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Certifying 3-Edge-Connectivity

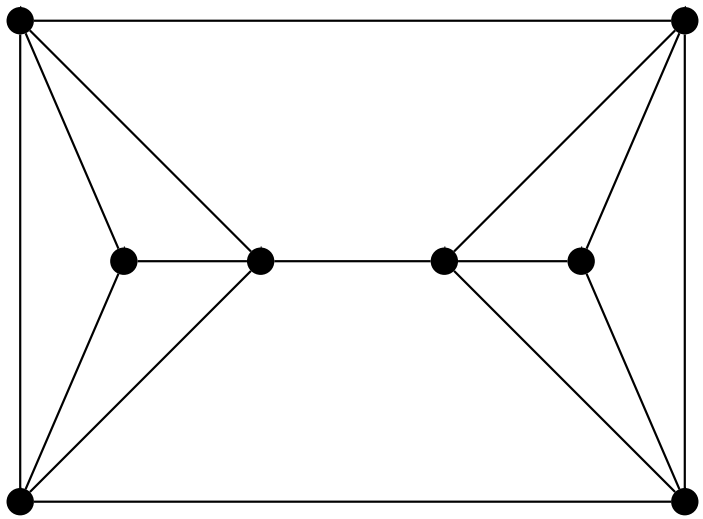
Kurt Mehlhorn Adrian Neumann Jens M. Schmidt

Max Planck Institute for Informatics

June 20, 2013

Problem

Given a (multi-)graph G, check whether there is a pair of edges e_1, e_2 such that $G - \{e_1, e_2\}$ is disconnected.



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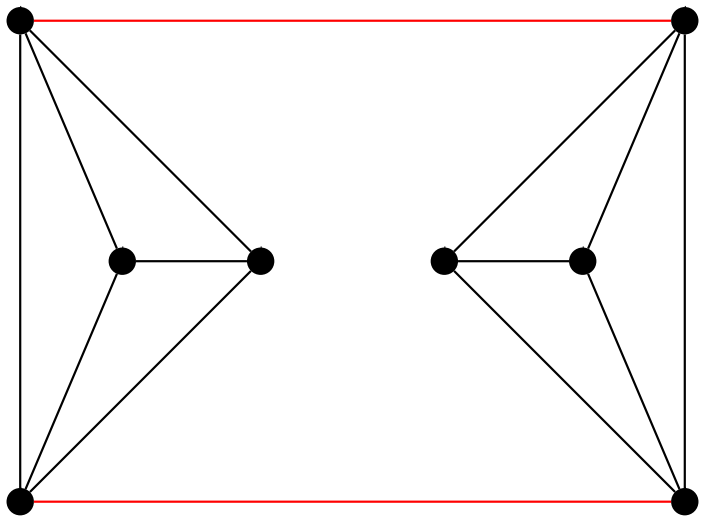

Related Work

3-edge-connectivity is a well studied problem. Many linear time solutions known, e.g.:

- 1992: Nagamochi and Ibaraki
- 1992: Taoka, Watanabe, and Onaga
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Why yet another algorithm?

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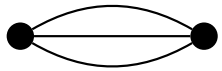
Goal

Make the checker implementation simple enough to allow formal verification. Then every answer is correct *by formal proof*.

Construction Sequence

For 3-edge-connectivity we use a *construction sequence* by Mader (1978). Start with a $K_2^3 =$ 

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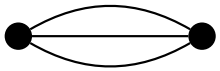
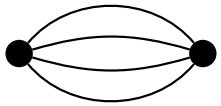
For 3-edge-connectivity we use a *construction sequence* by Mader (1978). Start with a $K_2^3 =$ 

- Add an edge 

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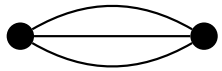
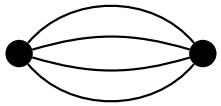
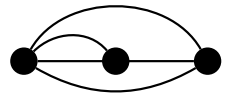
For 3-edge-connectivity we use a *construction sequence* by Mader (1978). Start with a $K_2^3 =$ 

- Add an edge 
- Split an edge, connect the new node with an old node 

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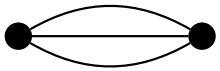
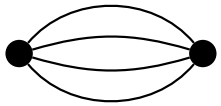
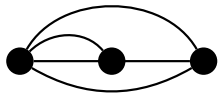
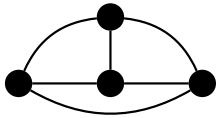
For 3-edge-connectivity we use a *construction sequence* by Mader (1978). Start with a $K_2^3 =$ 

- Add an edge 
- Split an edge, connect the new node with an old node 
- Split two edges and connect the new nodes 

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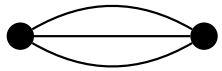
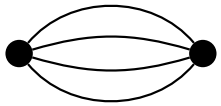
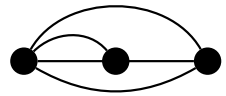
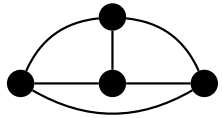
For 3-edge-connectivity we use a *construction sequence* by Mader (1978). Start with a $K_2^3 =$ 

- Add an edge 
- Split an edge, connect the new node with an old node 
- Split two edges and connect the new nodes 

Mader

Using these operations one can construct exactly the 3-edge-connected graphs.

Content

In This Talk

How to find a construction sequence for a given 3-connected graph.

Content

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How to find a construction sequence for a given 3-connected graph.

In the paper:

- Theorems for correctness.
- Details for the linear time algorithm.
- How to verify the certificate.
- How to find the 3-edge-connected components (upcoming journal version).

Subdivisions

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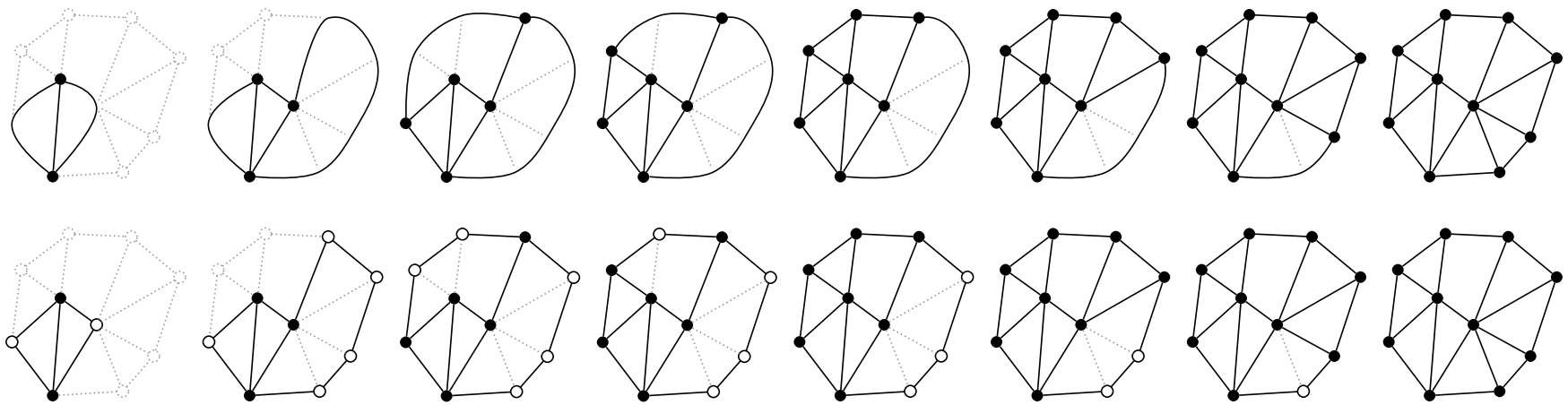

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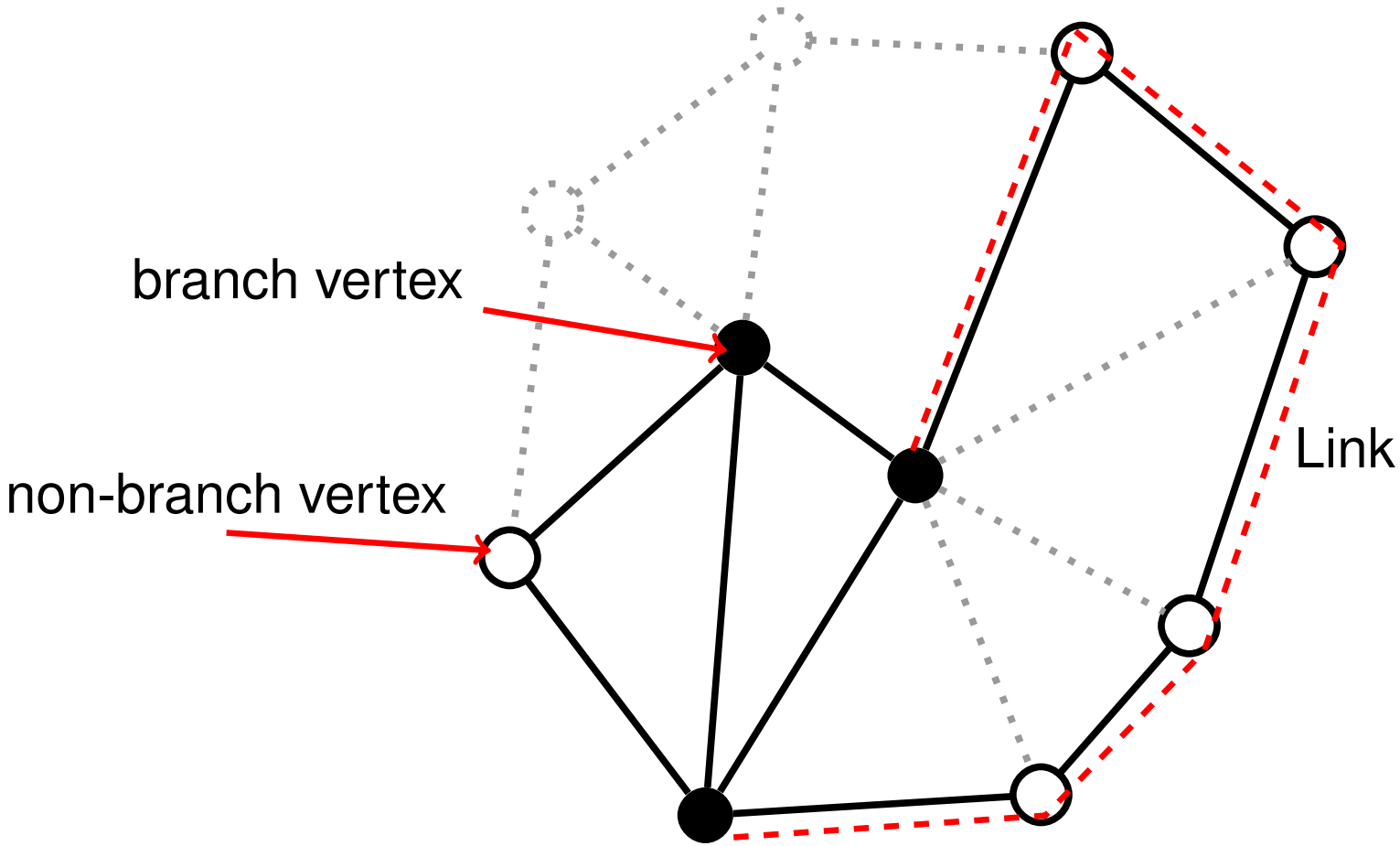

A First Algorithm

1. Find a K_2^3 subdivision. Initialize $G_c = K_2^3$
2. Find a path P in $G - G_c$ from a node u to a node v, such that
   a) at least one of $\{u, v\}$ has degree at least three, or
   b) v lies on a different link from u
3. Add P to the current subgraph
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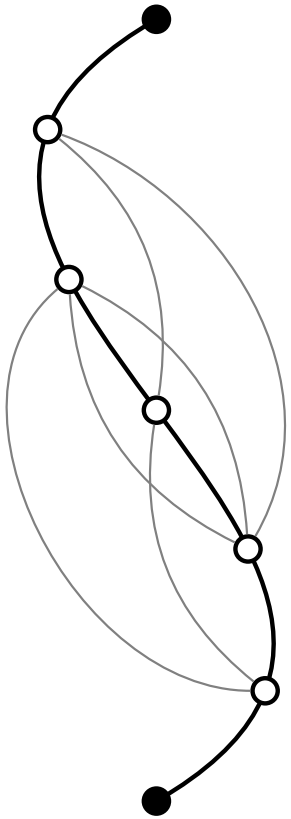

Proof by picture.

Chain Decomposition

A structure to help find a K_2^3 and subsequent paths. Very similar to an ear-decomposition.

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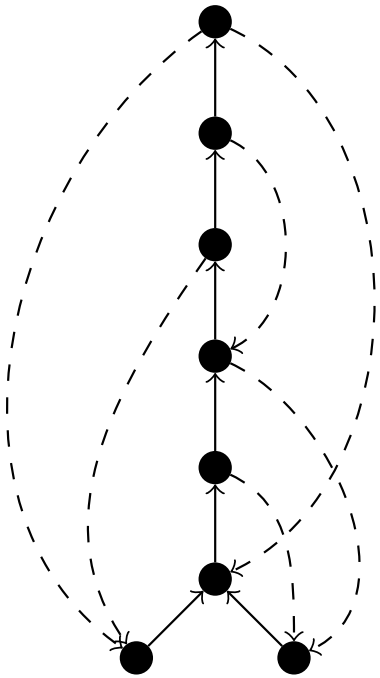

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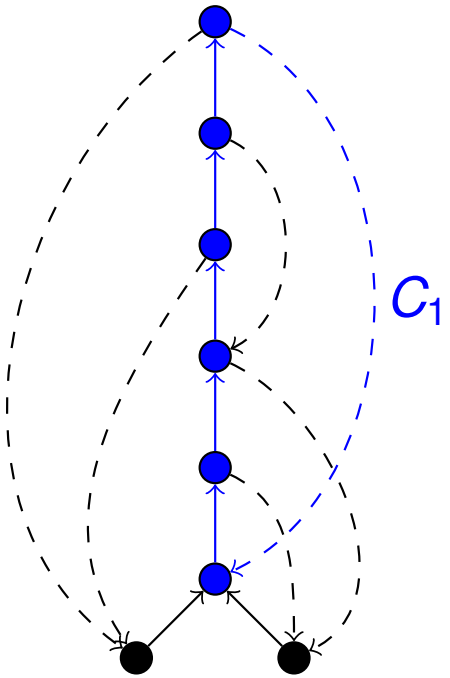

c_1

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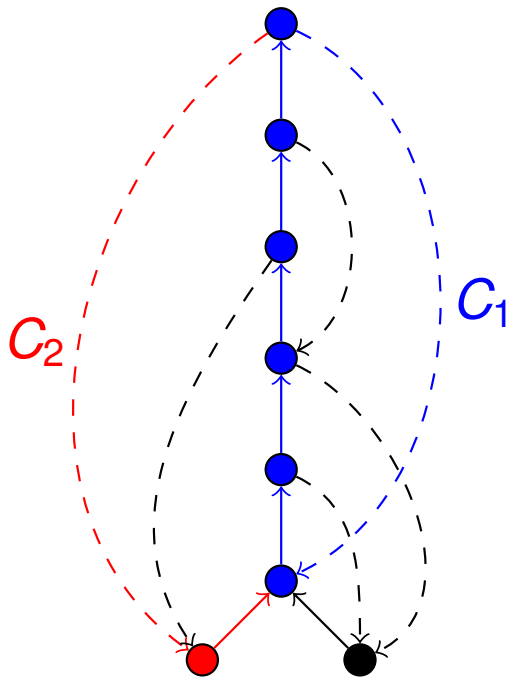

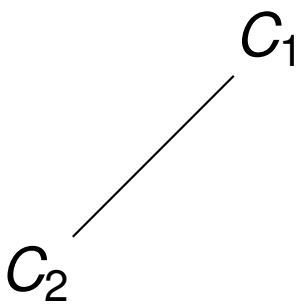

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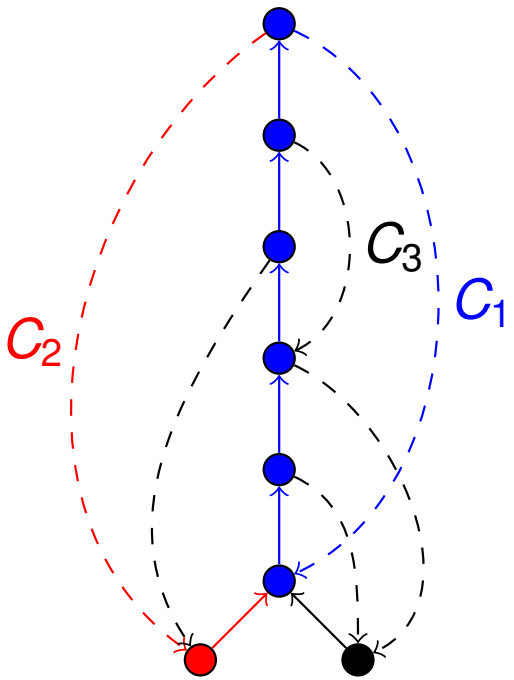

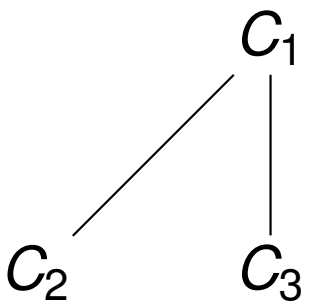

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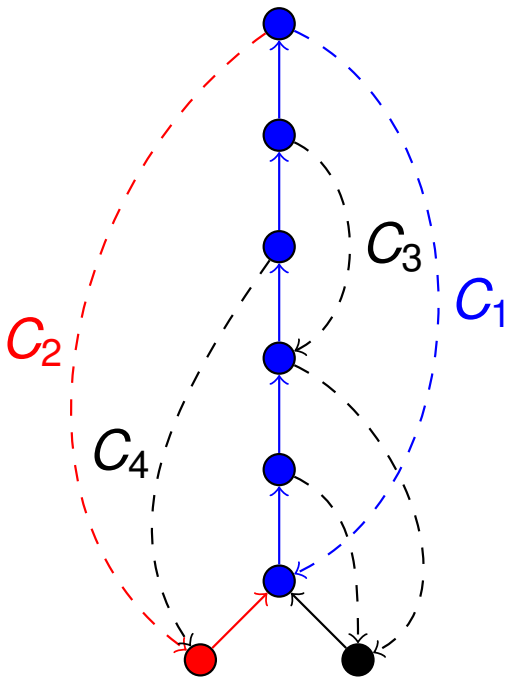

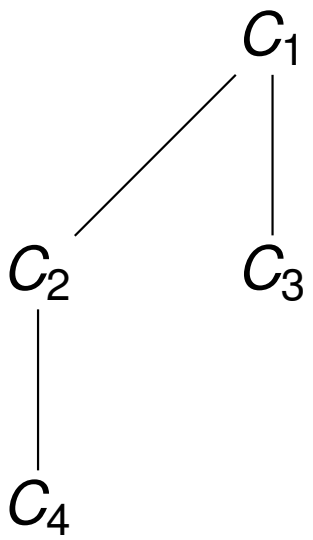

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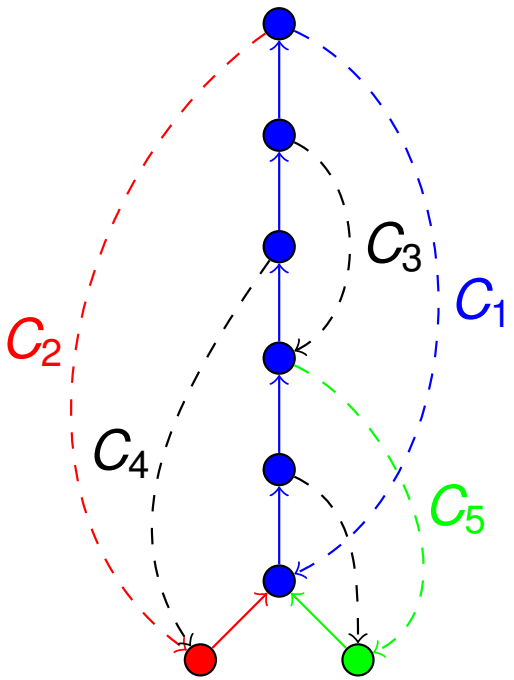

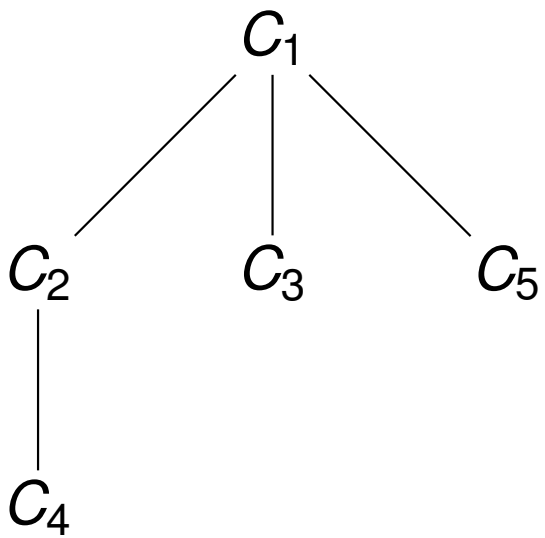

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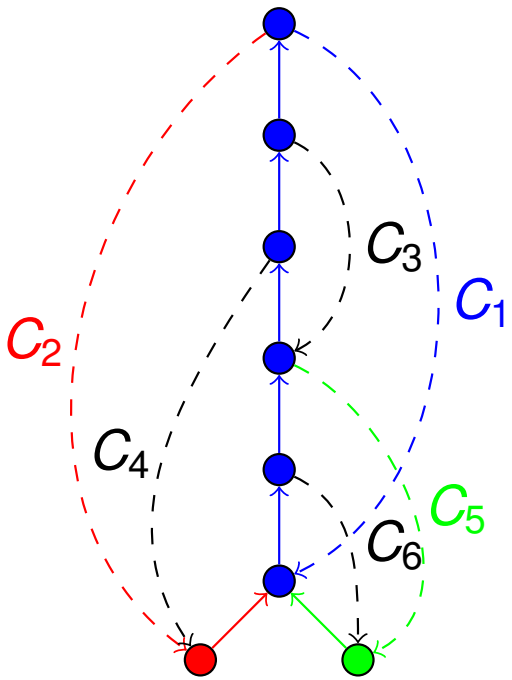

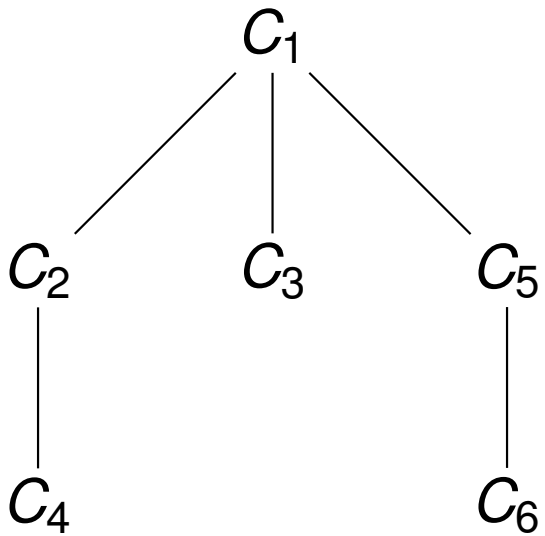

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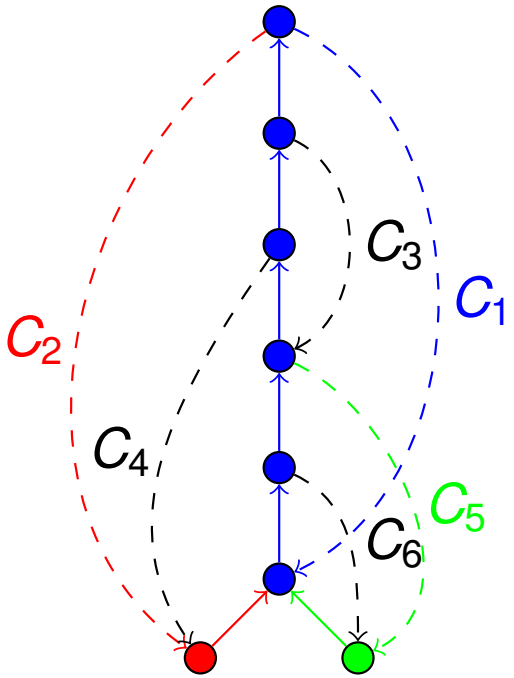

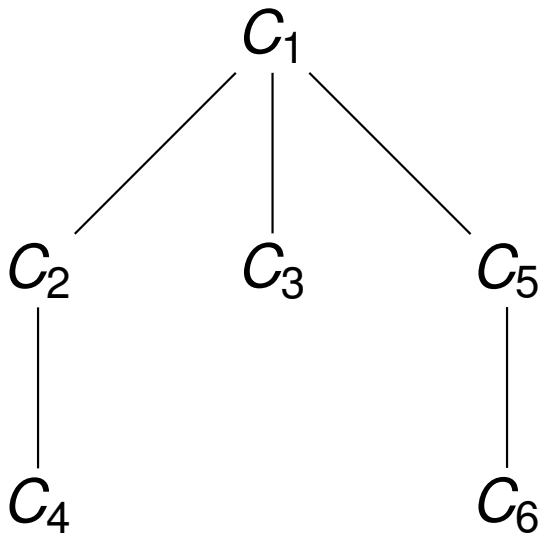

Also useful for certifying 2-vertex-connectivity, 2-edge-connectivity, and 3-vertex-connectivity!

An Improved Algorithm

Observations

- C_1 and C_2 can be chosen s.t. $C_1 \cup C_2 = K_2^3$
- A chain can be added to a graph if its endpoints lie on different links or one is a branch vertex.
- Adding a chain makes its endpoints branch vertices.

An Improved Algorithm

1. Compute a chain decomposition such that $C_1 \cup C_2 \sim K_2^3$
2. Initialize subgraph to $C_1 \cup C_2$.
3. Iterate over children C of C_1 and C_2. If C's endpoints are branching or on different links, add C to a list of addable chains.

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4. Take a chain C from the list of addable chains.
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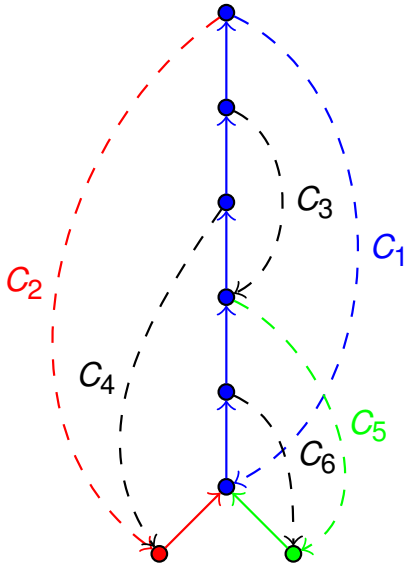
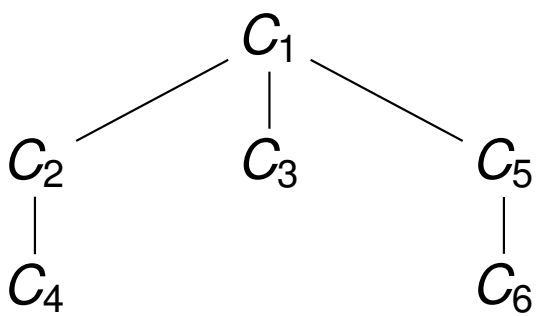
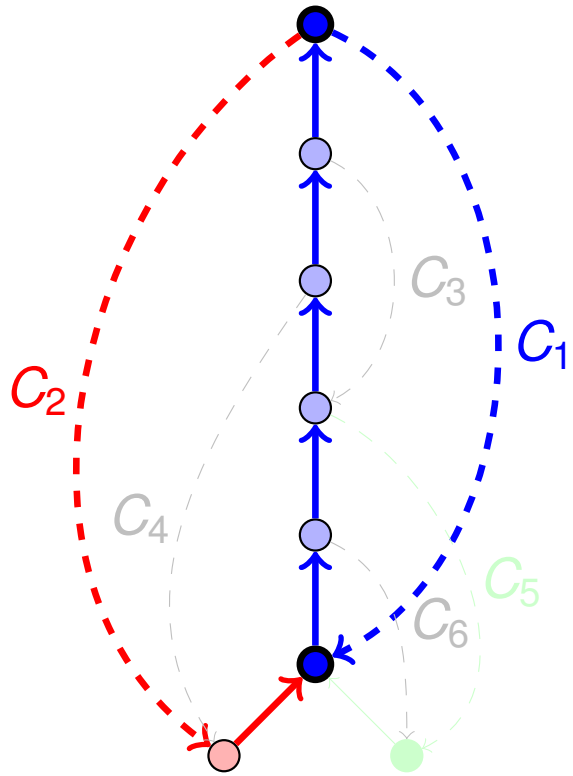
Can be implemented such that the runtime is in

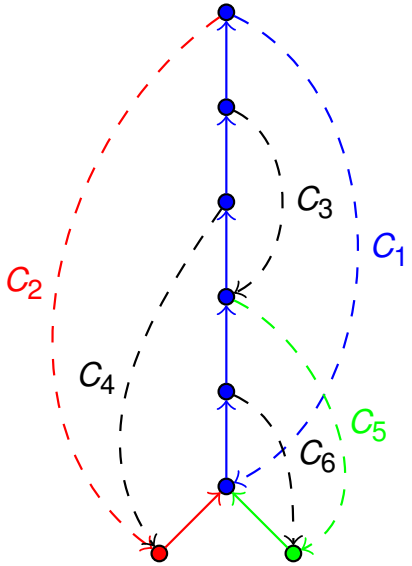
$$O((n+m)\log(n+m)).$$

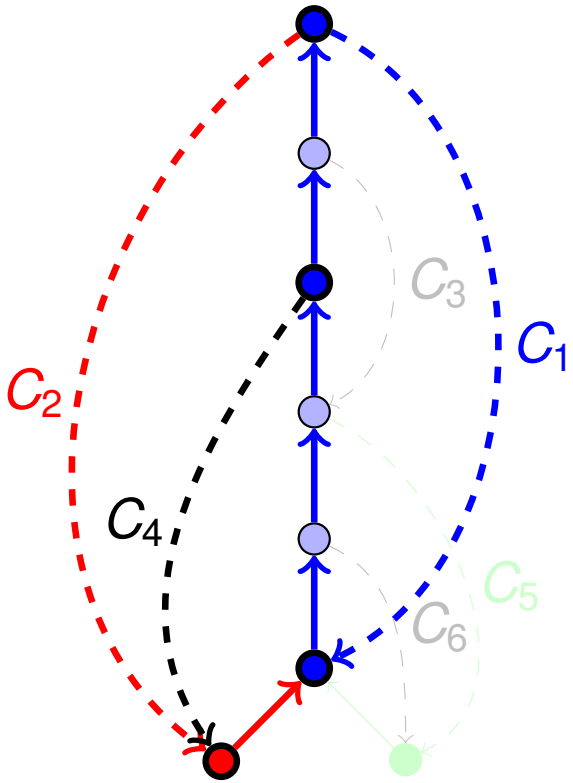






Addable: C_4, C_5





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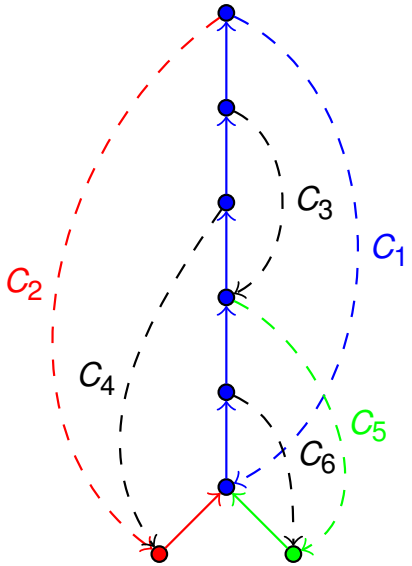
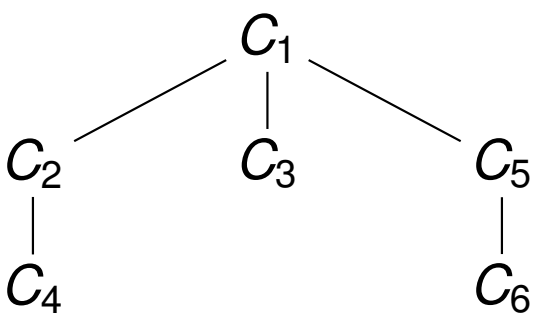
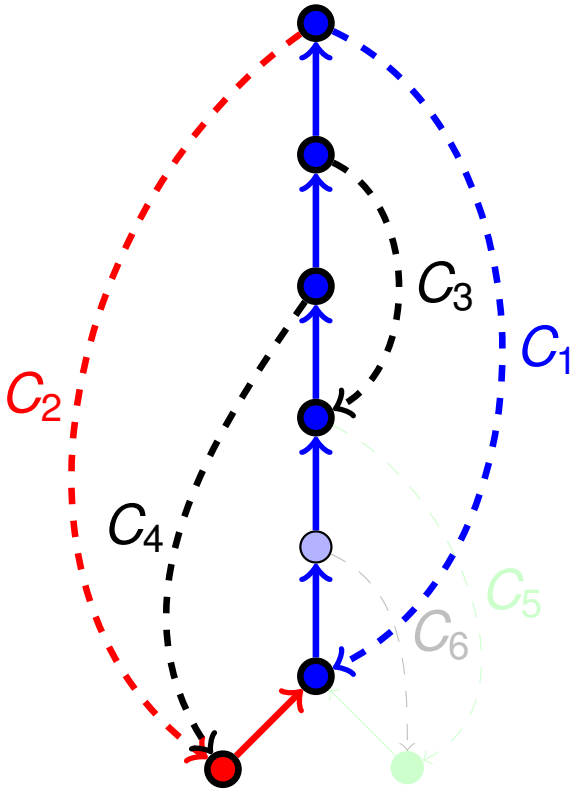
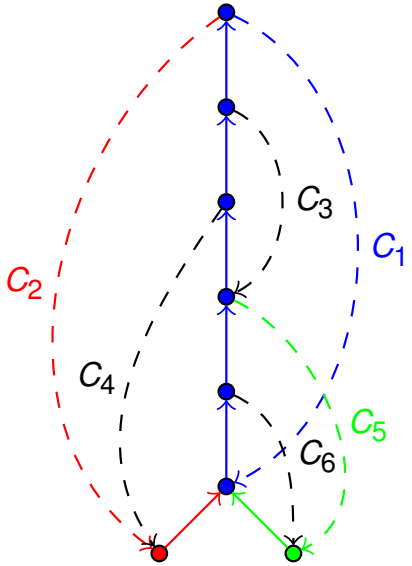
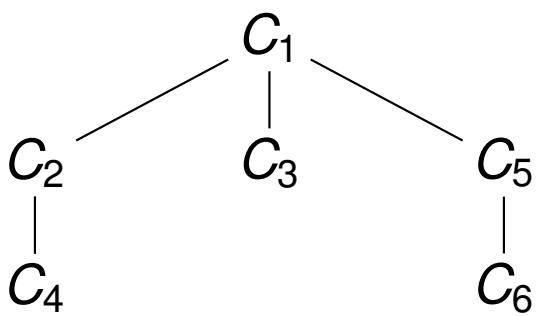


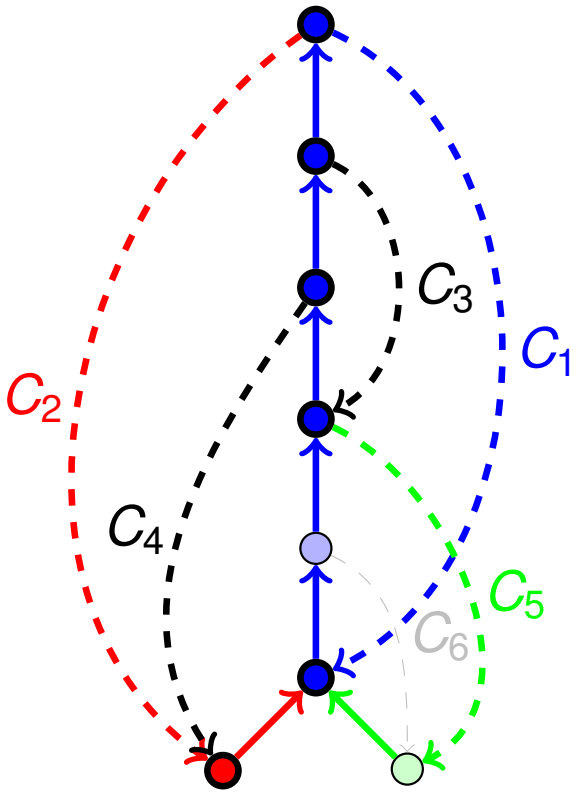




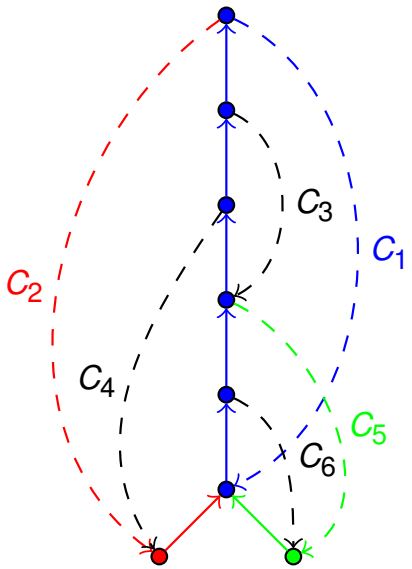
Addable: C_5







Addable: C_6


C_1
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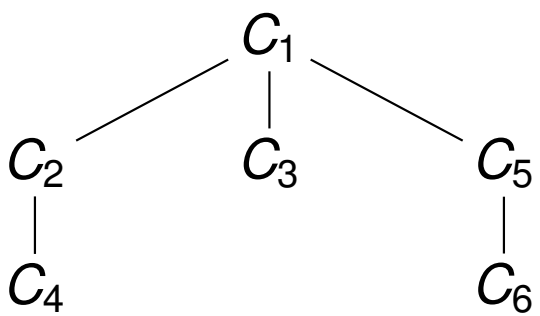
C_6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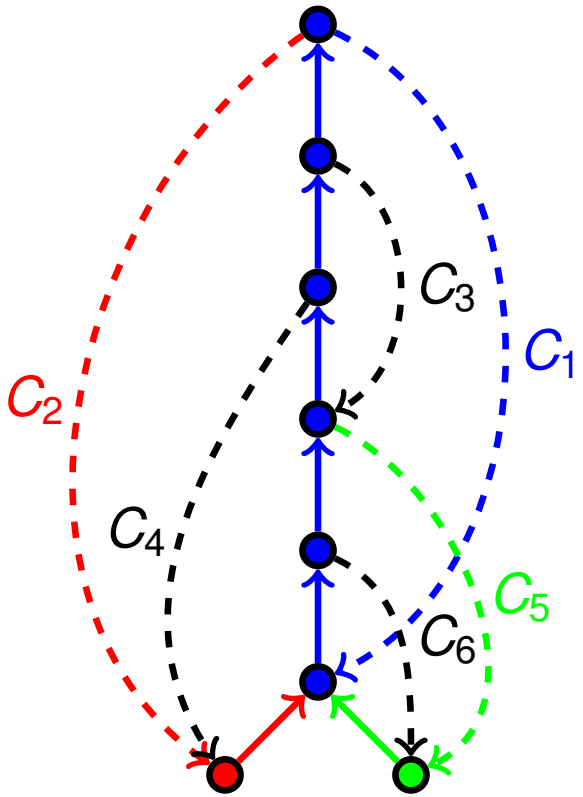
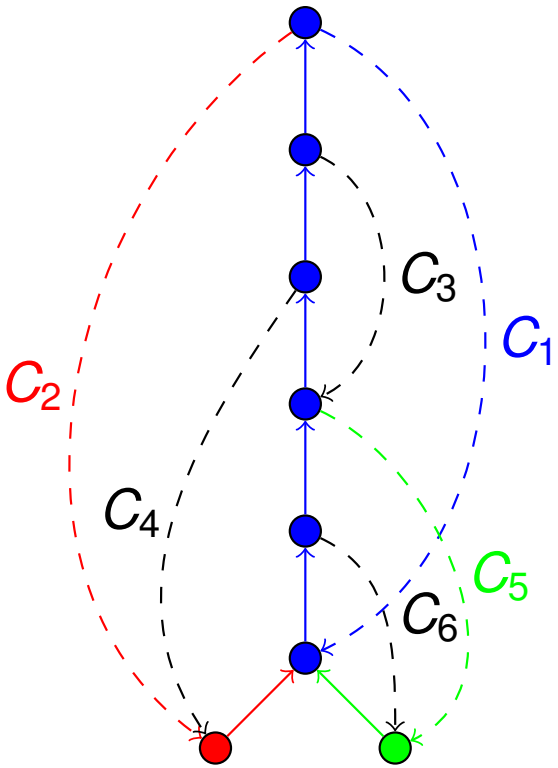
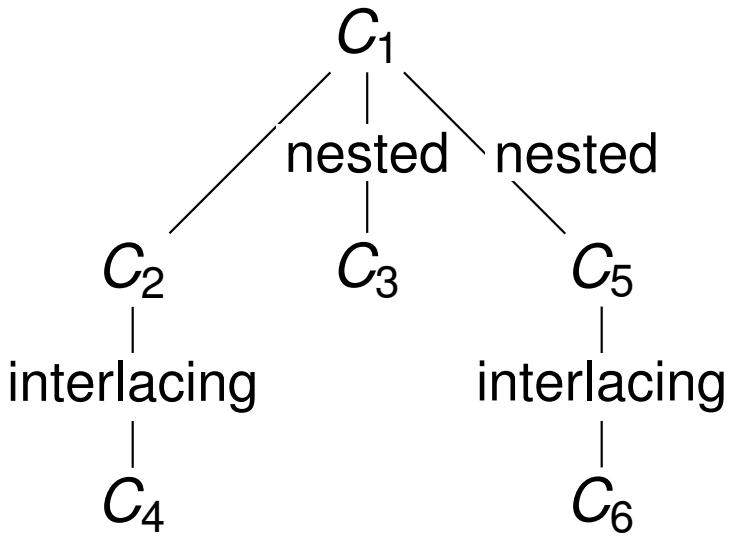

C_1
C_2
C_3
C_4
C_5
C_6

Classifying Chains





Observations

- Nested children can be added after their parent if there is a branch vertex on the tree path between their endpoints
- Interlacing children can be added immediately after their parent.

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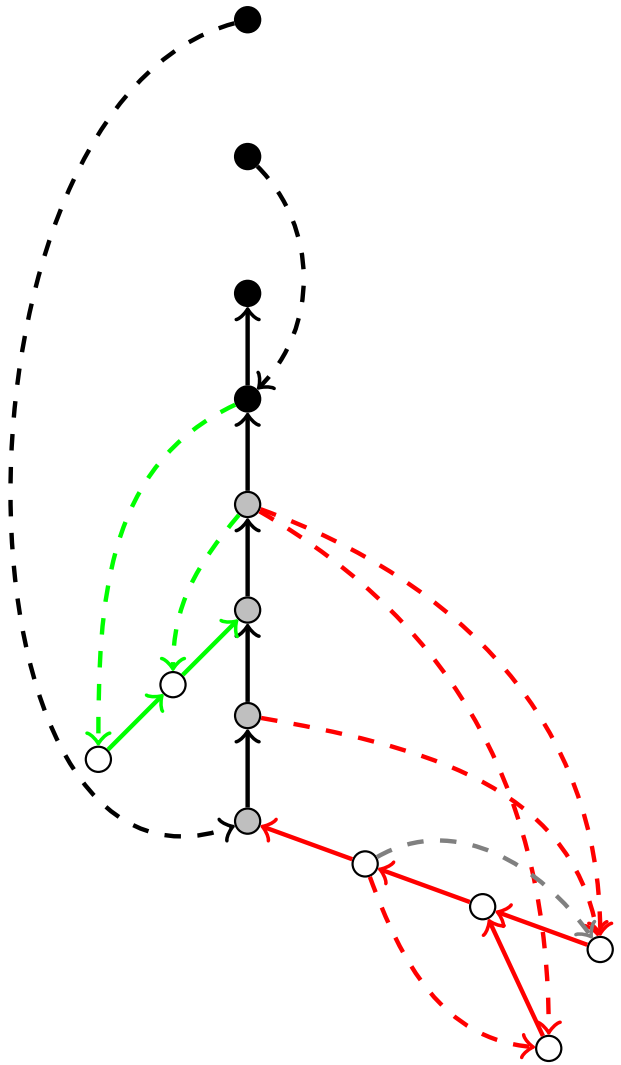
Idea

Add interlacing chains (and their interlacing offspring) as early as possible.
$\Rightarrow$ After adding a nested chain, add all interlacing offspring.

Segments

Segment: A nested child and all its interlacing offspring.

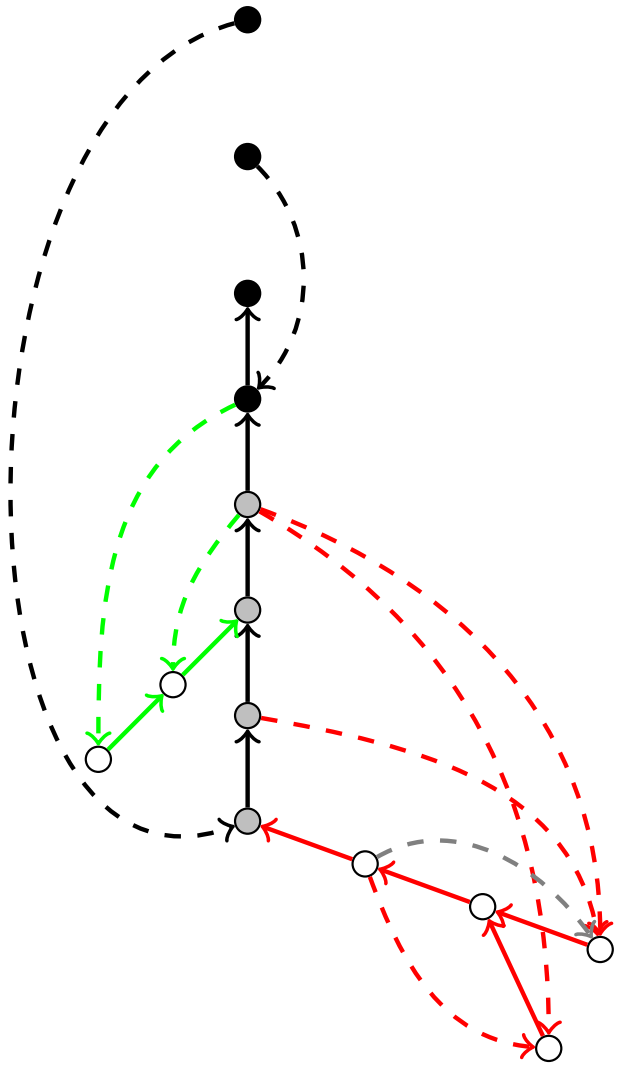
Can be added if there is a branch vertex between the endpoints of the nested chain.



Segments

Problem: Find an order on the segments.

Solution: Traverse the overlap graph.



A Linear Time Algorithm

Compute a chain decomposition $\{C_1, C_2, \ldots, C_{m-n+1}\}$
Initialize G_c to $C_1 \cup C_2$;
for i from 1 to $m - n + 1$ **do**
    Compute the segments w.r.t. C_i
    Find an insertion order $S_1, \ldots, S_k$ on the segments
    for j from 1 to k **do**
        Add the chains contained in S_j parent-first
    end for
end for

Wrap Up

- Certifying Algorithms: An important tool for reliable implementations.
- Construction Sequences: Certificates for connectivity properties.
- Chain decompositions: Structure for computing construction sequences.

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Open Problem

Can we find the construction sequence for k-(edge)-connected graphs efficiently?