Phase Transition of the 2-Choices Dynamics on Core-Periphery Networks

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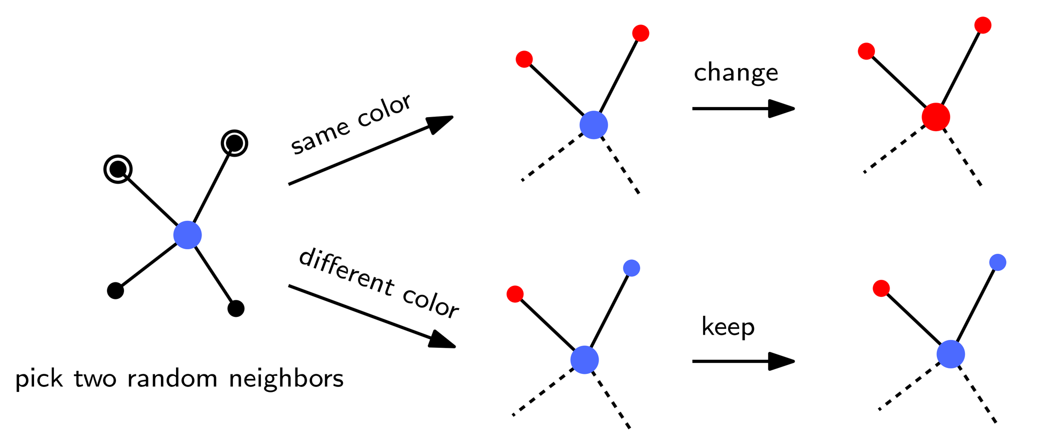
2-Choices Dynamics

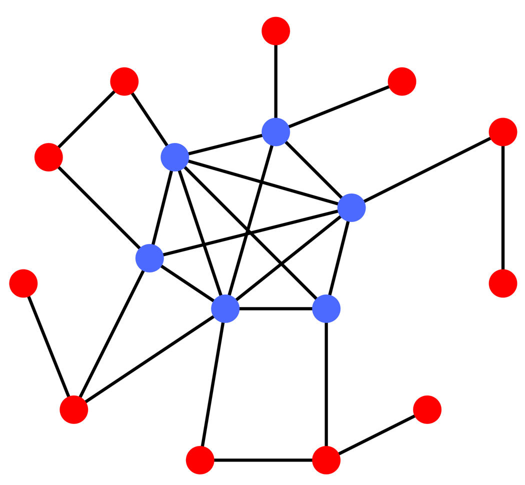
Given a graph $G = (V, E)$ and a coloring $V \to \{\text{red}, \text{blue}\}$, a **dynamics** is a process where in every synchronous round each node updates its color depending on its neighbors' colors according to a simple rule.

In the **2-Choices dynamics** the update rule is: *choose two random neighbors and if they have the same color, adopt it.*



Core-Periphery Network

A **core-periphery** network is a graph with a set $\mathcal{C}$ of densely connected nodes (the core) and the rest of the nodes $\mathcal{P}$ (the periphery) is loosely connected and dominated by the core.



We call $c_d = \frac{c(\mathcal{C},\mathcal{P})}{c(\mathcal{P},\mathcal{P})}$ the **dominance** and $c_r = \frac{c(\mathcal{C},\mathcal{C})}{c(\mathcal{C},\mathcal{P})}$ the **robustness**, where $c(A, B)$ is the number of cut edges between A and B.

Main Result

When running the 2-Choices dynamics on core-periphery networks where the core is initially blue and the periphery is initially red, there exists a universal constant $c^\star$ such that

- if $\mathbf{c_d} > \mathbf{c}^\star$ and c_r is large enough, then a configuration of **almost-consensus** on blue is reached in $\mathcal{O}(\log n)$ rounds with high probability
- if $\mathbf{c_d} < \mathbf{c}^\star$, then a **metastable** phase takes place where most of the nodes retain their initial opinion for $n^{\omega(1)}$ rounds with high probability

Selected Related Work

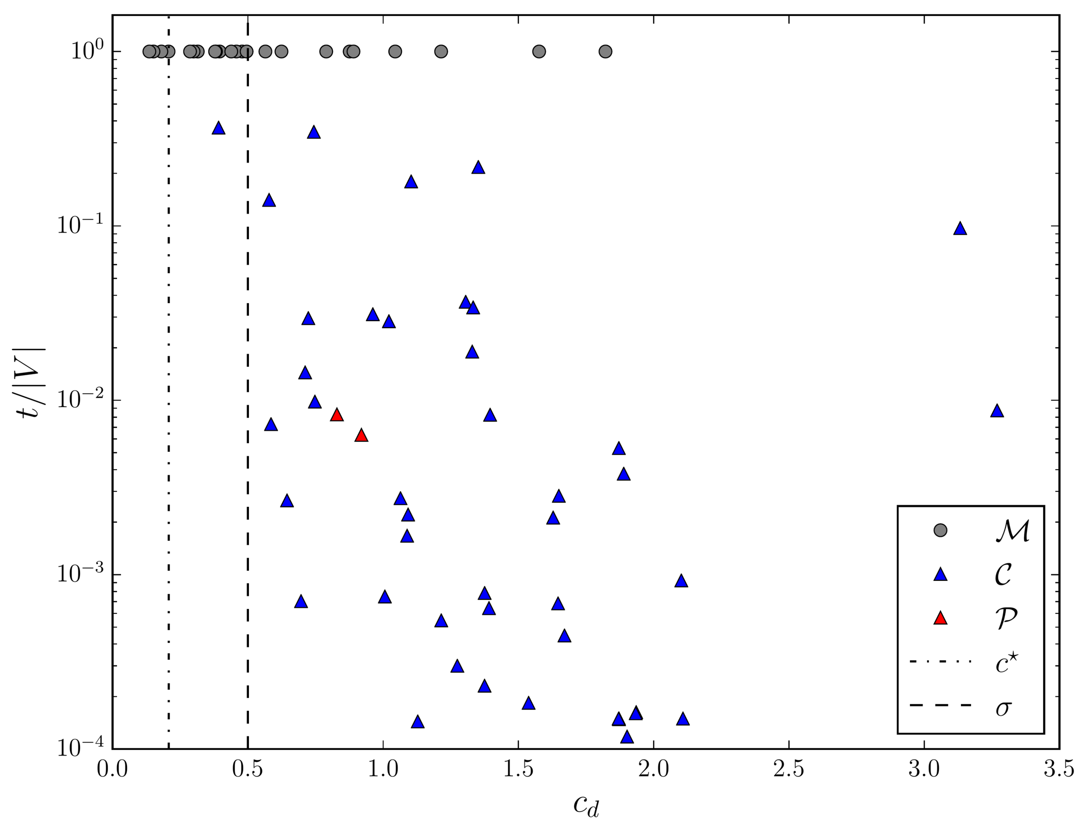
Two-choices dynamics: *The power of two choices in distributed voting*. Cooper, Elsässer, Radzik (ICALP 2014)

Core-periphery networks: *Distributed computing on core–periphery networks: Axiom-based design*. Avin, Borokhovich, Lotker, Peleg (Journal Of Parallel And Distributed Computing 2017)

Definition. The **volume** of a set of nodes $A \subseteq V$ in a graph is defined as $vol(A) = \sum_{v \in A} d_v$, where d_v is the degree of v.

Experimental Results

Tests conducted on **70 real-world networks**.



Metastability and almost-consensus of the experiments compared to the theoretical and empirical thresholds $c^\star$ and σ. In total, 86% of the runs are metastable when $c_d < \sigma$; and 81% of them lead to an almost-consensus if $c_d > \sigma$. (t = number of rounds until almost-consensus/metastability was declared; $\mathcal{M}$ = metastability)

Main Theorem

Theorem. Let $G = (V, E)$ be a graph where each node $v \in V$ is initially colored red and has degree $d_v \in \omega(\log n)$. Let $p \in [0, 1]$ be a constant. Then, if the nodes run a dynamics where in every round each node with probability p picks the color *blue* and with probability $1 - p$ runs the 2-choices dynamics, then there is a universal constant $p^\star$ such that

- **Almost-consensus:** If $\mathbf{p} > \mathbf{p}^\star$, then the agents reach a configuration such that the *volume* of nodes supporting the *blue* color is $(1 - o(1))vol(V)$ within $O(\log n)$ rounds with high probability.
- **Metastability:** If $\mathbf{p} < \mathbf{p}^\star$, then the *volume* of the *blue* nodes never exceeds $\frac{1-3p}{4(1-p)} vol(V)$ for any poly(n) number of rounds with high probability.

Proof idea.

- Give an upper and a lower bound on the expected number of red neighbors of any node $v \in V$ after one time step. Both can be expressed as a function whose behavior changes at $p^\star$.
- If $\mathbf{p} > \mathbf{p}^\star$, using a multiplicative form of the Chernoff bounds, show that the number of red neighbors of each node reduces by a constant with high probability. Using the union bound over the nodes, obtain the consensus result.
- If $\mathbf{p} < \mathbf{p}^\star$, show with the same tools that for each node it is very unlikely that the number of red neighbors decreases in a round. Using the union bound over all nodes and a superpolynomial number of rounds, conclude the metastability result.

Corollaries

Corollary (Consensus). Consider a core-periphery network where each node has $\omega(\log n)$ neighbors. If c_r is large and $c_d > c^\star$, then the nodes reach a configuration such that the *volume* of *blue* nodes is $(1 - o(1))vol(V)$ within $\mathcal{O}(\log n)$ rounds of the 2-Choices dynamics, w.h.p.

Corollary (Metastability). Consider a core-periphery network where each node has $\omega(\log n)$ neighbors. Then, with high probability, we have that for every round t and for any $poly(n)$ number of rounds of 2-Choices dynamics:

- if $c_r > \frac{1}{c^\star}$, then $vol(B^{(t)}) \geq \frac{3}{4} vol(\mathcal{C})$
- if $c_d < c^\star$, then $vol(R^{(t)}) \geq \frac{3}{4} vol(\mathcal{P})$

where $B^{(t)}$ are the *blue* agents and $R^{(t)}$ the *red* agents at time t.