Computing Real Roots of Real Polynomials ...
... and now For Real!

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Computing Real Roots ...

A classical approach

Computing Real Roots of Real Polynomials

Given a square-free polynomial

$$P(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0 \in \mathbb{R}[x]$$

and an open interval $\mathcal{I} = (a_0, b_0) \subset \mathbb{R}$,

find *isolating intervals* $I_1, \ldots, I_r \subset \mathcal{I}$ such that

- each I_j contains exactly one real root of P,
- the I_j are pairwise disjoint, and
- $\bigcup_j I_j$ covers all real roots of P in $\mathcal{I}$.

Subdivision approach

- *Queue* $\leftarrow \{\mathcal{I}\}$
  Isol $\leftarrow \emptyset$
- while *Queue* is not empty:
  - pop $I = (a, b)$ from *Queue*
  - if I contains no root: *(exclusion predicate)*
    discard I
  - else if I contains exactly one root: *(inclusion-and-isolation predicate)*
    add I to *Isol*
  - otherwise (or we-don't-know):
    - split I at $m = \frac{a+b}{2}$ *(or some other point)*
    - if $P(m) = 0$: add $\{m\}$ to *Isol*
    - add (a, m) and (m, b) to *Queue*

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Descartes' Rule of Signs

$r =$ # positive real roots of P

$v =$ # sign changes in coeffs of P

$$r \leq v \text{ and } r \equiv v \pmod 2$$

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localized version

$r_I =$ # real roots of P in $I = (a, b)$

$v_I =$ # sign changes of $(x+1)^n P\left(\frac{ax+b}{x+1}\right)$

$$r_I \leq v_I \text{ and } r_I \equiv v_I \pmod 2$$

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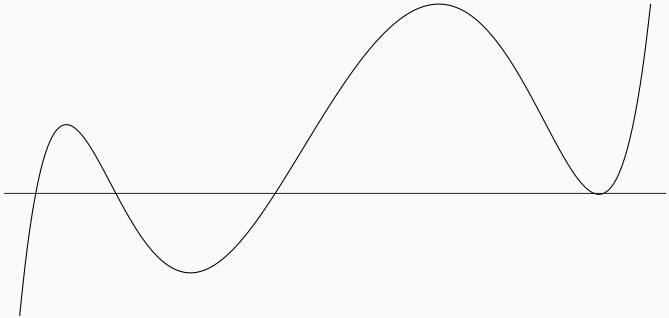
$r_I = v_I \in \{0, 1\}$ if other (complex) roots well-separated from I

Descartes method



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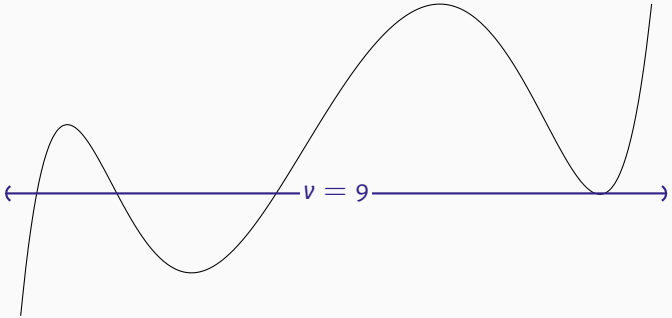

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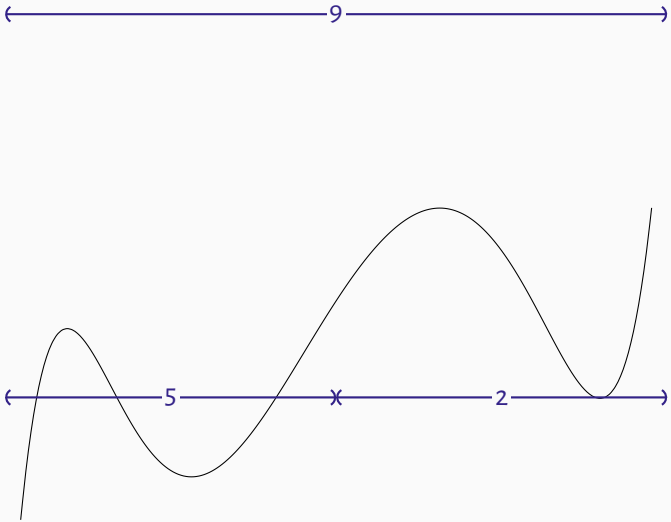

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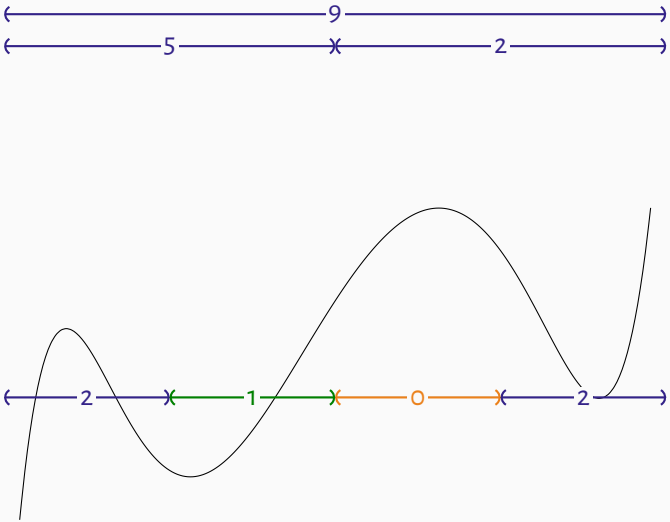

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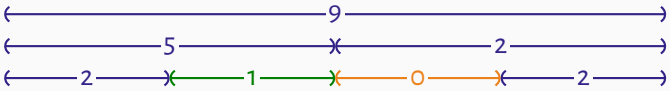

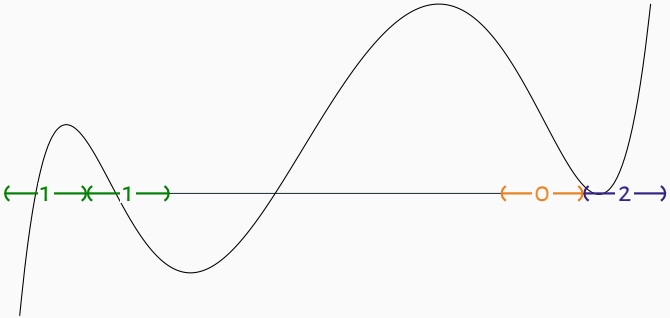

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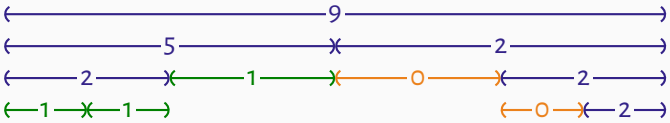

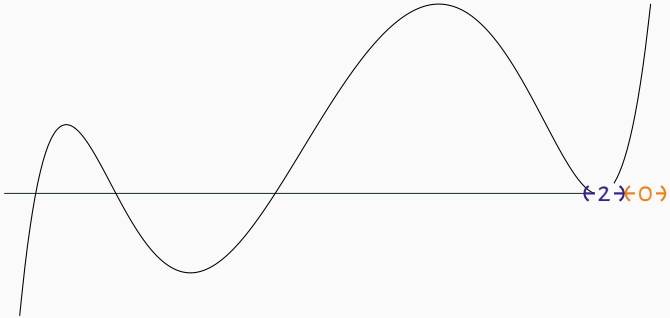

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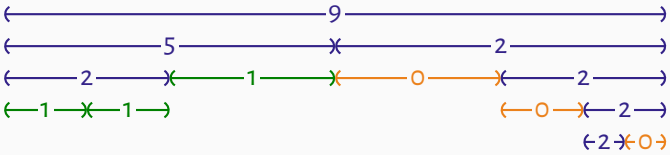

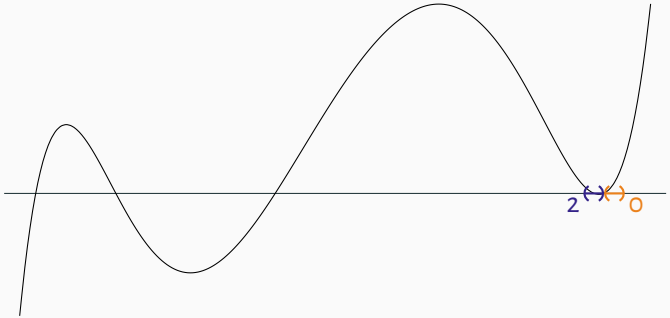

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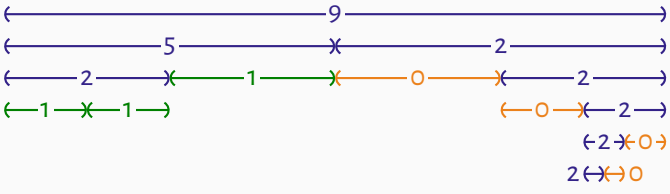

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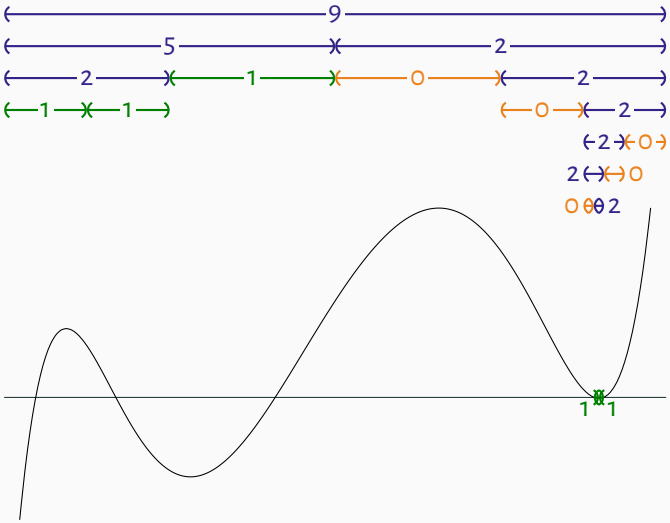

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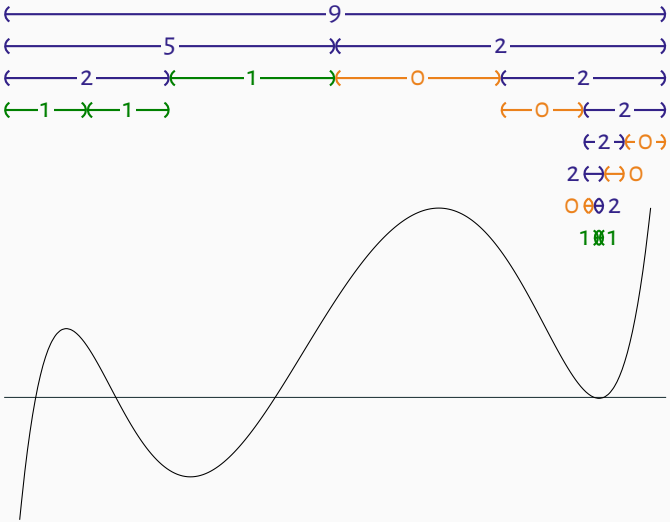

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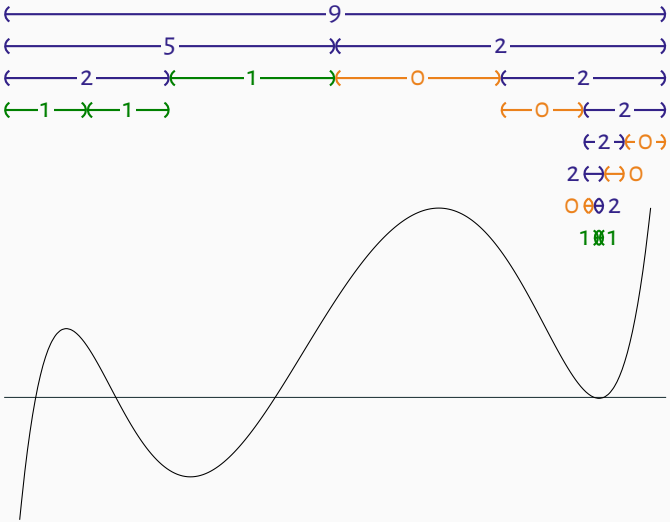

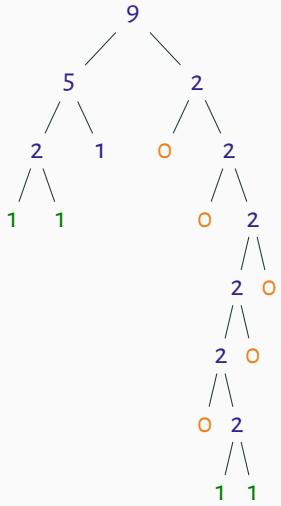

Complexity ("benchmark problem")

isolate *all* roots of an *integer* polynomial of degree n and coefficient bitsize τ

roots / tree width: $\leq n$ min. root separation: $\geq 2^{-\mathcal{O}(n\tau)}$

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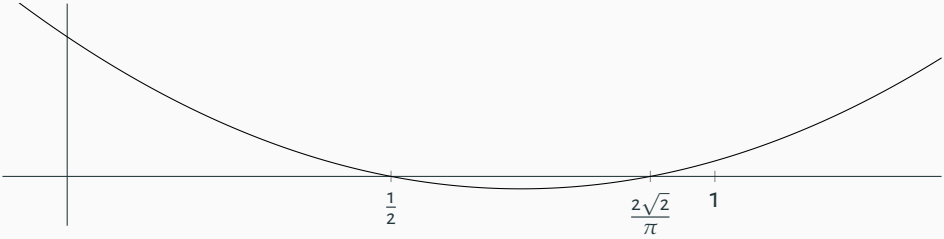
[1] [Eigenwillig, Sharma, Yap: ISSAC 2006] [Collins: JSC 2016]

… of Real Polynomials …

Deficiencies and remedies

$$P(x) = 2\pi x^2 - (\pi + 4\sqrt{2})x + 2\sqrt{2} = (\pi x - 2\sqrt{2})(2x - 1) \quad \in \mathbb{R}[x]$$
$$\approx 6.2831853072x^2 - 8.7984469031x + 2.8284271247$$



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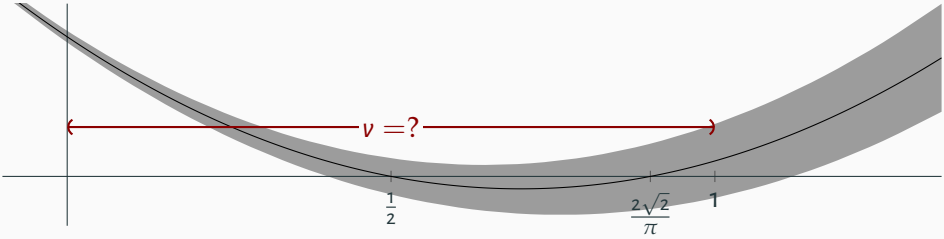
$$\approx \qquad 6....x^2 - \qquad 9....x + \qquad 3....$$



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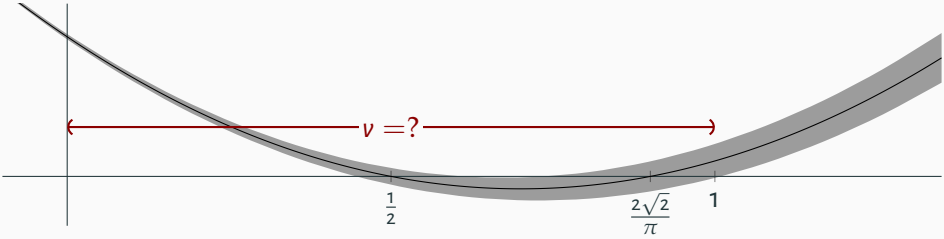
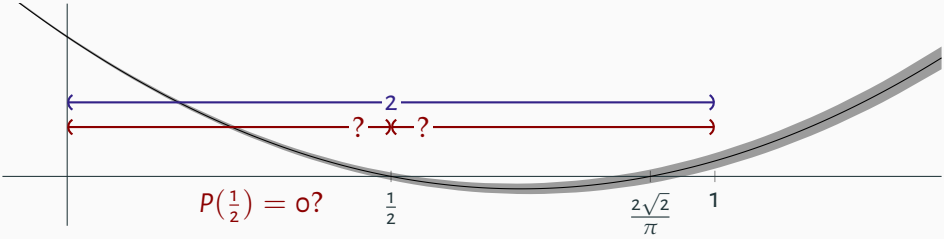
$$\approx \quad 6.3\ldots x^2 - \quad 8.8\ldots x + \quad 2.8\ldots$$



$$P(x) = 2\pi x^2 - (\pi + 4\sqrt{2})x + 2\sqrt{2} = (\pi x - 2\sqrt{2})(2x - 1) \quad \in \mathbb{R}[x]$$

$$\approx \quad 6.28\ldots x^2 - \quad 8.79\ldots x + \quad 2.83\ldots$$



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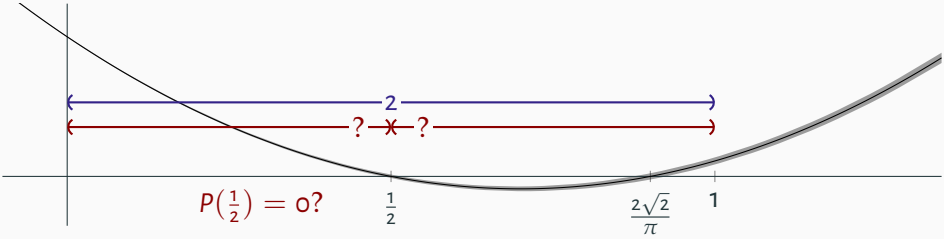

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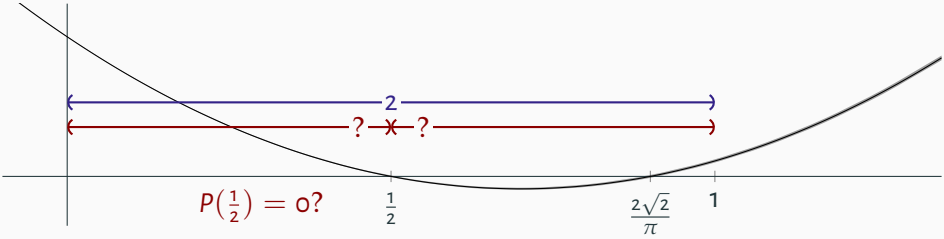

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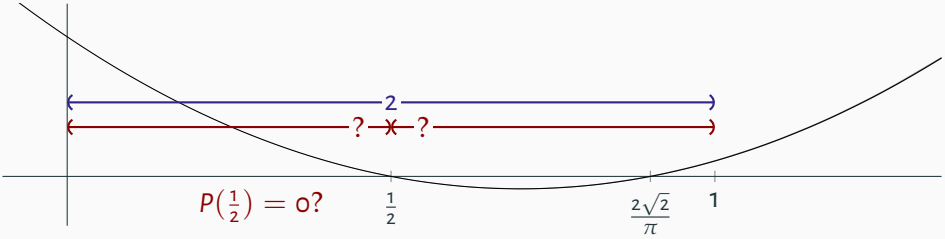

problems with arbitrarily approximable ("bitstream") coefficients

- *no termination* on subdivision on an exact root
- *precision demand* for processing $I = (a, b)$ depends on $|P(a)|$ and $|P(b)|$

ADsc (Approximate Descartes)

Instead of subdivision at $m \in I$:

- sample suitable points near m,
- choose an *admissible point* m^* with large value $|P(m^*)|$ among the samples
- and subdivide at m^*.

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In particular: $|P(m^*)| \neq 0$!

$\Rightarrow$ *guaranteed termination* for arbitrary (square-free) inputs

$\Rightarrow$ keeps *precision demand near theoretical optimum*

isolate *all* roots of an *integer* polynomial of degree n and coefficient bitsize τ

roots / tree width: $\leq n$ min. root separation: $\geq 2^{-\mathcal{O}(n\tau)}$

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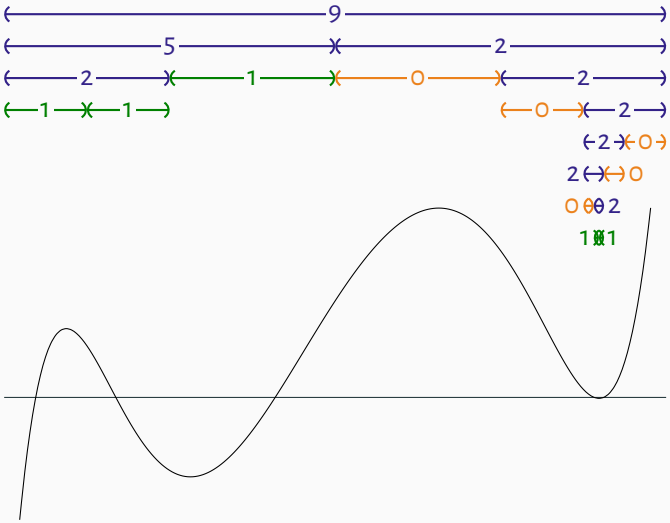
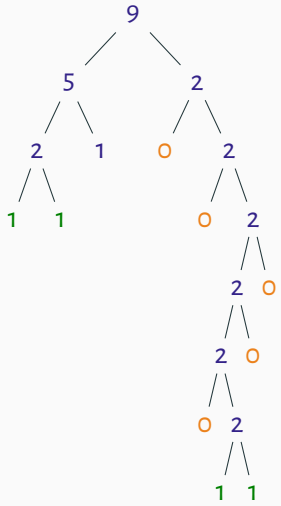
bisection, approximate	$\tilde{\mathcal{O}}(n\tau)$	$\tilde{\mathcal{O}}(n\tau)$[1]	$\tilde{\mathcal{O}}(n^3\tau^2)$[1]

[1] [Sagraloff: JSC 2014]

Clustered roots





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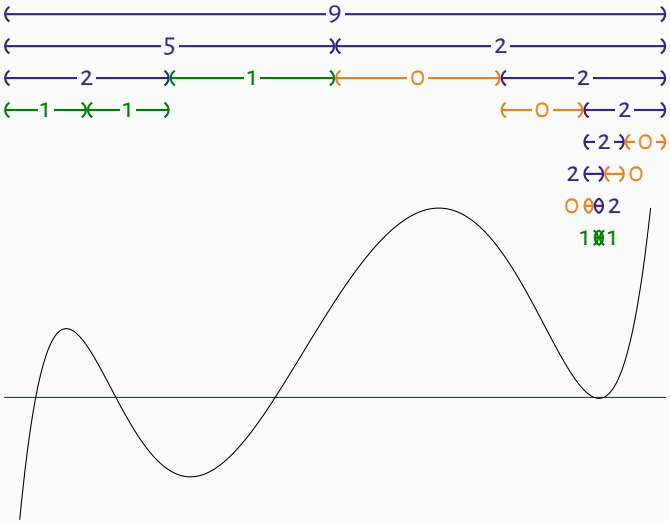

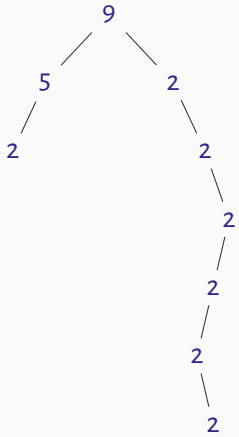

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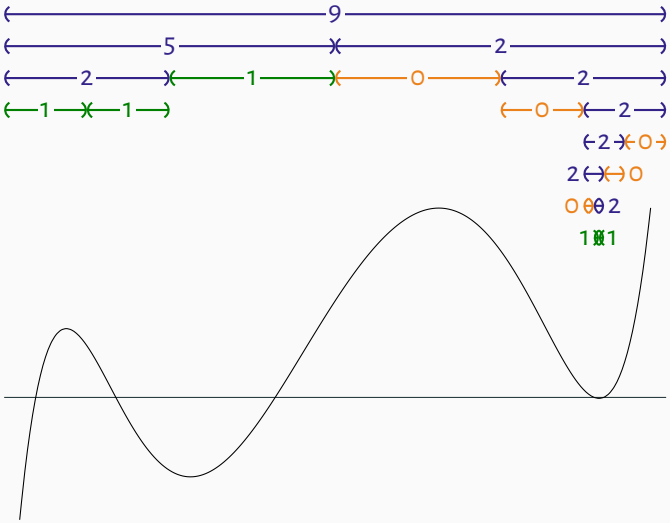

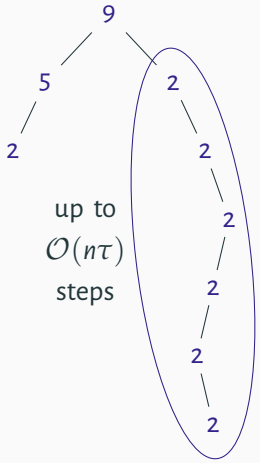

Combining Newton's and Descartes method

- inspired by (Approximate) Quadratic Interval Refinement[2] and the Brent-Dekker method[3]

[2] [Abbott: ISSAC 2006; ACM CCA 2014], [Kerber, Sagraloff: JCAM 2015]

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Combining Newton's and Descartes method

- inspired by (Approximate) Quadratic Interval Refinement[2] and the Brent-Dekker method[3]
- certified trial-and-error approach:
  - multiplicity-agnostic variant of *Newton iteration*
  - determine *candidate subinterval* of I containing a cluster
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- similar techniques in complex domain: upcoming talk by Juan Xu

[2] [Abbott: ISSAC 2006; ACM CCA 2014], [Kerber, Sagraloff: JCAM 2015]

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Complexity ("benchmark problem")

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Newton, exact over $\mathbb{Z}$ / $\mathbb{Q}$	$\tilde{\mathcal{O}}(n)$[1]	$\tilde{\mathcal{O}}(n^2\tau)^*$	$\tilde{\mathcal{O}}(n^3\tau)$[1]

*amortized over entire tree: $\tilde{\mathcal{O}}(n\tau)$

[1] [Sagraloff: ISSAC 2012]

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Newton, approximate	$\tilde{\mathcal{O}}(n)$	$\tilde{\mathcal{O}}(n\tau)^\dagger$	$\tilde{\mathcal{O}}(n^3 + n^2\tau)$[1]

*amortized over entire tree: $\tilde{\mathcal{O}}(n\tau)$

†amortized over entire tree: $\tilde{\mathcal{O}}(n + \tau)$

[1] [Sagraloff, Mehlhorn: JSC 2016]
best known: $\tilde{\mathcal{O}}(n^2\tau)$ [Pan: JSC 2002] [Mehlhorn, Sagraloff, Wang: JSC 2015]

… and now For Real!

Implementation results

RS

- C library for real root solving, refinement & more
- sophisticated implementation of classical Descartes
- tailored for approximate arithmetic
- high-performance general purpose solver
- default real root solver in Maple since version 11

[Rouillier, Zimmermann: JCAM 2003]

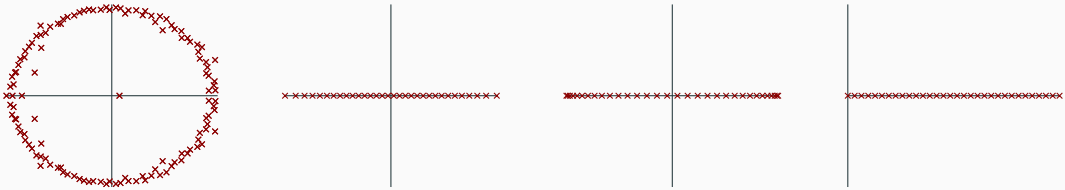
ANewDsc (Approximate Arithmetic Newton-Descartes)

- implemented on top of RS
- merged admissible point selection & Newton-Descartes
- heuristics to reduce / eliminate overhead on “easy” instances
- additional optimizations
- certified output (based on interval arithmetic using MPFI)
- matches theoretical worst-case bit complexity (Las Vegas)
  (assuming asymptotically fast polynomial arithmetic)
- to be integrated in Maple 201x?

[Sagraloff, Mehlhorn: JSC 2016]

Benchmark excerpt: tame (well-separated) instances

instance	degree	bitsize	RS		ANewDsc		speedup
			time [s]	nodes	time [s]	nodes	
random	16384	1024	275.1	46	279.0	40	0.99
Hermite	1024	9875	88.3	2237	79.6	1396	1.11
Legendre	1024	4640	118.2	2272	117.2	1325	1.01
Wilkinson	1024	8777	219.1	2064	398.8	1565	0.73

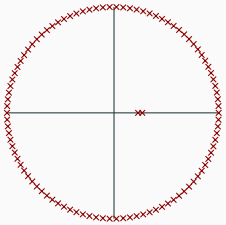


Benchmark excerpt: hard (clustered) instances

instance	degree	bitsize	RS		ANewDsc		speedup
			time [s]	nodes	time [s]	nodes	
monic random	4096	1024	713.1	3212	14.6	37	48.84
random$^2 - 1$	1024	1029	288.5	6648	5.2	419	55.48

Benchmark excerpt: hard (clustered) instances

instance	degree	bitsize	RS		ANewDsc		speedup
			time [s]	nodes	time [s]	nodes	
Mignotte polynomials: $x^n - (ax-1)^2$							
$a = 2^{31}$	1025	63	3.6	66	1.4	27	2.57
$a = 5$	1025	5	402.6	2384	0.7	43	575.14
$a = 2^{31} + 1$	1025	63	13338.4	31810	1.2	53	11115.3

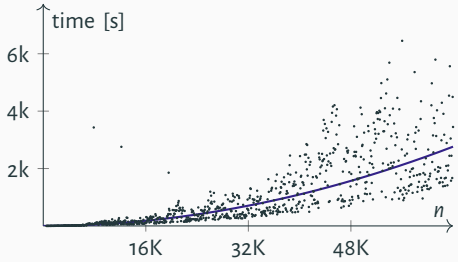


two clustered roots near $1/a$
with separation of appx. $a^{n/2}$

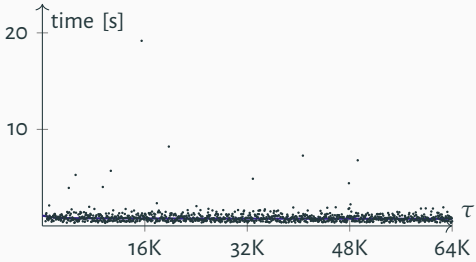
Comparison to other solvers

instance	n	τ	MPSolve	CF	Sage	RS	ANewDsc
random	16384	1024	578.9	376.6	3825.1	275.1	279.0
Hermite	1024	9875	2169.9	140.5	11.9	88.3	79.6
Legendre	1024	4640	5061.3	79.0	9.7	118.2	117.2
Wilkinson	1024	8777	5308.1	13.7	14.1	219.1	398.8
random$^2 - 1$	512	1029	33.5	339.2	3.1	71.5	1.3
	1024	1029	141.8	> 7200	8.2	288.5	5.2
random monic	1024	1024	2.6	0.7	4115.0	38.8	1.2
	16384	1024	579.3	279.2	> 7200	> 7200	143.1
$x^n - (2^a x - 1)^2$	2047	5	87.2	1.2	21.1	1.0	1.2
$x^n - (ax - 1)^2$	2047	5	131.5	1.5	21.7	1.6	1.6
$x^n - (ax^2 - 1)^2$	2047	5	99.1	> 7200	26.5	3472.0	8.3
nested 4-fold	260	2560	69.6	348.8	5.6	636.7	2.1
	1028	260	186.2	3993.7	20.9	430.4	7.4

Bit complexity estimation: tame instances



estimated exponent for degree: 1.97



estimated exponent for bitsize: -0.03

theoretical vs. estimated bit complexity:

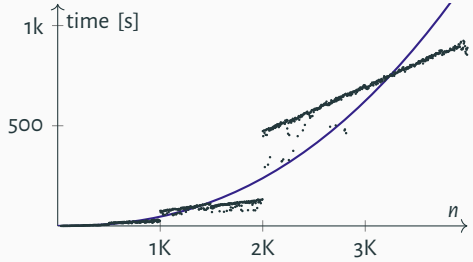
$$\tilde{\mathcal{O}}(n^3 + n^2\tau) \text{ vs. } \mathcal{O}(n^{1.97}\tau^{-0.03})$$

---

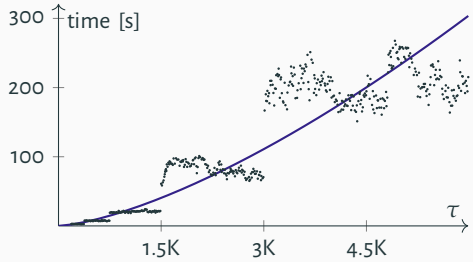
degree n, integer coefficients in range $(-2^\tau, 2^\tau)$ chosen uniformly at random

left: $\tau = 1024$; right: $n = 1024$

Bit complexity estimation: hard instances



estimated exponent for degree: 2.37



estimated exponent for bitsize: 1.45

theoretical vs. estimated bit complexity:

$$\tilde{\mathcal{O}}(n^3 + n^2\tau) \text{ vs. } \mathcal{O}(n^{2.37}\tau^{1.45})$$

Mignotte-like: $x^n - (ax^2 - 1)^3$ (cluster of multiplicity 3 around irrational center)
left: $a = 2^{256} - 1$; right: $n = 1024$

ANewDsc: Approximate Arithmetic Newton-Descartes

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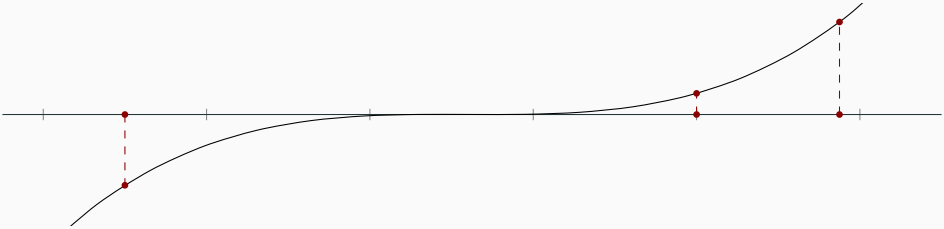
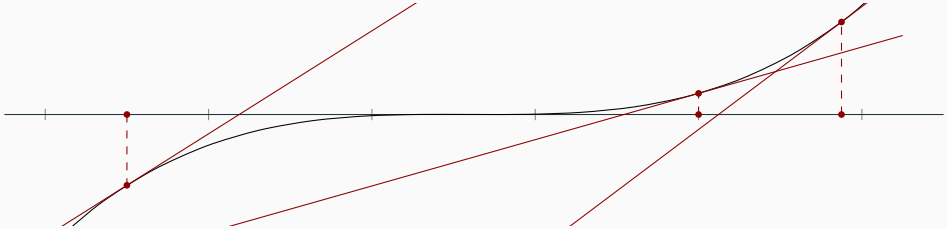
Thank you for your attention!

Newton-Descartes



- choose samples x_i

Newton-Descartes



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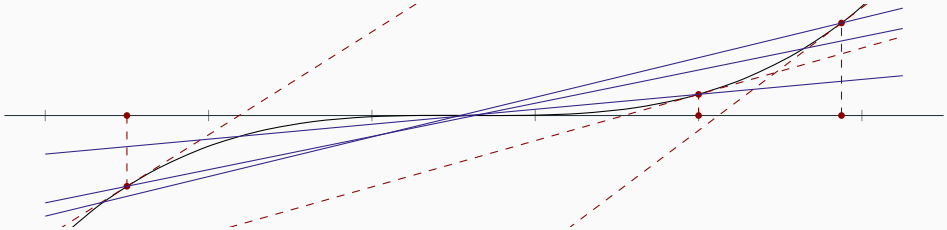

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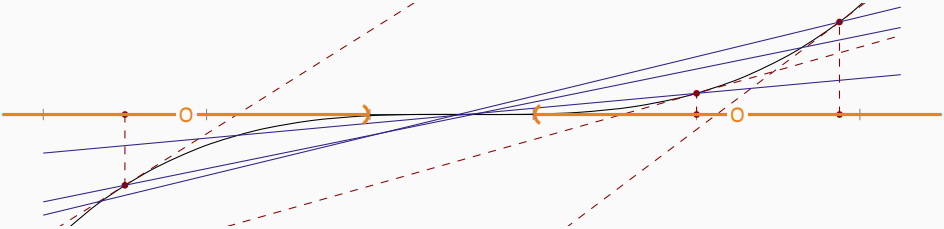

- choose samples x_i
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- guess multiplicity k s.t. $x_i - k \cdot N(x_i)$ is approximately the same
- verify that no roots are lost; otherwise, resort to bisection

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