Rewrite Techniques in Theorem Proving
Part III: Redundancy

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Overview

We discuss superposition as a special case of subterm chaining with equality relations.

Furthermore, we outline how saturated sets of formulas may be constructed and use the notion of fairness to establish a corresponding connection between dynamic and static aspects of the theorem proving process.

Superposition

We may use subterm chaining as the basic inference rule for dealing with equality.

(Positive) Superposition

$$\frac{C \vee \boxed{\boxed{s} \approx t} \qquad D \vee \boxed{\boxed{u[s']} \approx v}}{(C \vee D \vee u[t] \approx v)\sigma}$$

(Negative) Superposition

$$\frac{C \vee \boxed{\boxed{s} \approx t} \qquad D \vee \boxed{\neg(\boxed{u[s']} \approx v)}}{(C \vee D \vee u[t] \approx v)\sigma}$$

where (i) σ is a most general unifier of s and s' and (ii) s' is not a variable.

Example

Suppose $a \succ b \succ c \succ d$ and consider the following clauses:

$$\boxed{c} \not\approx d$$

$$\boxed{b} \approx d$$

$$a \not\approx d \vee \boxed{a} \approx c$$

$$\boxed{a} \approx b \vee a \approx d$$

By superposition we obtain

$$a \not\approx d \vee a \approx d \vee b \approx c$$

If we disregard tautologies, no other clauses can be deduced by superposition, yet the given clause set is inconsistent.

Equality factoring

Transitivity resolution can be replaced by a considerably more restrictive inference rule that also subsumes factoring.

Equality Factoring

$$\frac{C \vee \boxed{u'} \approx v' \vee \boxed{\boxed{u} \approx v}}{(C \vee v \approx v' \vee u' \approx v')\sigma}$$

where σ is a most general unifier of u and u'.

Example

Continuing with the example above, we see that there is also an equality factoring inference

$$\frac{a \approx b \vee a \approx d}{b \not\approx d \vee a \approx d}$$

We may use the conclusion of this inference for superposition, in

$$\frac{b \not\approx d \vee a \approx d \quad a \not\approx d \vee a \approx c}{a \not\approx d \vee b \not\approx d \vee c \approx d}$$

and again in

$$\frac{b \not\approx d \vee a \approx d \quad a \not\approx d \vee b \not\approx d \vee c \approx d}{d \not\approx d \vee b \not\approx d \vee b \not\approx d \vee c \approx d}$$

and eventually derive a contradiction.

Resolution and factoring

Positive factoring is essentially a special case of equality factoring. For example, we have

$$\frac{C \vee s \approx t \vee s \approx t}{C \vee t \not\approx t \vee s \approx t}$$

Similarly, resolution is a special case of superposition, if we encode non-equality atoms A as equations $A \approx \top$.

For example,

$$\frac{C \vee A \approx \top \quad D \vee \neg(A \approx \top)}{C \vee D \vee \top \not\approx \top}$$

Redundancy of Clauses

Let $\mathcal{E}$ be the set of all equality axioms (reflexivity, symmetry, transitivity, congruence).

Recall that a ground clause C is $\mathcal{E}$-redundant with respect to a set N if there exist ground instances $C_1, \dots, C_n$ of N such that

$$C_1, \dots, C_n, \mathcal{E} \models C \text{ and}$$
$$C \succ C_1, \dots, C_n.$$

A clause is $\mathcal{E}$-redundant in N if all its ground instances are.

Examples

The following clauses are all $\mathcal{E}$-redundant:

$$C \vee A \vee \neg A \qquad \text{(tautologies)}$$

$$C \vee t \not\approx t$$

$$s \not\approx v \vee t \not\approx v \vee s \approx t$$

Furthermore *subsumed* clauses are redundant. That is, a clause $C \vee D$ is redundant in the presence of C.

Redundancy of Inferences

A ground superposition inference

$$\frac{C \quad D}{D'}$$

where $D \succ C$ is $\mathcal{E}$-redundant with respect to N if either some premise is redundant or else there exist ground instances $C_1, \ldots, C_n$ of N such that

$$C_1, \ldots, C_n, \mathcal{E} \models D' \text{ and}$$
$$D \succ C_1, \ldots, C_n.$$

An inference is $\mathcal{E}$-redundant if all its ground instances are.

Superposition into variables

We have explicitly excluded superposition on variable subterms. This restriction is a consequence of our notion of redundancy.

For example, if $u \succ v$ and $f(u,u) \succ g(u)$, then the superposition inference

$$\frac{u \approx v \quad f(x,x) \approx g(x)}{f(u,v) \approx g(u)}$$

is redundant. Any ground instance has as second premise an instance of $f(u,u) \approx g(u)$. But the latter formula logically follows from the smaller formulas $u \approx v$ and $f(v,v) \approx v$ and the equality axioms; and hence is redundant.

Note that superposition *from* a (unshielded) variable is still required.

Saturation up to Redundancy

A set N is saturated up to redundancy if all inferences from N are redundant (with respect to N).

THEOREM.

> Let N be a set of clauses that is saturated up to $\mathcal{E}$-redundancy under superposition and equality factoring. Then N is unsatisfiable if and only if it contains the empty clause.

REMARKS.

> A superposition inference is redundant if its conclusion is already present.
>
> Equality factoring can be replaced by merging paramodulation.

Refutational theorem proving

Theorem provers typically consist of two components:

- A deductive inference system that is used to generate new formulas, and
- techniques for simplifying formulas and eliminating redundancies.

Example.

> Inference system: Hyper-resolution
> Redundancy criteria: Tautology deletion and subsumption

Often a distinction is made between forward and backward simplification.

Theorem proving derivations

Let $\mathcal{J}$ be an inference system and $\mathcal{R}$ be a redundancy criterion.

A theorem proving derivation is a sequence

$$N_0 \vdash N_1 \vdash N_2 \vdash \cdots$$

of sets of formulas, such that for all i

$$N_{i+1} \setminus N_i \subseteq \mathcal{J}(N_i) \text{ and}$$
$$N_i \setminus N_{i+1} \subseteq \mathcal{R}(N_{i+1}).$$

The limit of a derivation is the set

$$N_\infty = \bigcup_i \bigcap_{j \geq i} N_j$$

of all persisting formulas.

Redundancy criteria

Let $\mathcal{RF}(N)$ and $\mathcal{RI}(N)$ be the sets of formulas and inferences, respectively, that are deemed to be redundant in the context of a set N.

We say that $\mathcal{R} = (\mathcal{RF}, \mathcal{RI})$ is a redundancy criterion if

1. if $N \subseteq N'$, then $\mathcal{R}(N) \subseteq \mathcal{R}(N')$
2. if $N' \subseteq \mathcal{RF}(N)$, then $\mathcal{RI}(N) \subseteq \mathcal{RI}(N \setminus N')$

Fairness

A derivation is fair if every non-redundant inference from N_∞ is redundant with respect to some set N_j, i.e.

$$\mathcal{J}(N_\infty) \setminus \mathcal{R}\mathcal{J}(N_\infty) \subseteq \bigcup_j \mathcal{R}\mathcal{J}(N_j).$$

THEOREM.

> The limit of a fair derivation is saturated up to redundancy

We say that $\mathcal{J}$ is compatible with $\mathcal{R}$ if an inference is redundant whenever its conclusion is already present.

PROPOSITION.

> If $\mathcal{J}$ is compatible with $\mathcal{R}$, then a derivation is fair if the conclusion of every non-redundant inference from N_∞ is contained in some set N_k.

Simplification

Simplification is the combination of deduction and deletion.

EXAMPLE.

Let N be $N_0 \cup \{A \vee B, A \vee \neg B \vee C\}$.

Then

$$N \vdash N \cup \{A \vee C\}$$

by resolution plus factoring.

Also,

$$N \cup \{A \vee C\} \vdash N_0 \cup \{A \vee B, A \vee C\}$$

by subsumption.

These two steps are often combined into resolution with subsumption:

$$N \vdash N_0 \cup \{A \vee B, A \vee C\}.$$

Simplification by rewriting

Suppose the set N contains equations

$$-x * y + y * x \approx y * 0 \text{ and}$$

$$z * 0 \approx 0$$

We may deduce

$$-x * y + y * x \approx 0$$

(by paramodulation, not superposition) and then delete

$$-x * y + y * x \approx y * 0$$

The combination of these derivation steps is known as simplification by rewriting or demodulation.

Fundamental theorem proving concepts and techniques

Deduction
Redundancy (simplification)
Constraints

Selection
Orderings
Extension by definition
Polarity

Commutation
Chaining
Basic strategy

Literature

L. Bachmair and H. Ganzinger, Rewrite-based equational theorem proving with selection and simplification. *Journal of Logic and Computation 4* (1994):217–247.

- describes superposition with selection
- proposes two notions of redundancy
- describes simplification techniques
- refutations in saturated theories