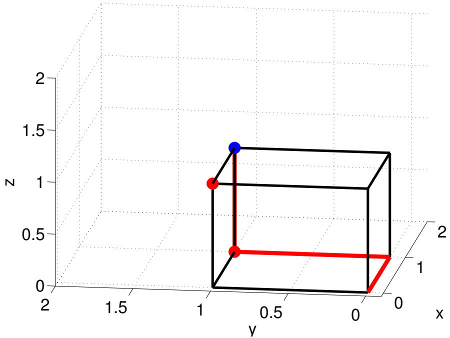
- The remaining corner is presented as a sum of the selected corners.



$$\begin{pmatrix} \color{blue}{1} \\ \color{blue}{1} \\ \color{blue}{1} \end{pmatrix} \approx \begin{pmatrix} \color{red}{1} & \color{red}{0} \\ \color{red}{1} & \color{red}{1} \\ \color{red}{0} & \color{red}{1} \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

BCX Visualized

- The remaining corner is presented as a **Boolean** sum of the selected corners.

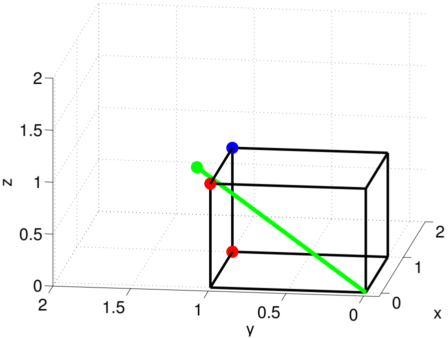


$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

BCX Visualized

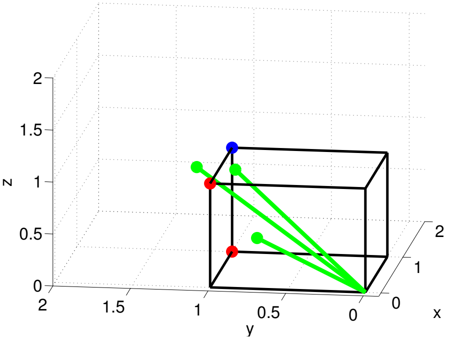
- The remaining corner is presented as a **Boolean** sum of the selected corners.
- Green line represents rank-2 SVD approximation of the blue corner.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.8536 \\ 1.2071 \\ 0.8536 \end{pmatrix}$$



BCX Visualized

- The whole rank-2 SVD approximation is the following.



$$\mathbf{U\Sigma V}^T = \begin{pmatrix} 1.1036 & 0.8536 & 0.1036 \\ 0.8536 & 1.2071 & 0.8536 \\ 0.1036 & 0.8536 & 1.1036 \end{pmatrix}$$

Two Subproblems To Solve 1: Basis Usage

Problem (Basis Usage, BU)

Given matrices $\mathbf{A} \in \{0,1\}^{n \times m}$ *and* $\mathbf{C} \in \{0,1\}^{n \times k}$, *find a matrix* $\mathbf{X} \in \{0,1\}^{k \times m}$ *minimizing* $d_1(\mathbf{A}, \mathbf{C} \circ \mathbf{X}) = \sum_{i,j} \left|(\mathbf{A})_{ij} - (\mathbf{C} \circ \mathbf{X})_{ij}\right|$.

Few notes:

1. General definition: $\mathbf{C}$ does **not** have to have $\mathbf{A}$'s columns.
2. Each column of $\mathbf{X}$ is independent!

Thus, an equivalent problem is:

Problem

Given a vector $\mathbf{a} \in \{0,1\}^{n}$ *and a matrix* $\mathbf{C} \in \{0,1\}^{n \times k}$, *find a vector* $\mathbf{x} \in \{0,1\}^{k}$ *minimizing* $\sum_i \left|a_i - (\mathbf{C} \circ \mathbf{x})_i\right|$.

Two Subproblems To Solve 2: Mixing Matrix

Problem (Mixing Matrix, MM)

Given matrices $\mathbf{A} \in \{0,1\}^{n \times m}$, $\mathbf{C}$ *of* k *columns of* $\mathbf{A}$, *and* $\mathbf{R}$ *of* r *rows of* $\mathbf{A}$, *find a matrix* $\mathbf{U} \in \{0,1\}^{k \times r}$ *minimizing* $d_1(\mathbf{A}, \mathbf{C} \circ \mathbf{U} \circ \mathbf{R}) = \sum_{i,j} \left| (\mathbf{A})_{ij} - (\mathbf{C} \circ \mathbf{U} \circ \mathbf{R})_{ij} \right|$.

- Now $\mathbf{C}$ and $\mathbf{R}$ are restricted to column and row subsets.
- No element of $\mathbf{U}$ is independent.

$$(\mathbf{C} \circ \mathbf{U} \circ \mathbf{R})_{ij} = \bigvee_{h=1}^{k} \bigvee_{l=1}^{r} c_{ih} \wedge u_{hl} \wedge r_{lj}.$$

$\Rightarrow$ Element u_{hl} can change $(\mathbf{C} \circ \mathbf{U} \circ \mathbf{R})_{ij}$ only when $c_{ih} = r_{lj} = 1$.

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Complexity of the BU Problem (1/3): Background

The Positive–Negative Partial Set Cover problem ($\pm$PSC):

- Cover as many of the positive elements as possible while minimizing the number of covered negative elements.

BU and $\pm$PSC problems are essentially the same.

1. BU with A having only 1 column is no easier than other instances.
2. C = incidence matrix of the set system; a = positive ($a_i = 1$) and negative ($a_i = 0$) elements; x selects the sets to the cover.

Complexity of the BU Problem (2/3): The Negative Side

Theorem

1. *Unless* P $=$ NP, *then for any* $\varepsilon > 0$ *there exists no poly-time approximation algorithm for BU with ratio*

$$\Omega\left(2^{\log^{1-\varepsilon}(k^4)}\right).$$

2. *Unless* NP $\subseteq$ DTIME$(n^{\text{polylog}(n)})$, *then for any* $\varepsilon > 0$ *there exists no poly-time approximation algorithm for BU with ratio*

$$\Omega\left(2^{\log^{1-\varepsilon} f}\right),$$

where f *is the maximum number of* 1*s in* $\mathbf{A}$*'s columns.*

- **N.B.** $2^{\log^{1-\varepsilon} n}$ is superpolylogarithmic and subpolynomial.

Complexity of the BU Problem (3/3): The Positive Side

Theorem

There is a poly-time approximation algorithm with ratio $2\sqrt{(k+f)\log f}$.

The algorithm needs to solve the classical Set Cover multiple times with inflated input instances.

Complexity of the MM Problem

Theorem

The MM problem can be reduced to the $\pm$PSC *problem in an approximation-preserving way.*

Theorem

The $\pm$PSC *problem can be reduced to the MM problem preserving the approximability up to constant factors.*

- The results for the BU problem hold for the MM problem.
- Caveat! The parameters have changed
  - no meaningful counterpart to f
  - k becomes to $\max\{k, r\}/2$.

Complexity of the BCX Problem

- The hardness of BU does not automatically mean that BCX is hard.
- Nevertheless, via a reduction from BU we get that (the decision version of) BCX is NP-complete.
  - This reduction is **not** approximation-preserving.
- The complexity of the BCUR problem is an open question.

Linear and Boolean Worlds: A Comparison

Linear world

- Finding $\mathbf{x}$ to minimize $\|\mathbf{C}\mathbf{x} - \mathbf{a}\|$ (i.e. least-squares fitting) is poly-time.
- Finding $\mathbf{U}$ to minimize $\|\mathbf{C}\mathbf{U}\mathbf{R} - \mathbf{A}\|$ is poly-time.
- Complexity of the Column Subset Selection problem is unknown.

Boolean world

- Finding $\mathbf{x}$ to minimize $\|\mathbf{C} \circ \mathbf{x} - \mathbf{a}\|$ (i.e. the BU problem) is hard even to approximate.
- Finding $\mathbf{U}$ to minimize $\|\mathbf{C} \circ \mathbf{U} \circ \mathbf{R} - \mathbf{A}\|$ (i.e. the MM problem) is hard even to approximate.
- The BCX problem is NP-hard.

Outline

Finding $\mathbf{C}$ and $\mathbf{R}$

Local-search heuristic $\texttt{Loc}$:

1. start with random columns in $\mathbf{C}$
2. **while** reconstruction error decreases **do**
   1. swap a column of $\mathbf{C}$ with a column of $\mathbf{A}$ not in $\mathbf{C}$ if this reduces the reconstruction error most
3. **return** $\mathbf{C}$

- Find $\mathbf{R}$ by running $\texttt{Loc}$ to $\mathbf{A}^T$.
- We need to know **some** $\mathbf{X}$ to know how good a swap is.
  - $\Rightarrow$ Use greedy *cover* function: column $\mathbf{c}^i$ is used to cover column $\mathbf{a}^j$ (i.e. $x_{ij} = 1$) if $\mathbf{c}^i$ covers more **yet uncovered 1s** of $\mathbf{a}^j$ than it covers **uncovered 0s**.

Finding $\mathbf{X}$ and $\mathbf{U}$

- `Loc & ±PSC`: Use the ±PSC algorithm to find $\mathbf{X}$.
  - Practically infeasible to $\mathbf{U}$.
- `Loc & IterX`: Iteratively update rows of $\mathbf{X}$ using the *cover*-function.
- `Loc & IterU`: Start with empty $\mathbf{U}$ and flip u_{hl} if it decreases the error; iterate untill convergence.
- `Loc & Maj`: For each a_{ij} mark which u_{hl} should be set to 1 or 0, and select u_{hl} to be the (weighted) majority of the opinions.
  - Recall: u_{hl} can change the value of $(\mathbf{C} \circ \mathbf{U} \circ \mathbf{R})_{ij}$ only if $c_{ih} = r_{lj} = 1$.

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Other Algorithms

For general CX and CUR decompositions:

- $\mathrm{8\,4\,4}$ by Berry, Pulatova, and Stewart (ACM Trans. Math. Softw. 2005)
- DMM by Drineas, Mahoney, and Muthukrishnan (ESA, APPROX, and arXiv 2006–07)
  - based on sampling, approximates SVD within $1+\varepsilon$ w.h.p., but needs lots of columns in C.

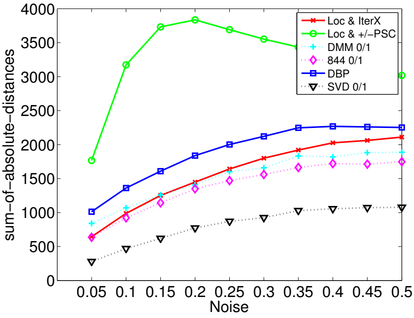
For general decompositions:

- SVD
  - lower bound for linear methods; in practice also a lower bound to all methods

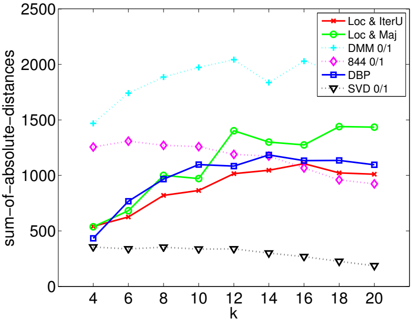
For general Boolean matrix decompositions:

- DBP by Miettinen et al.
  - theoretical lower bound for Boolean methods

Synthetic Data



[B]CX decomposition, noise varies



[B]CUR decomposition, k varies

- Results of continuous methods are rounded for improved accuracy.

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Outline

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Conclusions

- Boolean CX and CUR decompositions are potential tools for data mining.
- The problems are hard even to approximate, somewhat contrast to linear decompositions.
- Open questions: approximability of BCX, complexity of BCUR.
- Simple algorithms work up to some level, better ones are sought.

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Conclusions

- Boolean CX and CUR decompositions are potential tools for data mining.
- The problems are hard even to approximate, somewhat contrast to linear decompositions.
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Thank You!

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