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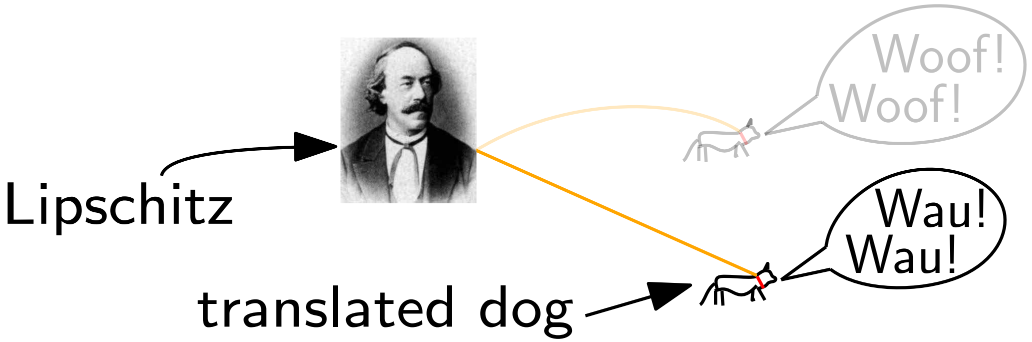
When Lipschitz Walks Your Dog: Algorithm Engineering of the Discrete Fréchet Distance under Translation

Karl Bringmann, Marvin Künnemann, and **André Nusser**



Teaser

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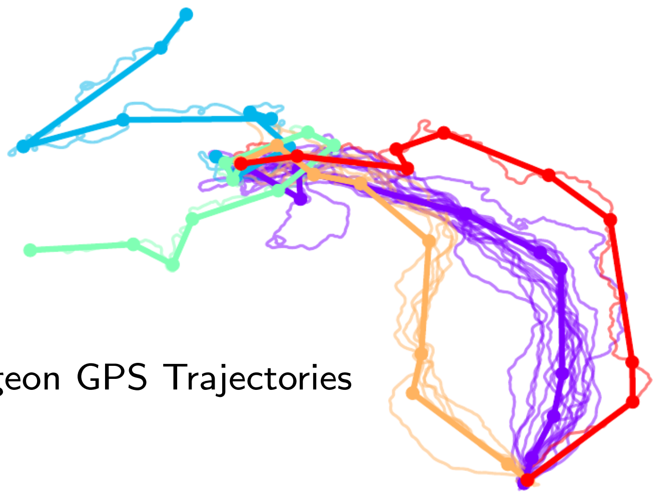
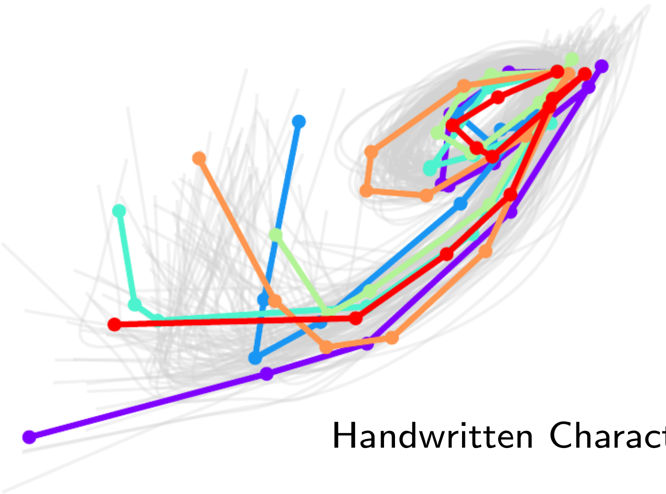
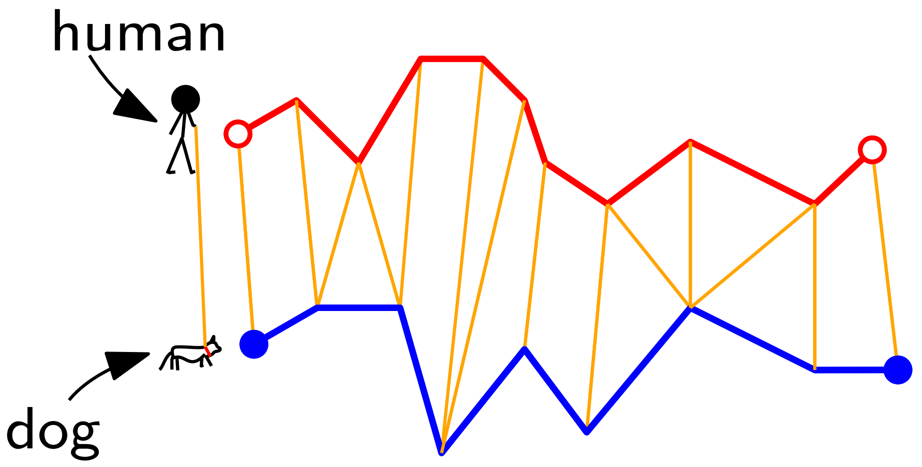
Trajectory Similarity:





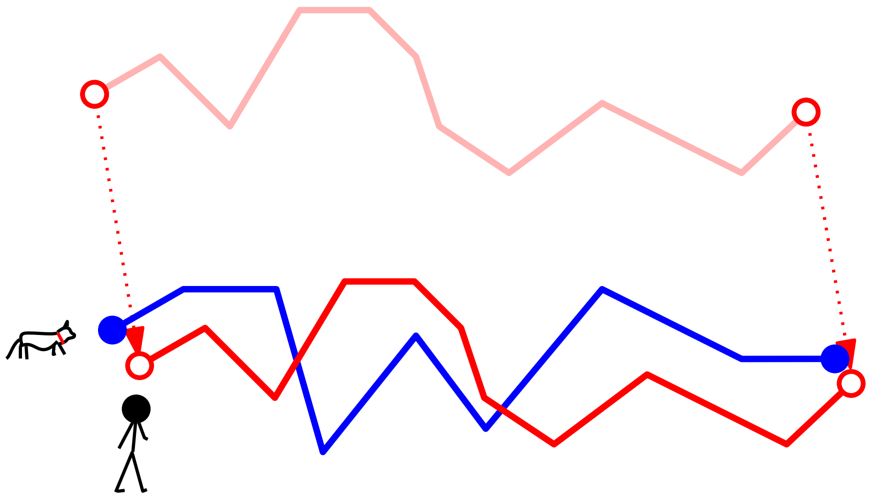
Fréchet Distance:

- traversal based
- fast in practice



Fréchet Distance Under Translation:

- traversal based
- only impractical algorithms (before)
- translation invariant



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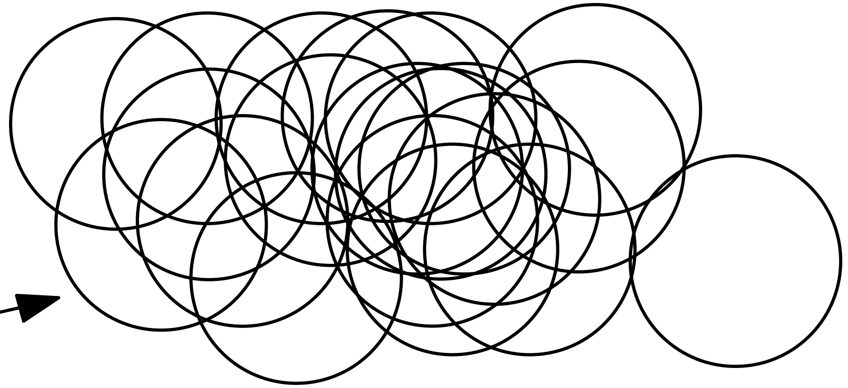
Teaser

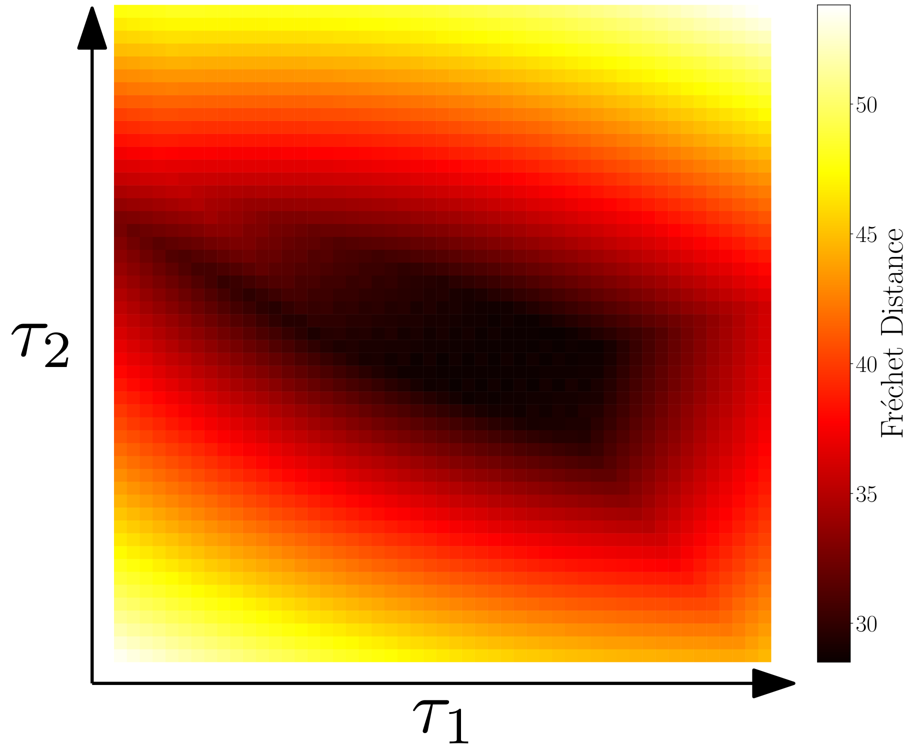
Best algorithm: $\mathcal{O}(n^{4.66})$

Conditional lower bound: $n^{4-o(1)}$

↳ All algorithms build $\mathcal{O}(n^4)$ arrangement!



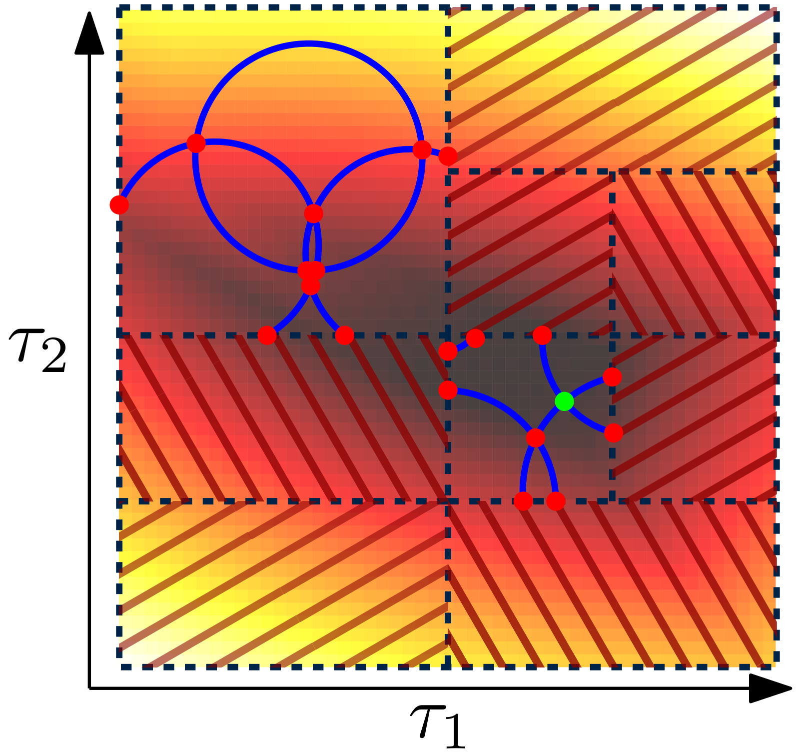
Fréchet under translation is 1-Lipschitz in τ!



Use continuous optimization:

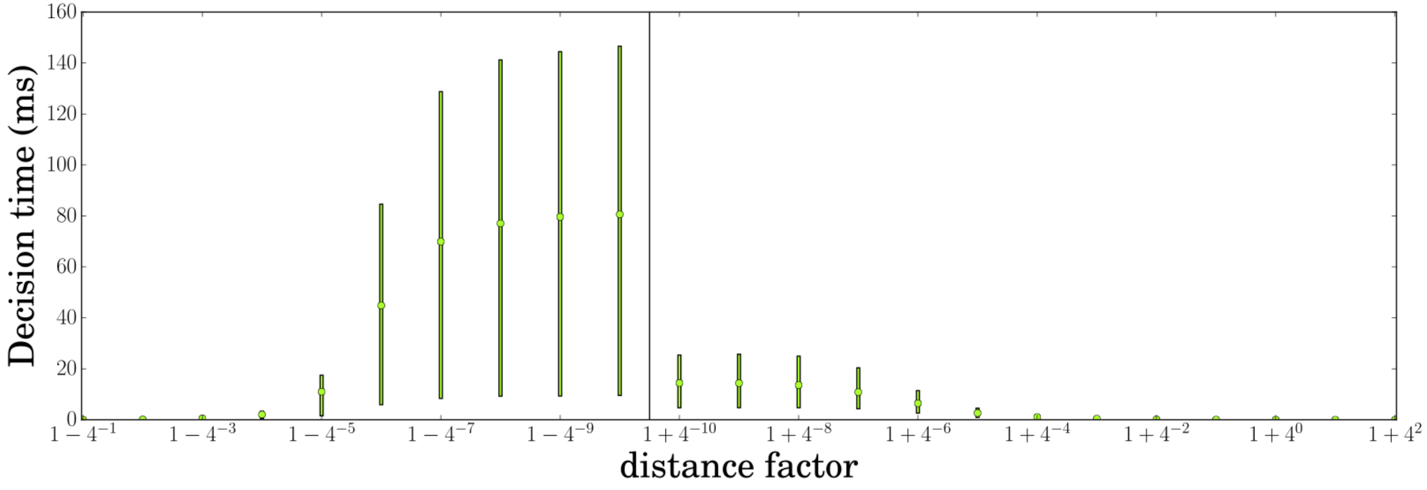
- branch & bound!

Lipschitz Meets Fréchet:



Teaser

SAME-CHARACTERS:



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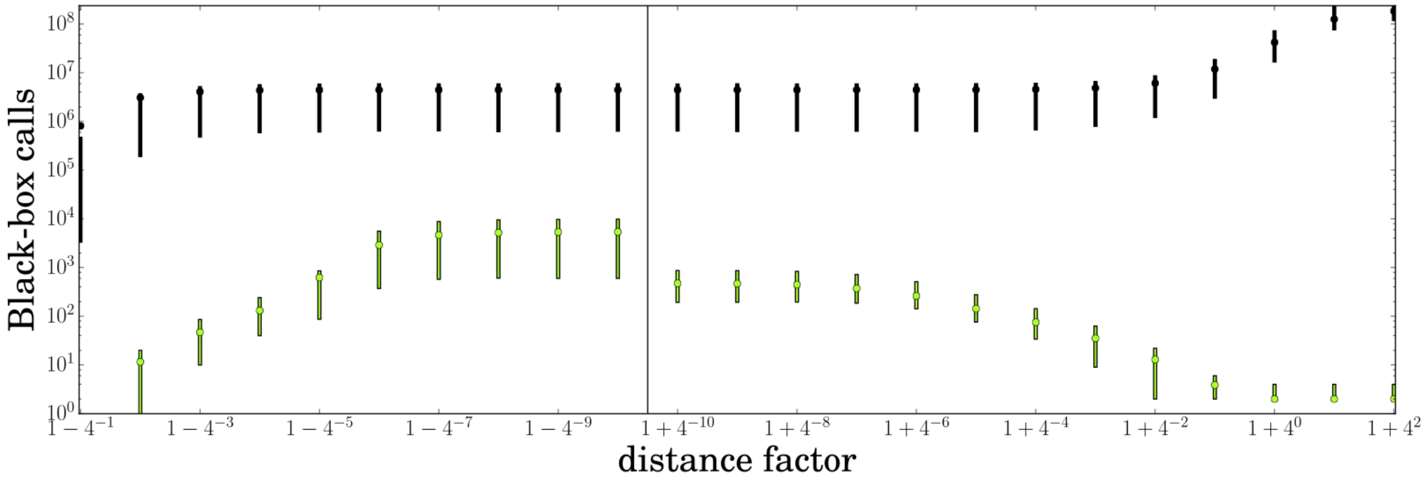
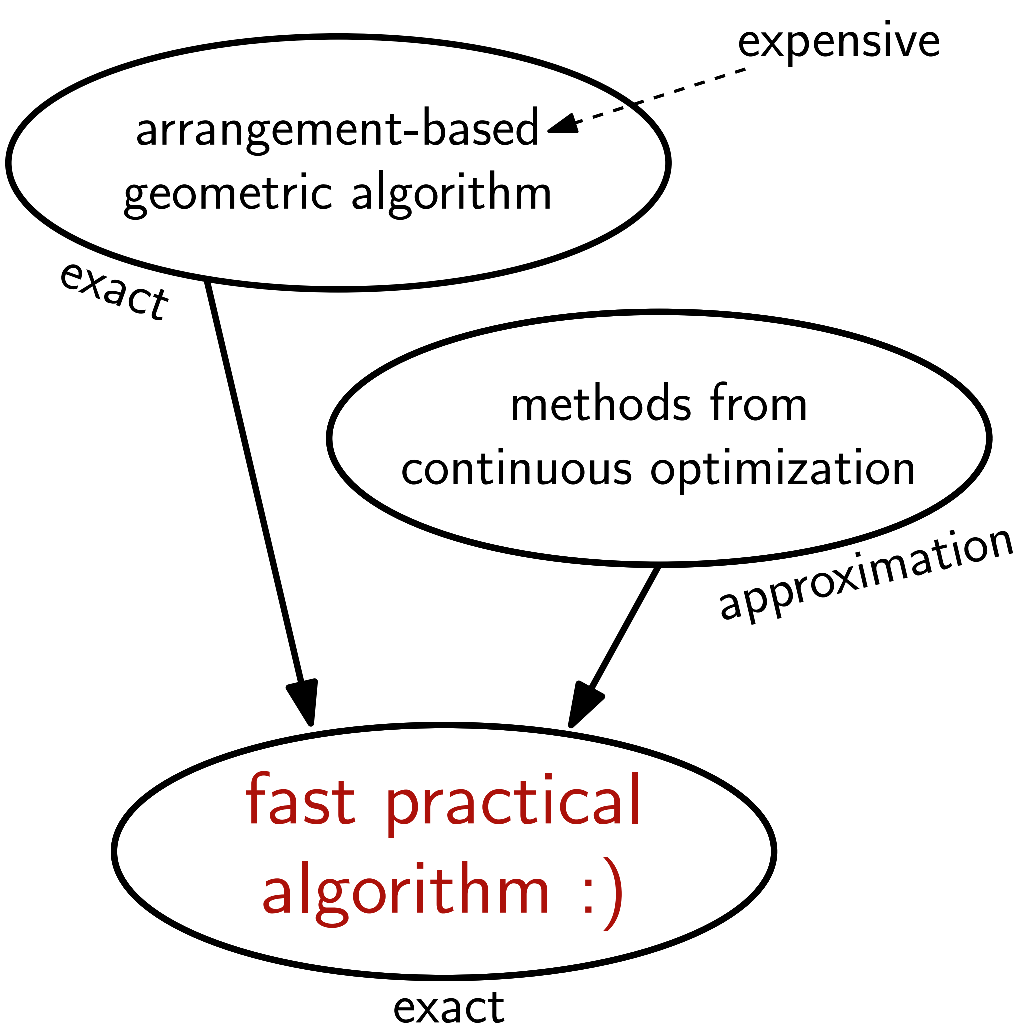


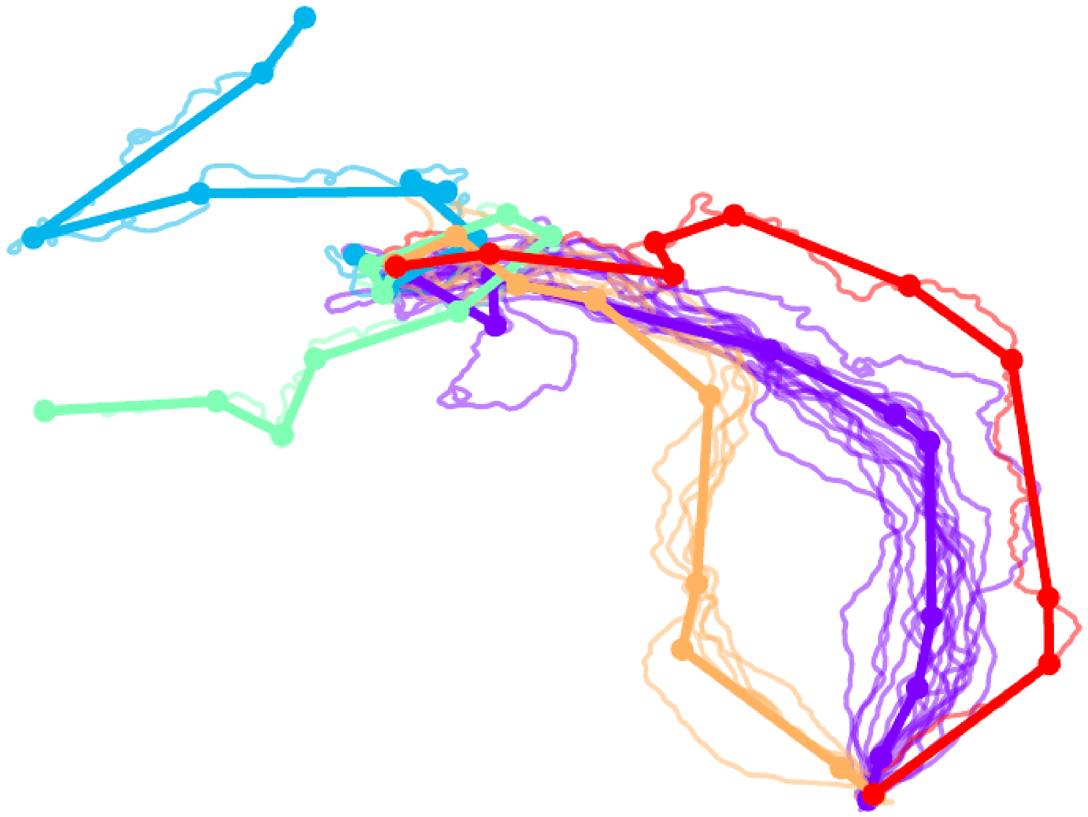
Take-Home Message:



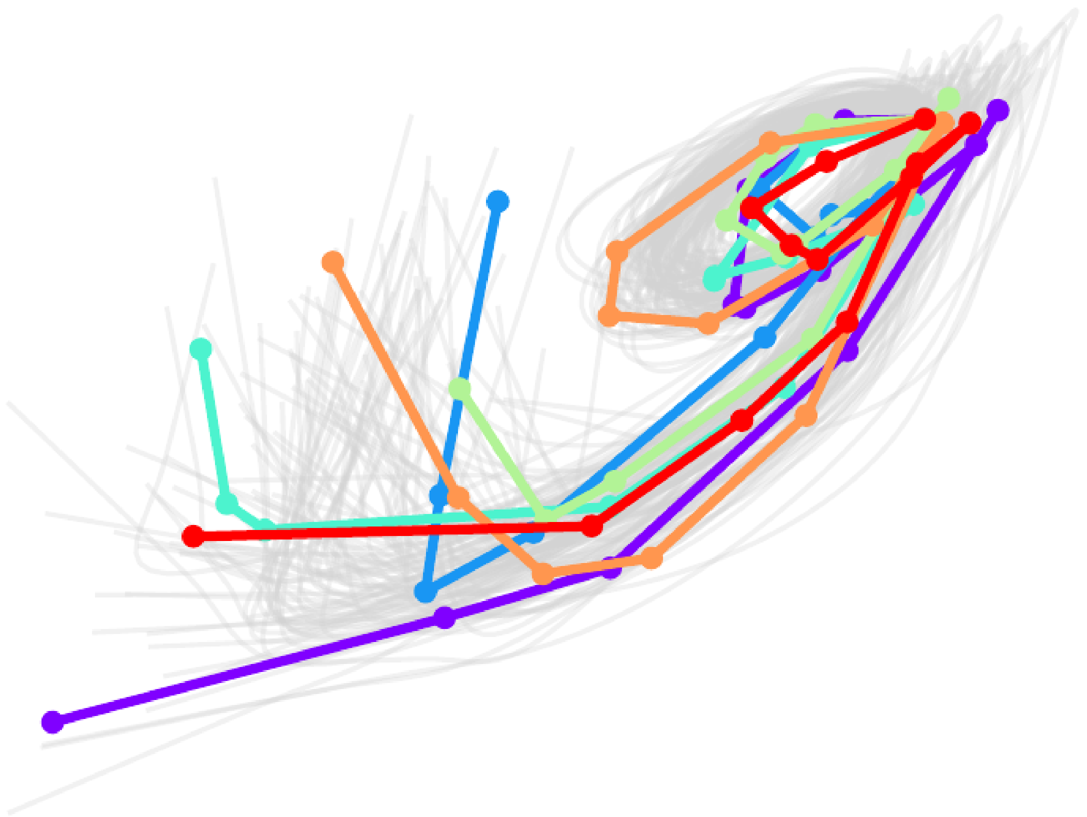
End of Teaser

Why Trajectory Similarity?

Pigeons' GPS Trajectories:

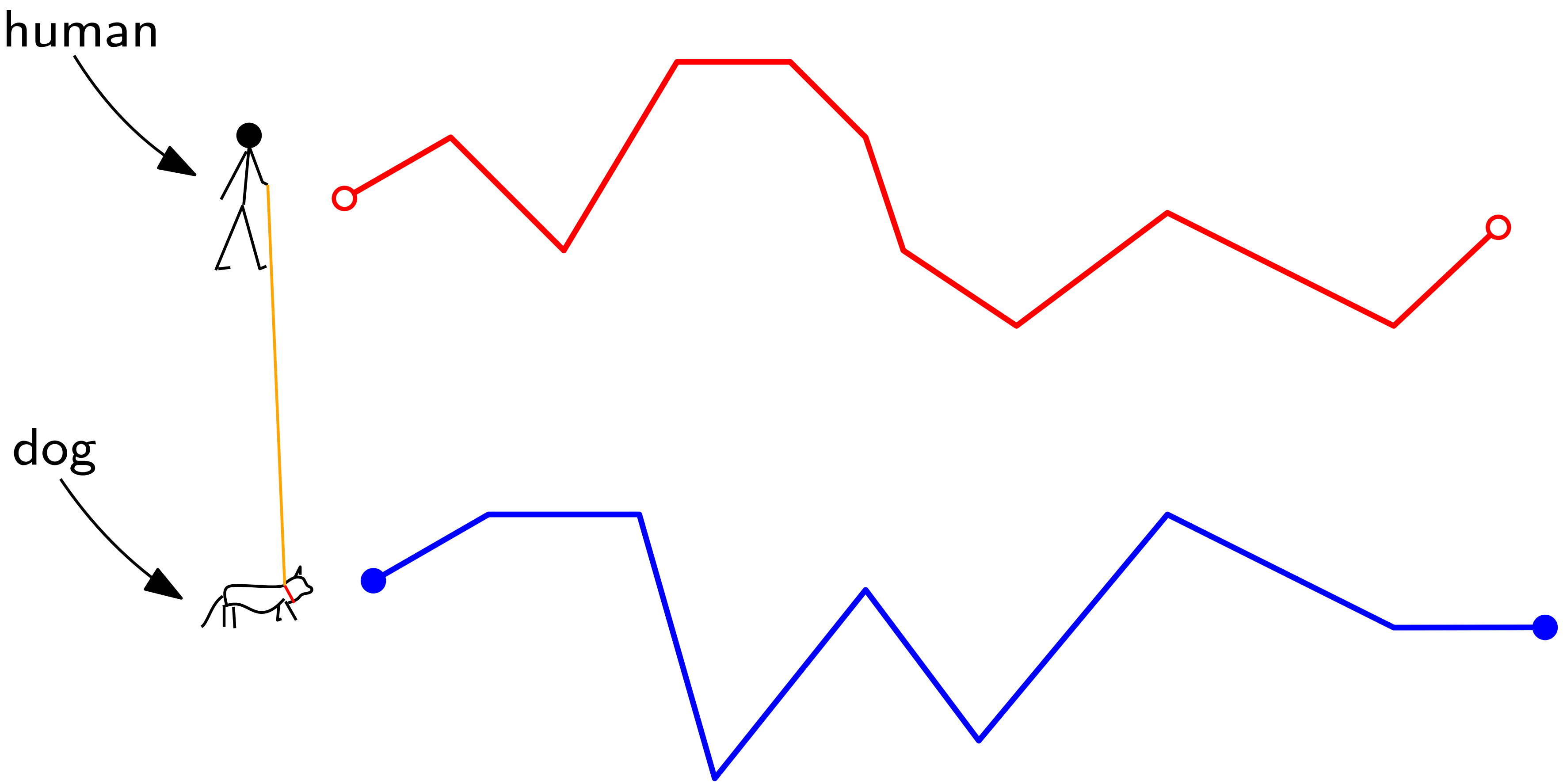


Handwritten Character Trajectories:



Discrete Fréchet Distance

Intuition



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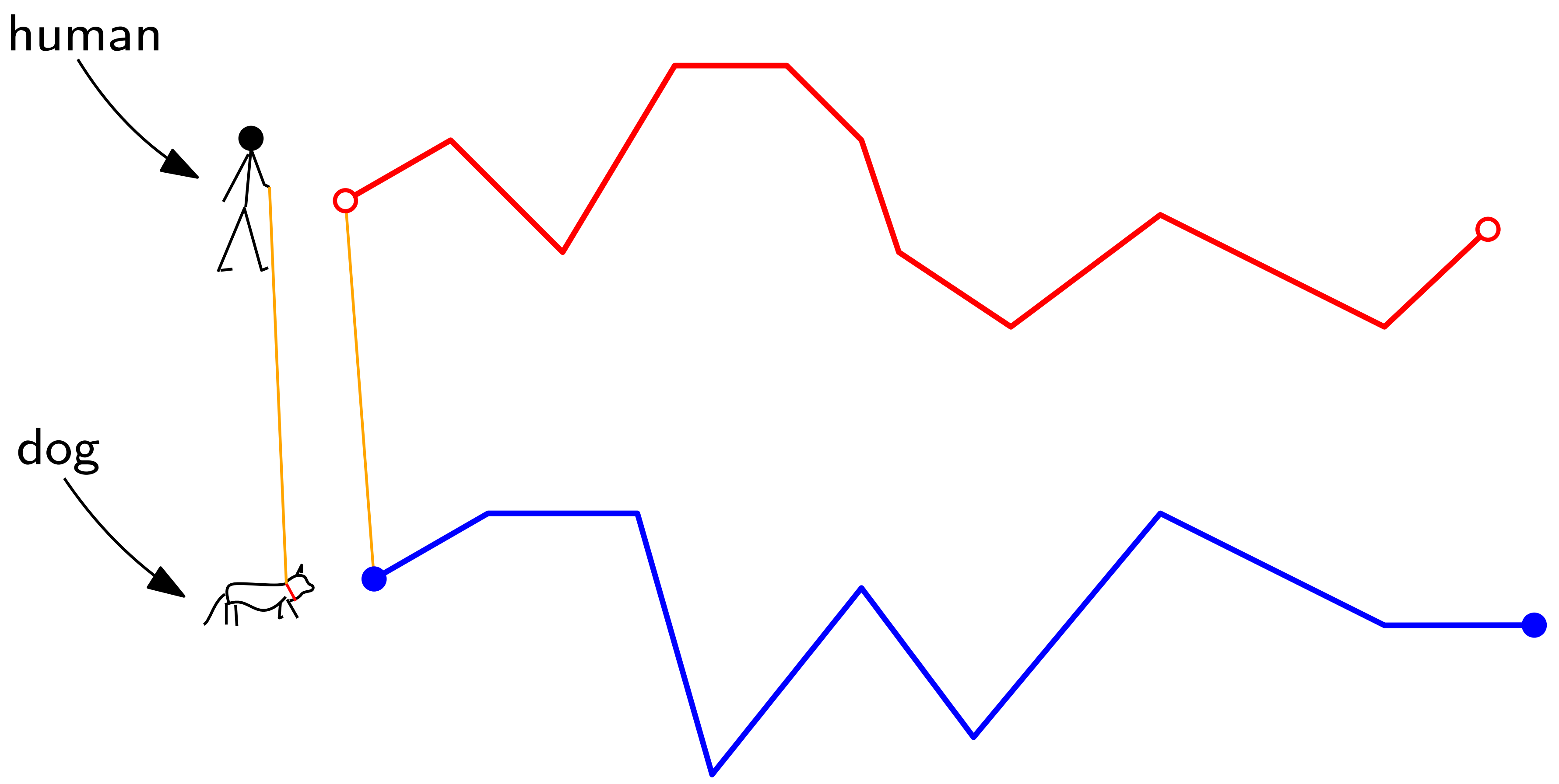

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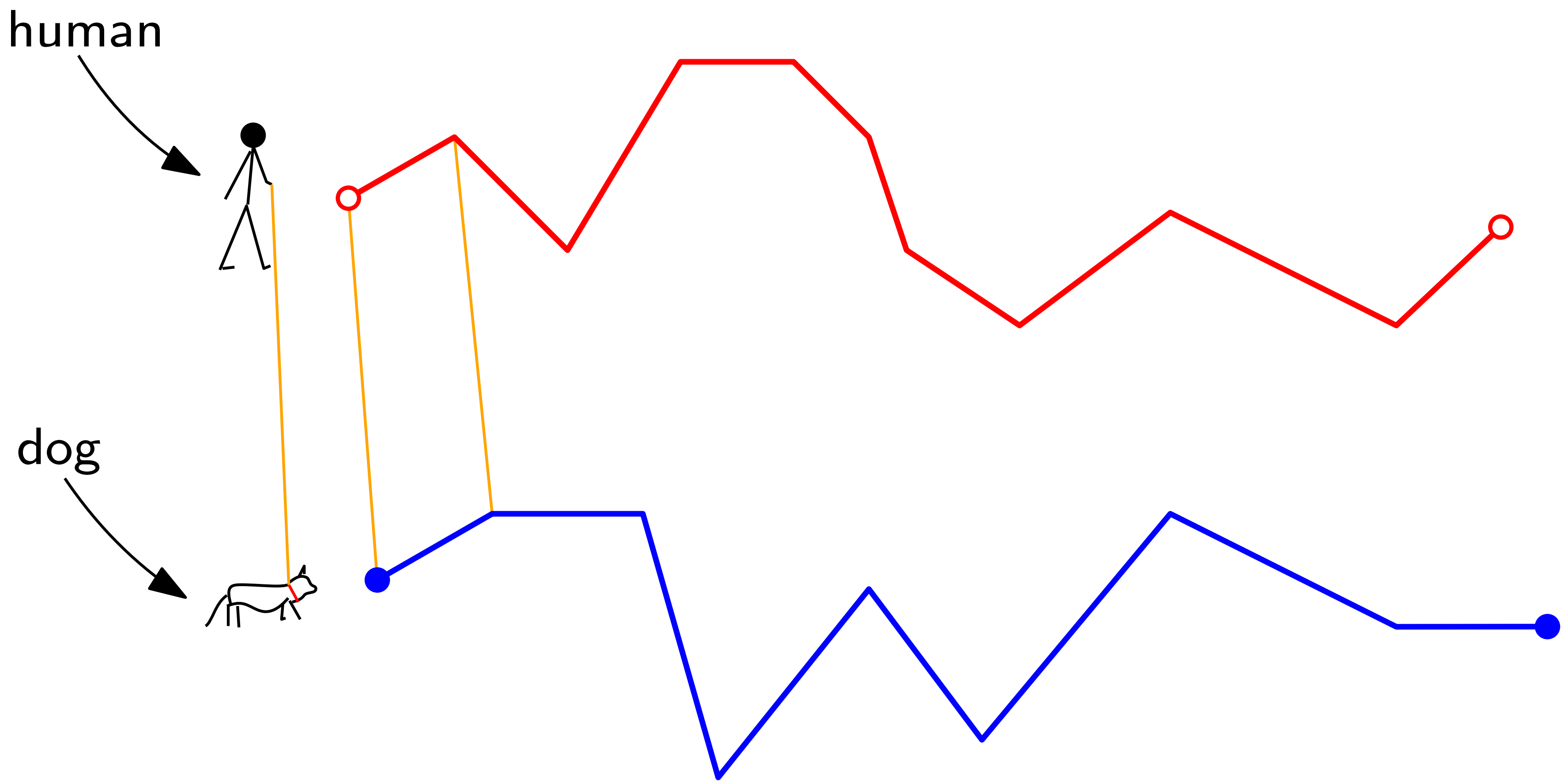

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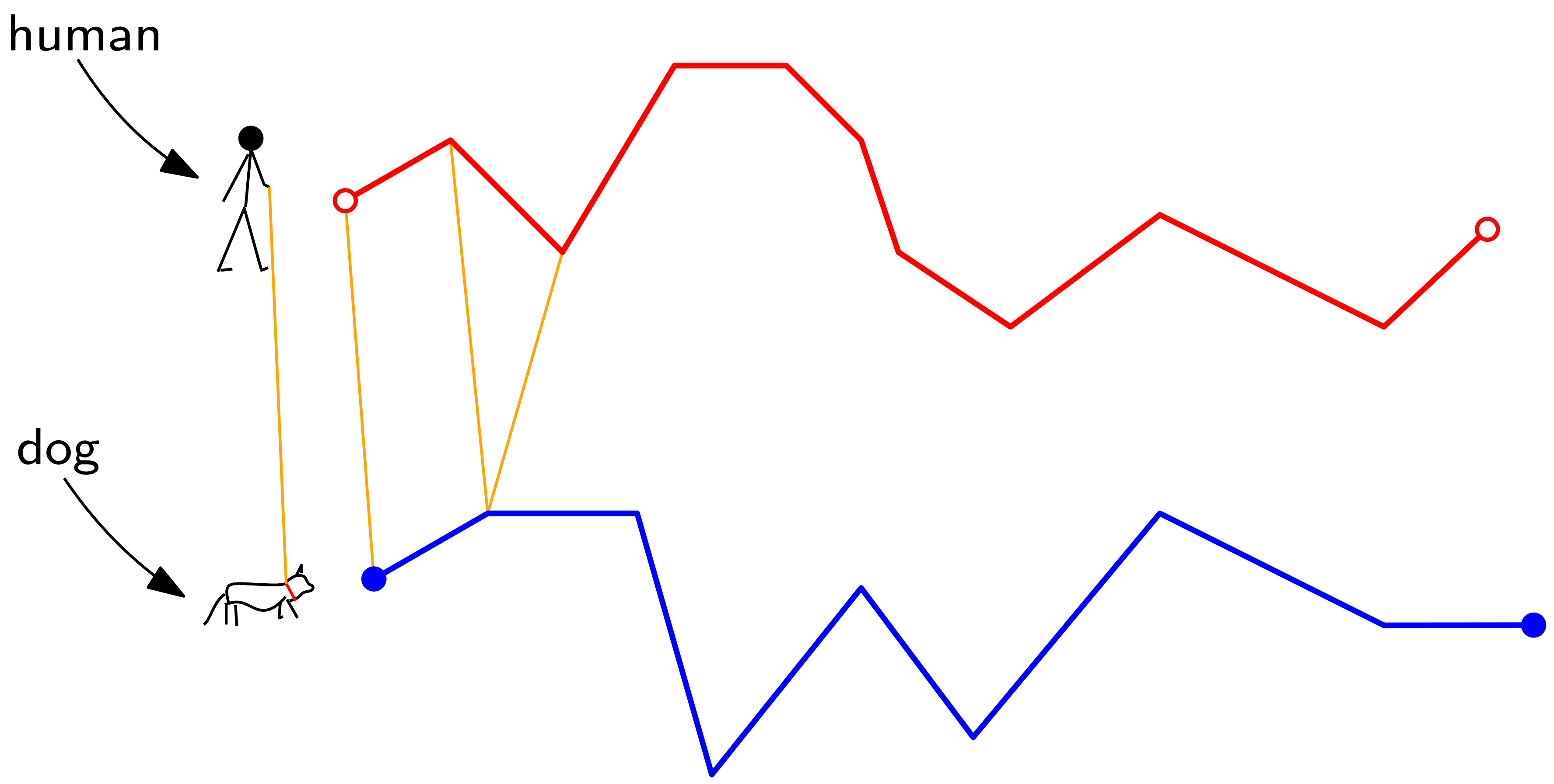

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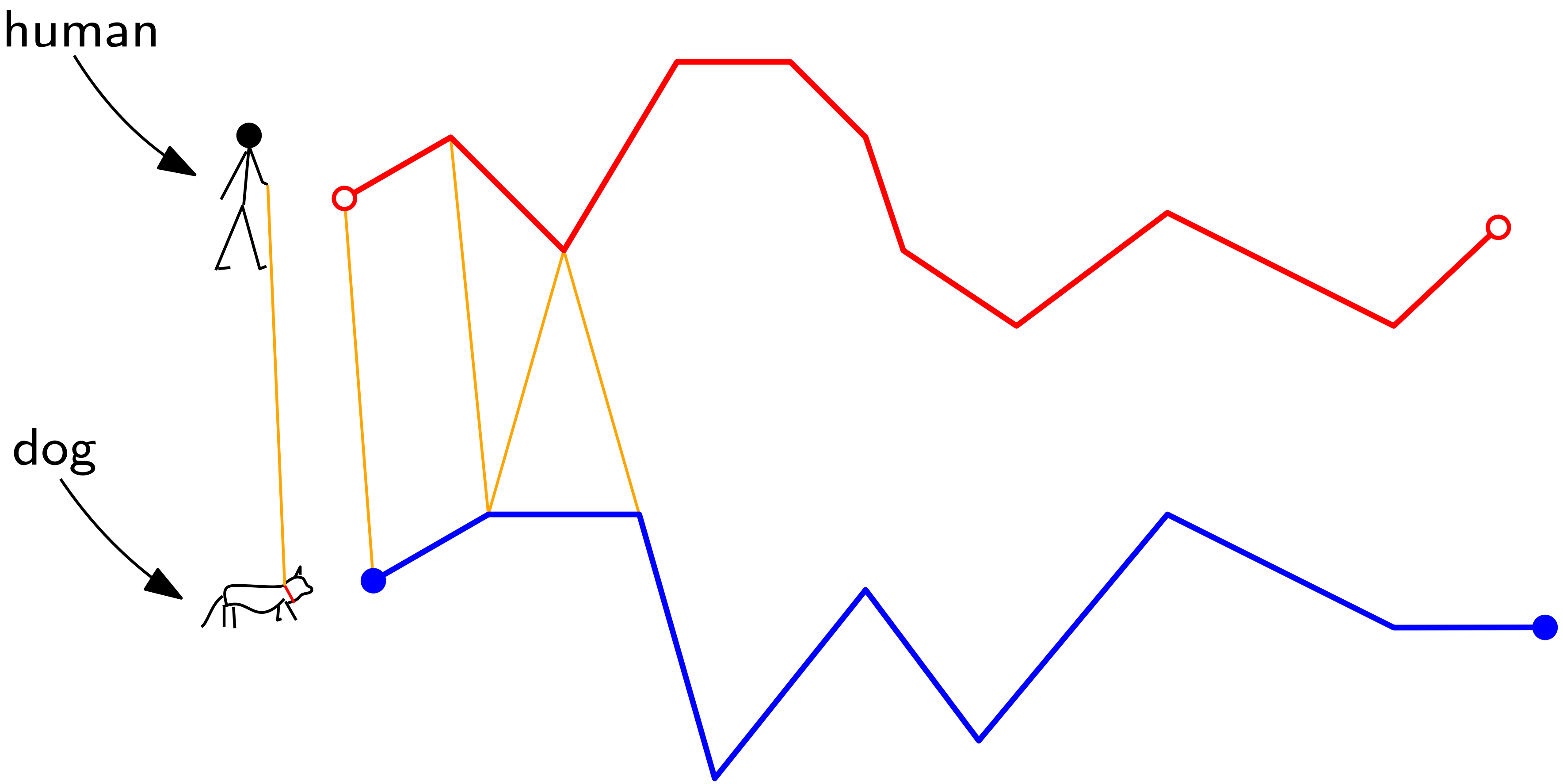

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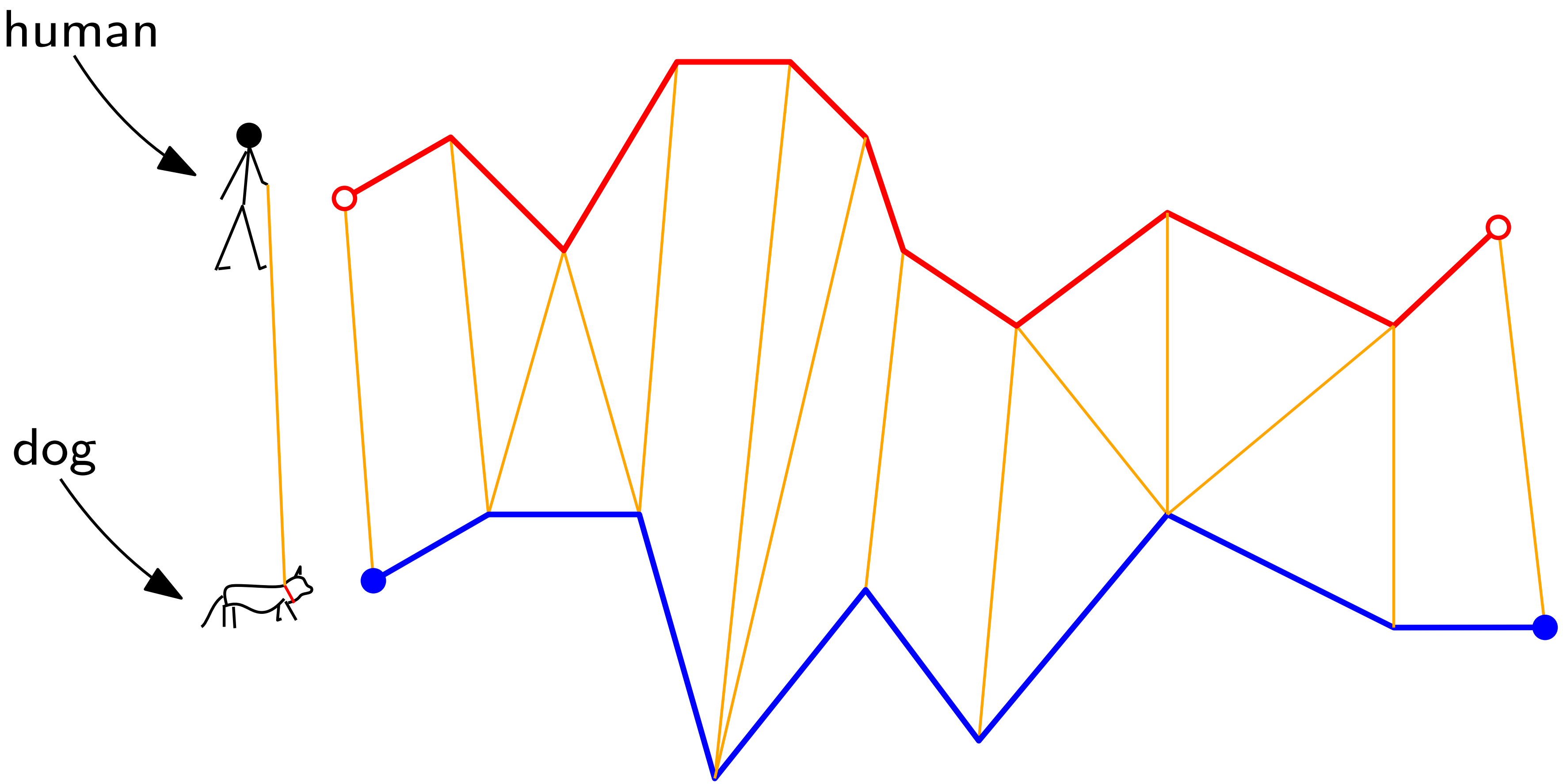

Question: What is the traversal that achieves the shortest leash length?

Discrete Fréchet Distance

Formal Definition

$$\delta_F(\pi, \sigma) := \min_{f,g \in \mathcal{T}} \max_{t \in [0,1]} \left\| \pi_{f(t)} - \sigma_{g(t)} \right\|$$

$\pi, \sigma =$ polygonal curves of length n
$\mathcal{T} =$ set of monotone and surjective functions from $[0,1]$ to $\{1, \ldots, n\}$

Discrete Fréchet Distance under Translation

Definition

Intuition: Allow arbitrary translations $\tau \in \mathbb{R}^2$ of curve σ.

$$\delta_T(\pi, \sigma) := \min_{\tau \in \mathbb{R}^2} \delta_F(\pi, \sigma + \tau)$$

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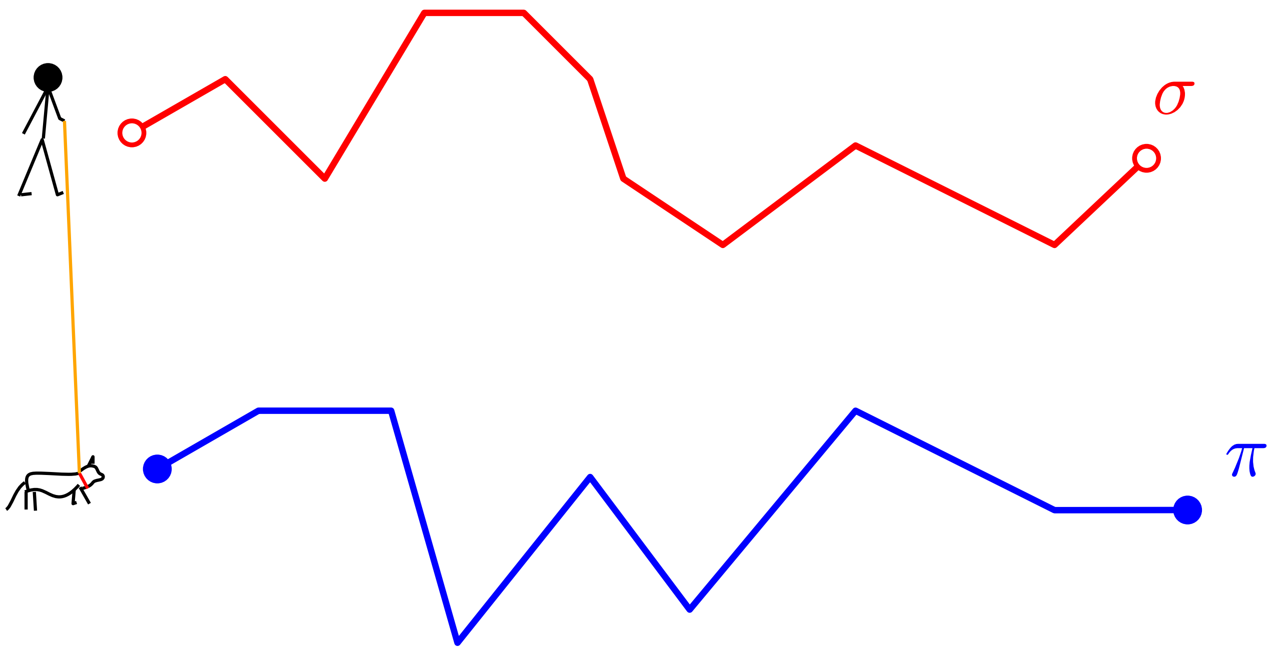

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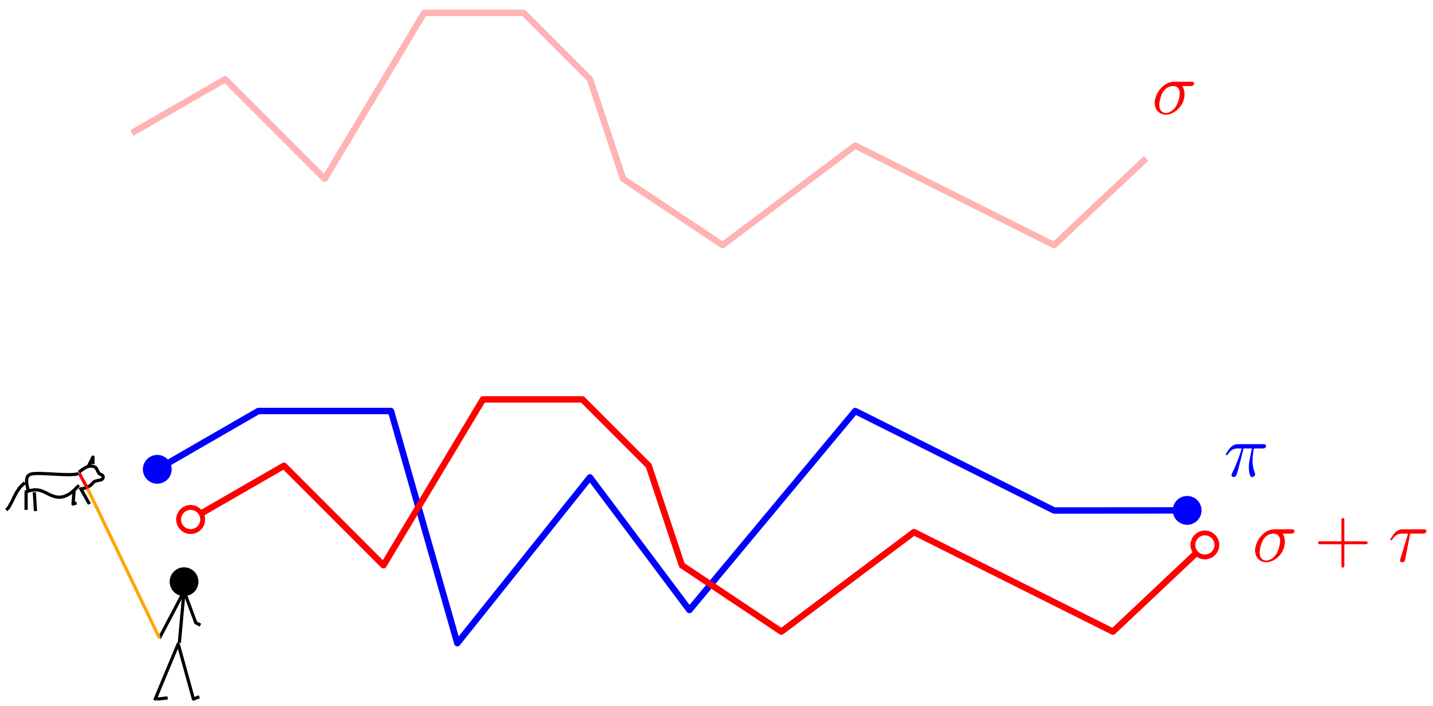

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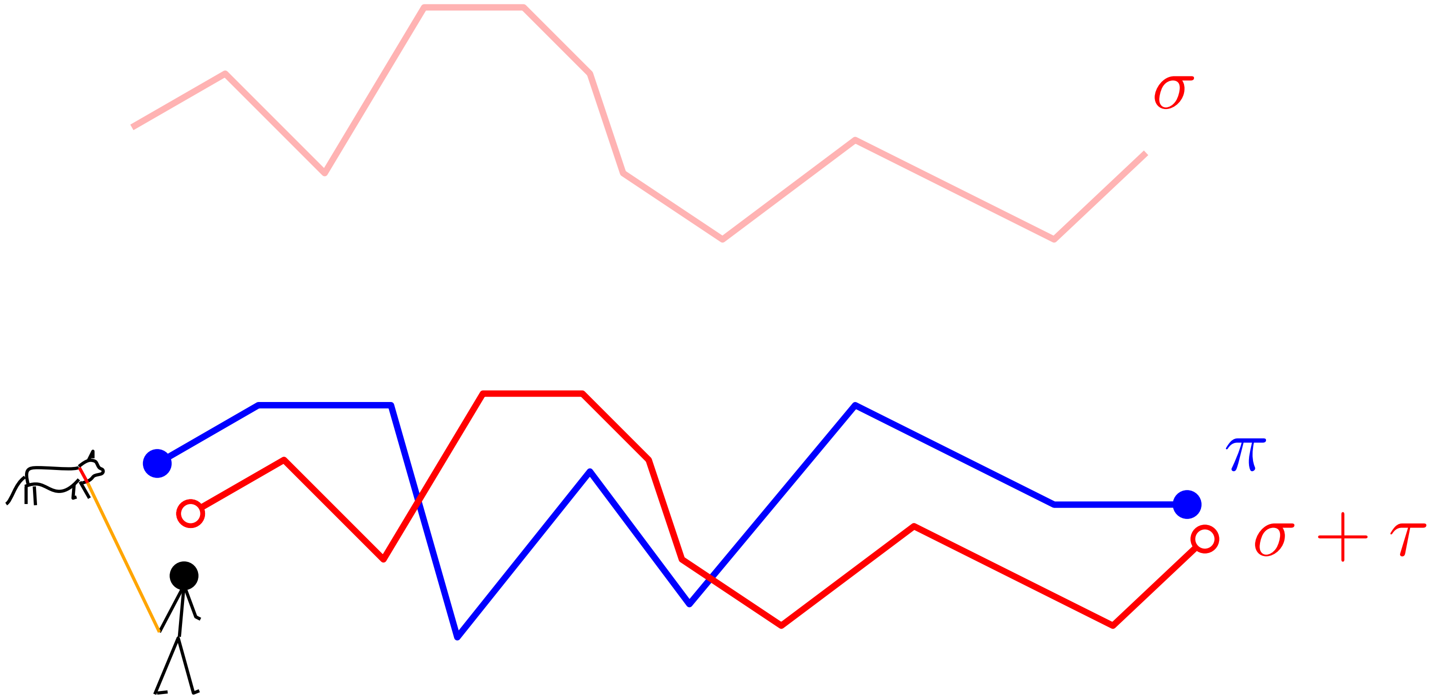

Decision Problem:

- Given π, σ, δ
- $\delta_T(\pi, \sigma) \leq \delta$?

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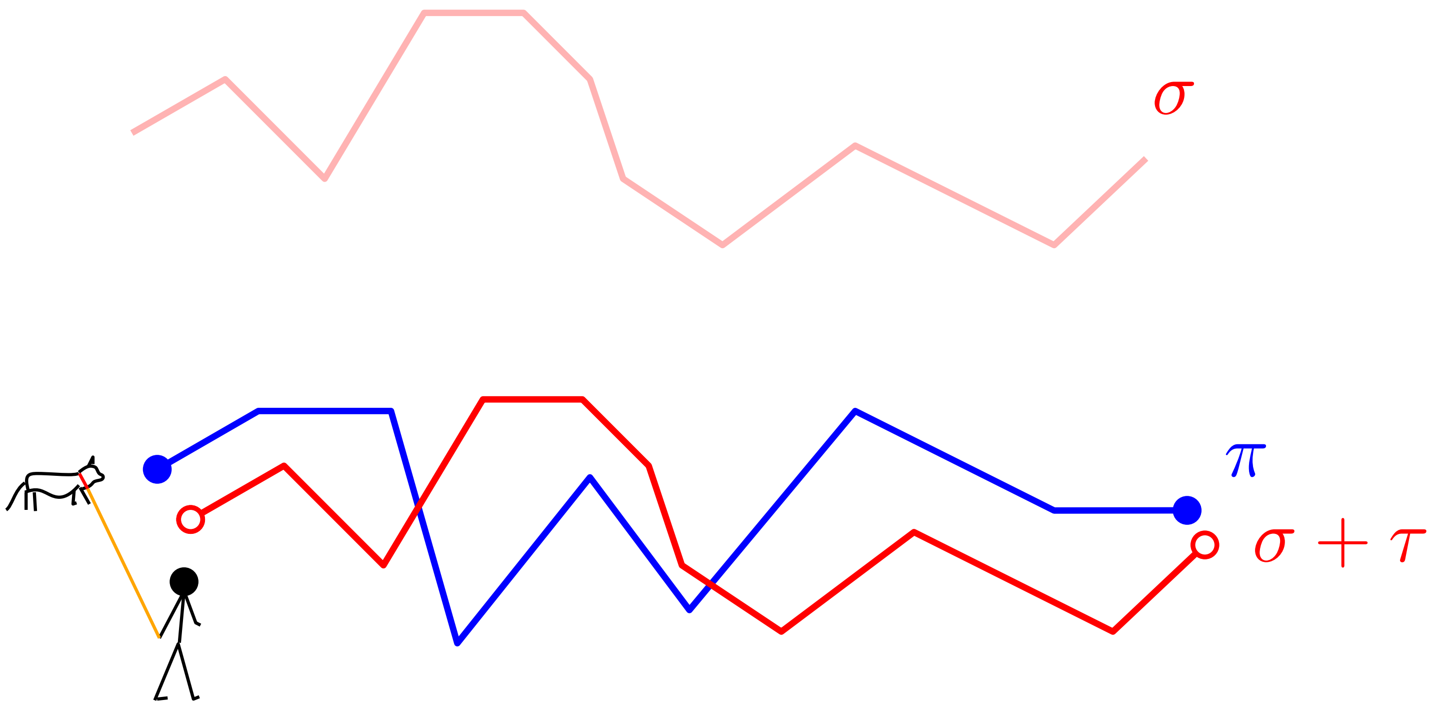

Decision Problem:

- Given π, σ, δ
- $\delta_T(\pi, \sigma) \le \delta$?

Focus on this in the talk!

Goal:

Performant implementation
computing the discrete Fréchet distance under translation
on practical inputs.

Related Work

Theory:

curve length

- Discrete Fréchet distance under translation in $\tilde{\mathcal{O}}(n^5)$
  [Agarwal, Ben Avraham, Kaplan, Sharir arXiv'15]
- Discrete Fréchet distance under translation in $\tilde{\mathcal{O}}(n^{4.66})$
  [Bringmann, Künnemann, N. SODA'19]
- SETH based lower bound of $n^{4-o(1)}$ for discrete Fréchet distance under translation
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Related Work

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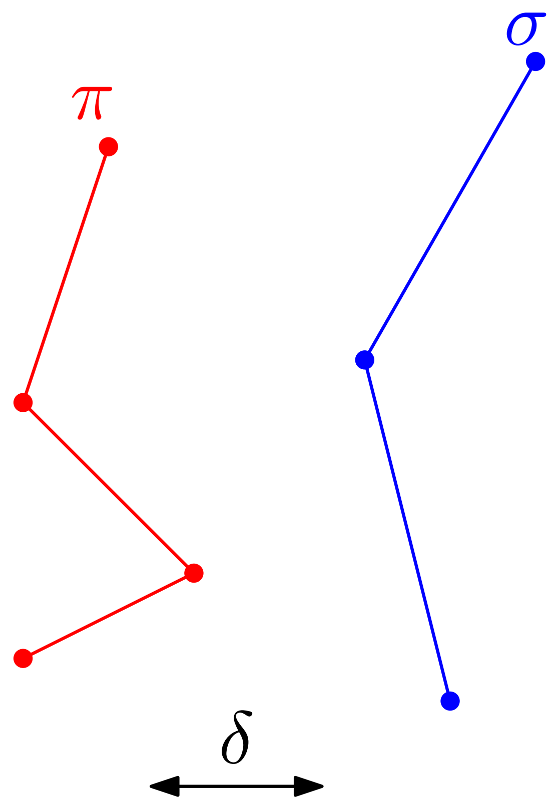
Practice:

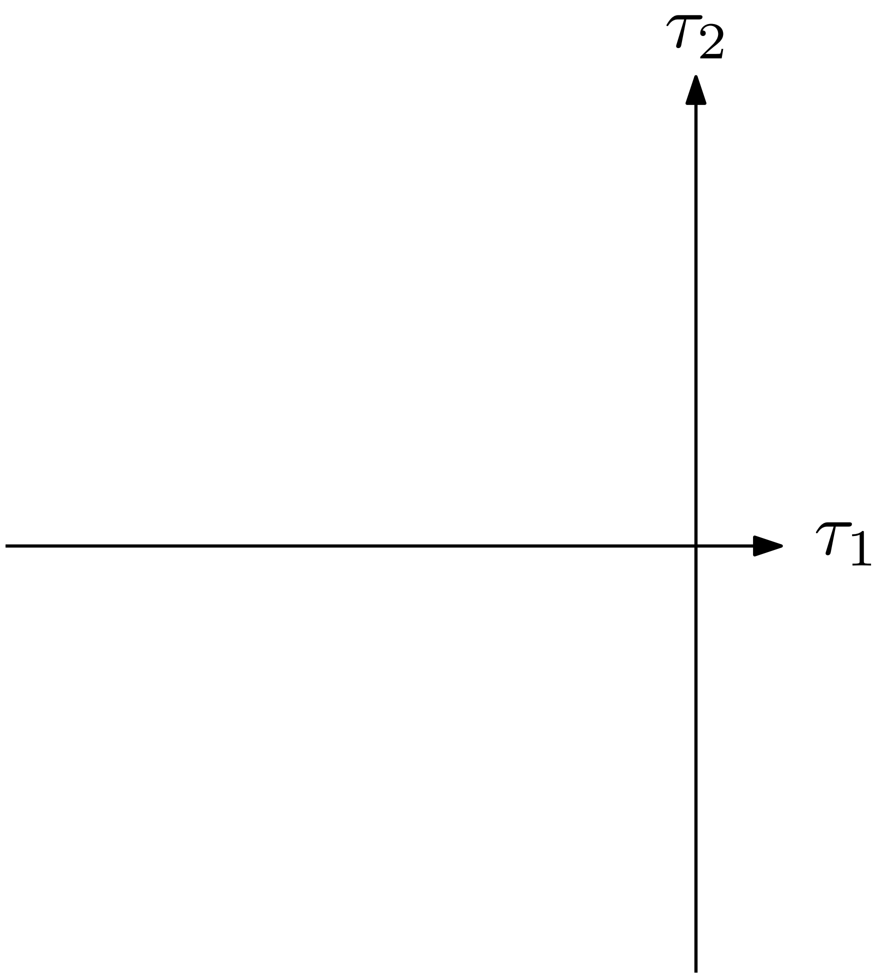
- GIS Cup on (fixed-translation) Fréchet distance near neighbors search
  [Werner, Oliver; Baldus et al.; Buchin et al.; Dütsch et al. SIGSPATIAL'17]
- State of the art (fixed-translation) Fréchet distance implementation
  [Bringmann, Künnemann, N. SoCG'19]

Approach I: Discrete Algorithms

Arrangement

- Idea: Partition the plane into equivalent regions.





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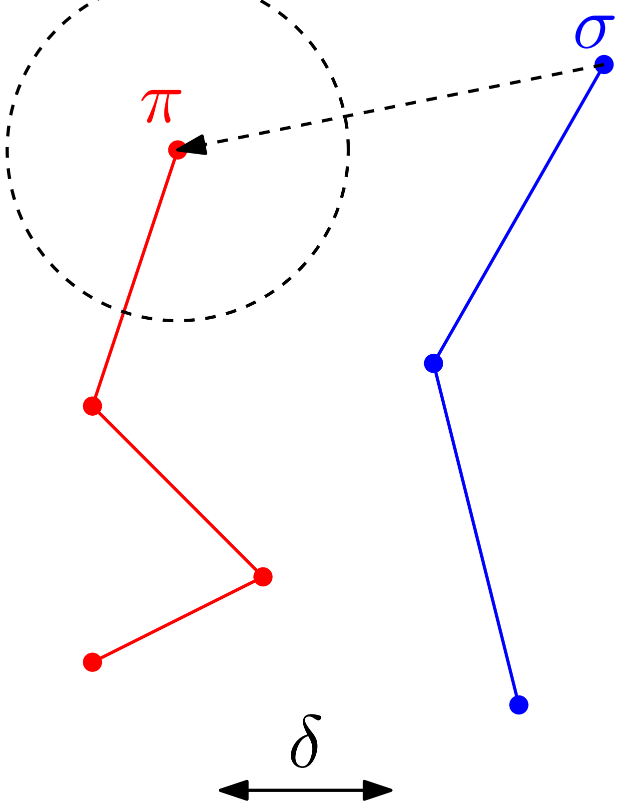

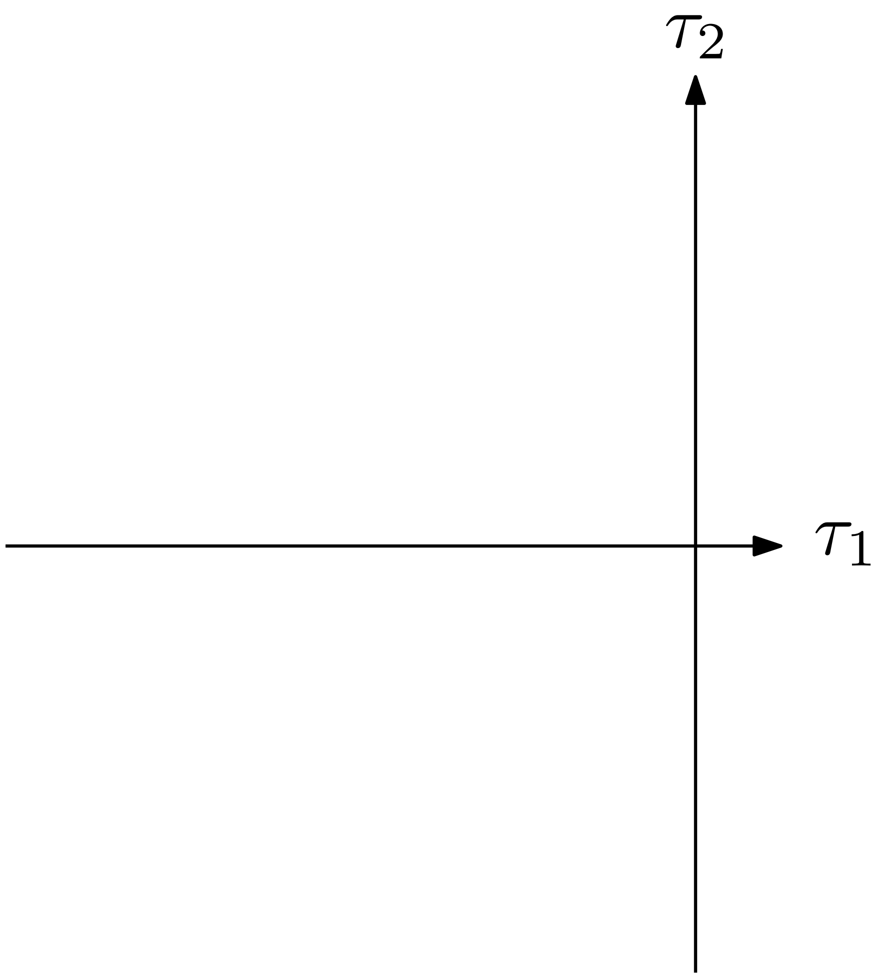

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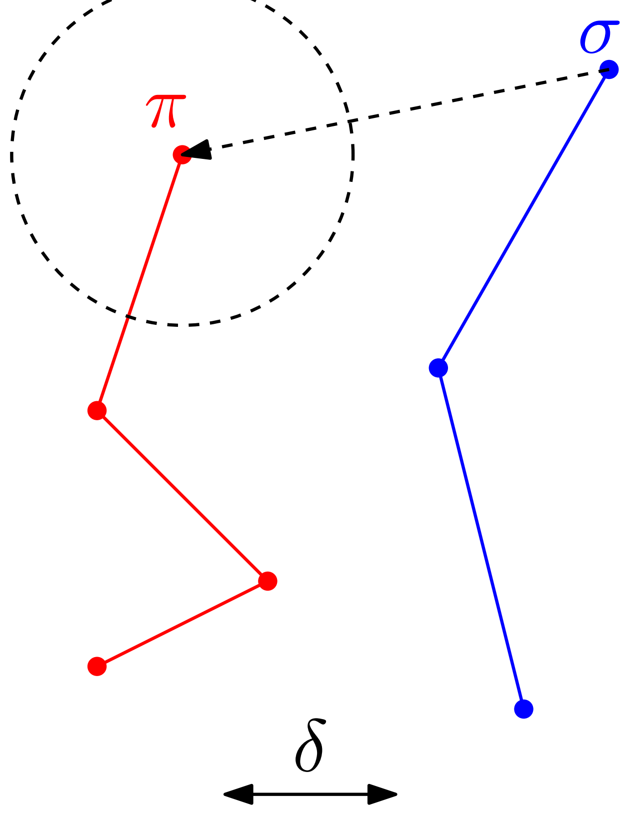

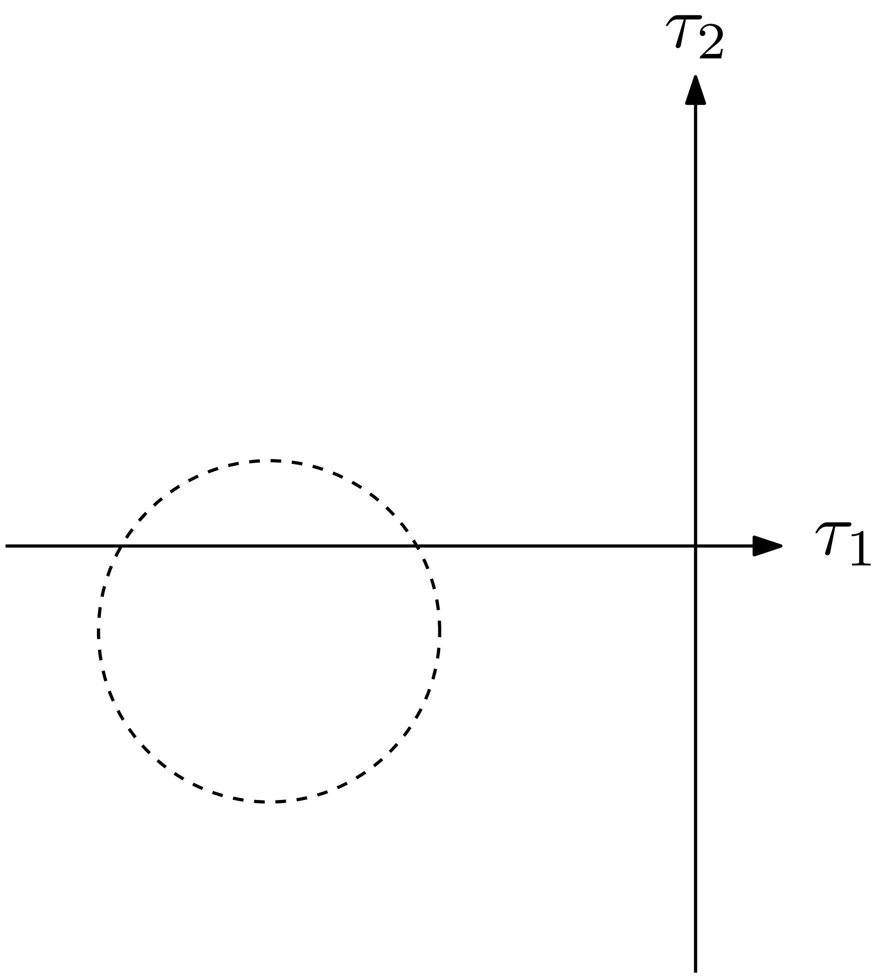

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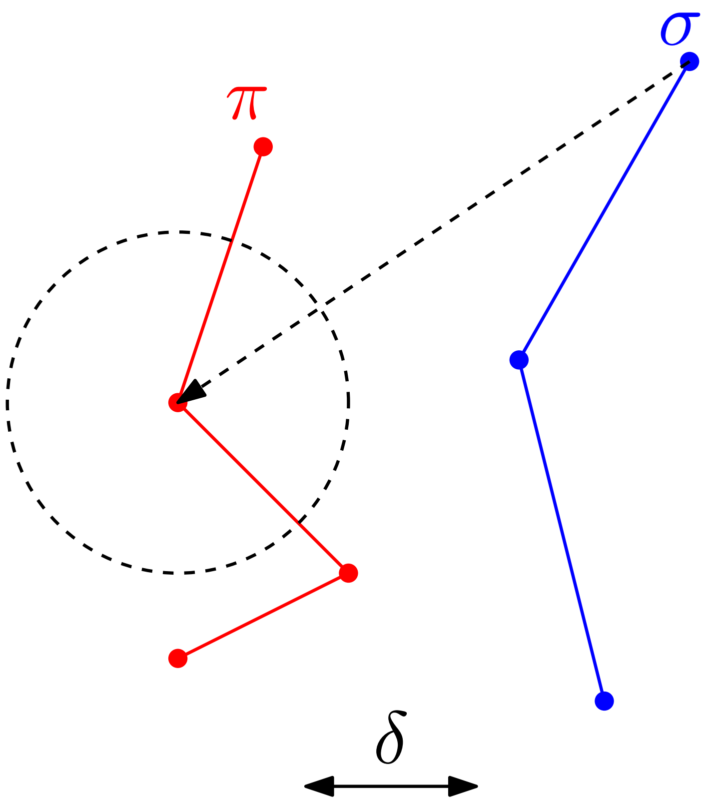

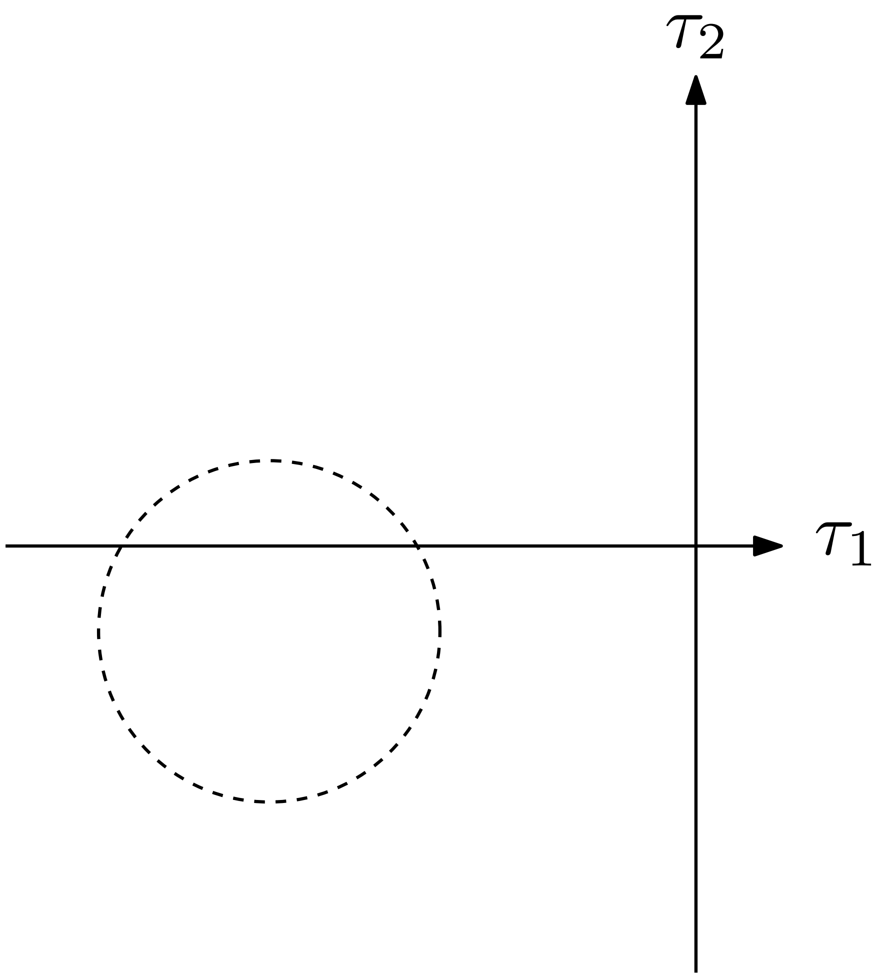

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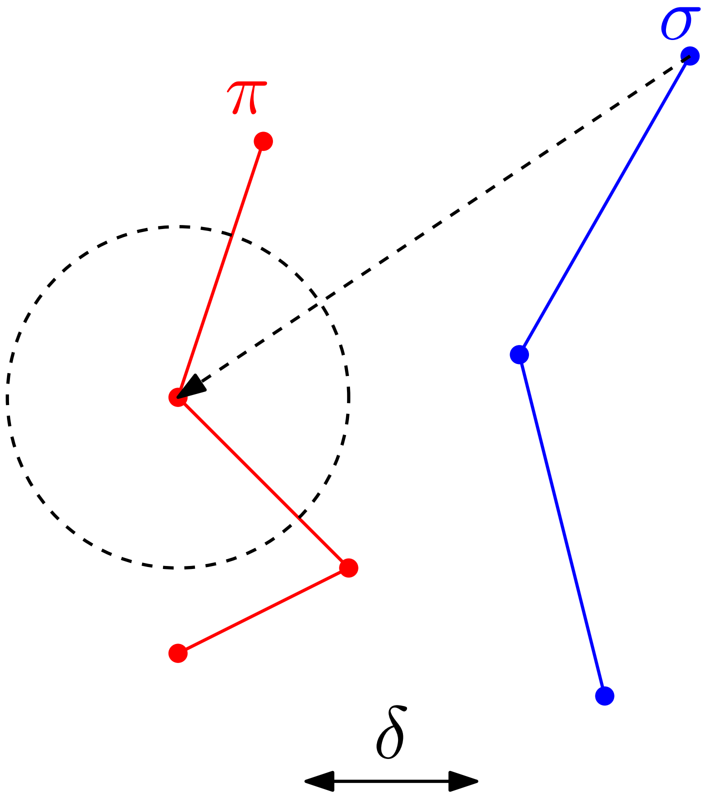

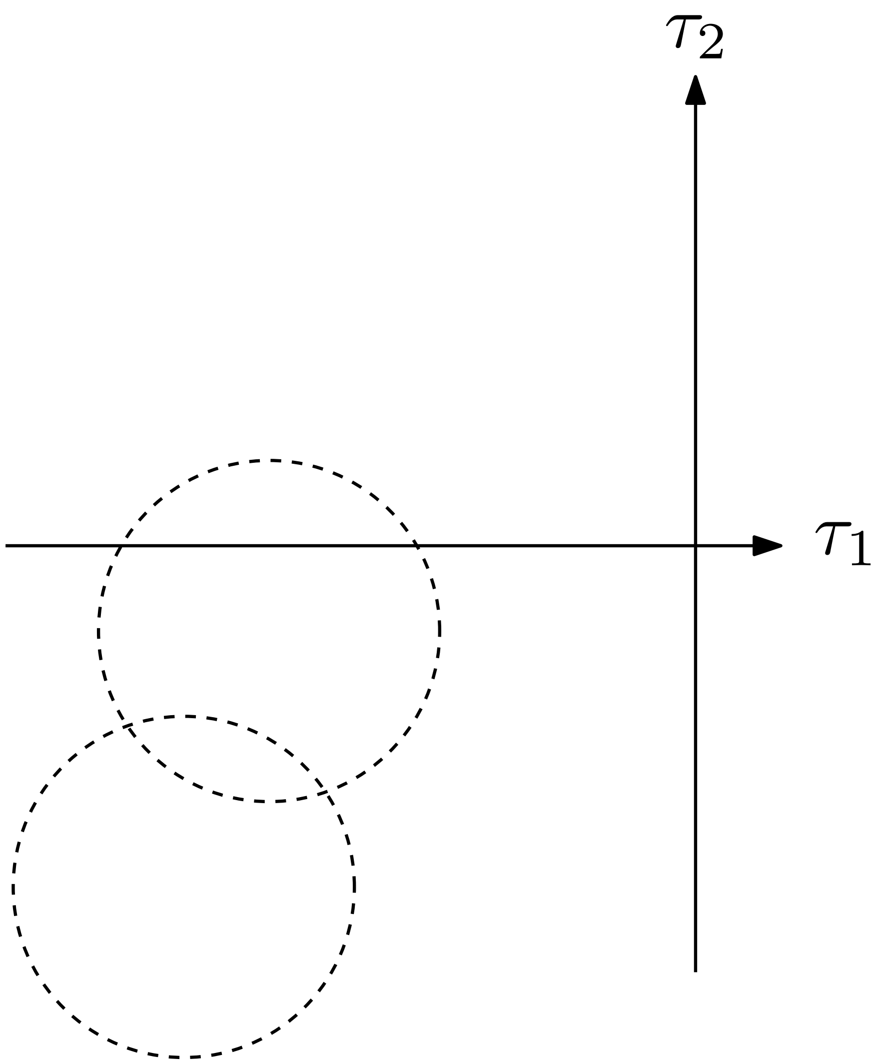

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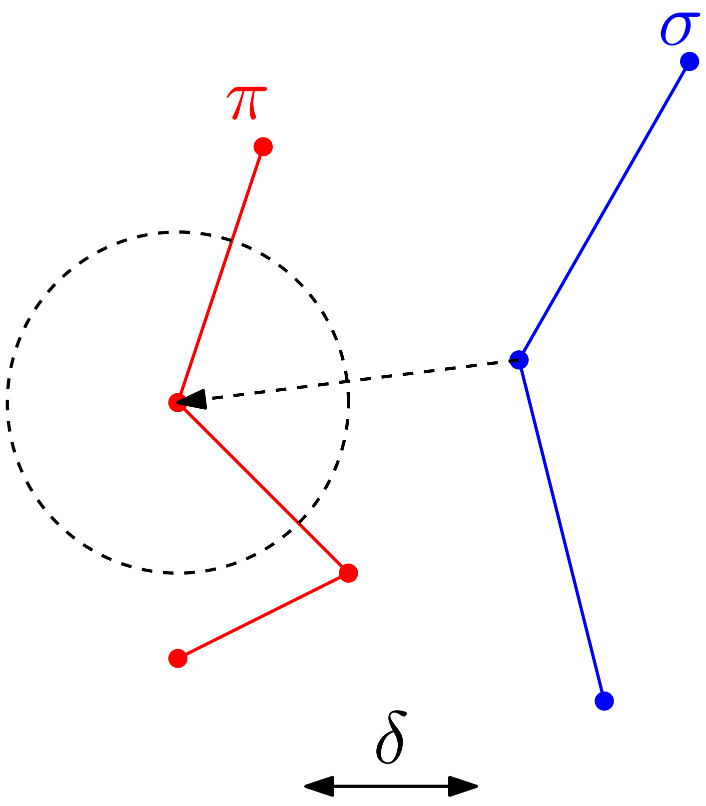

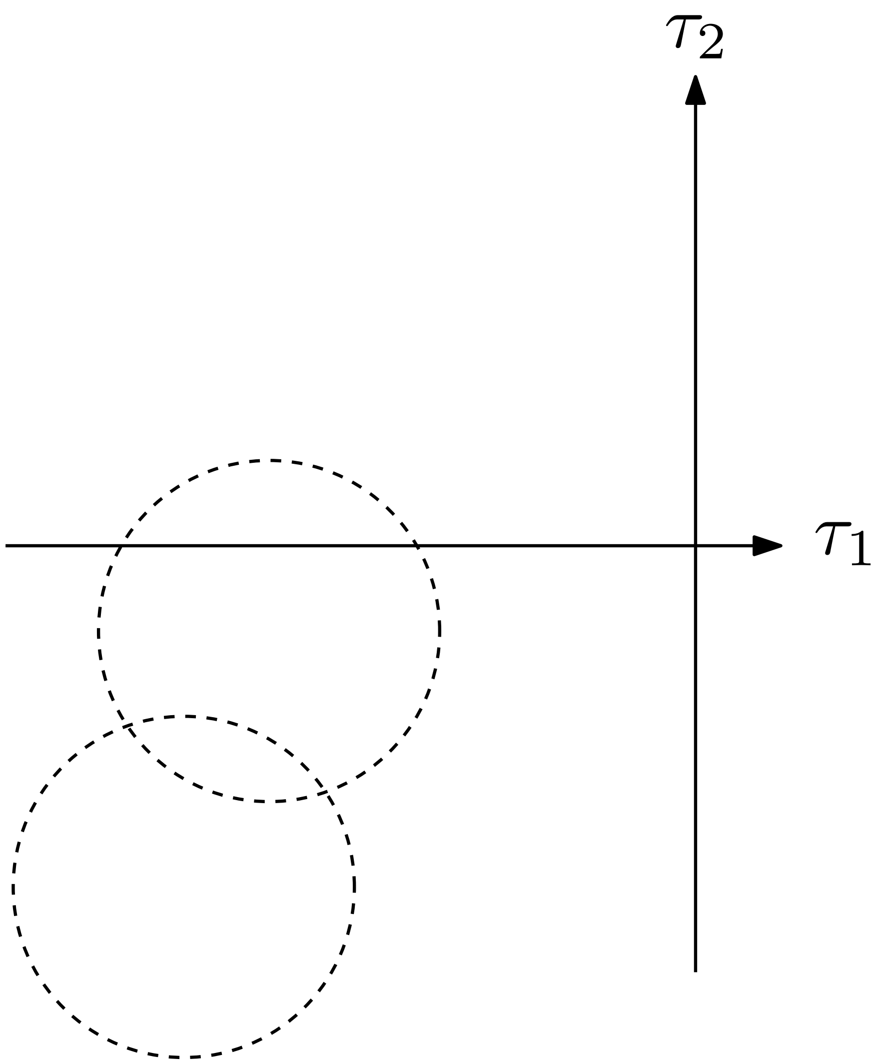

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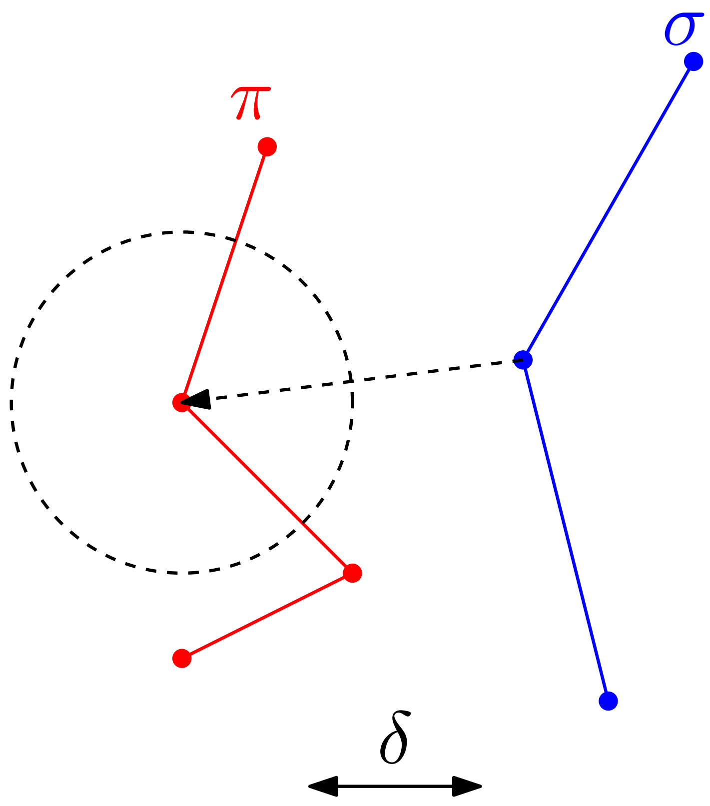

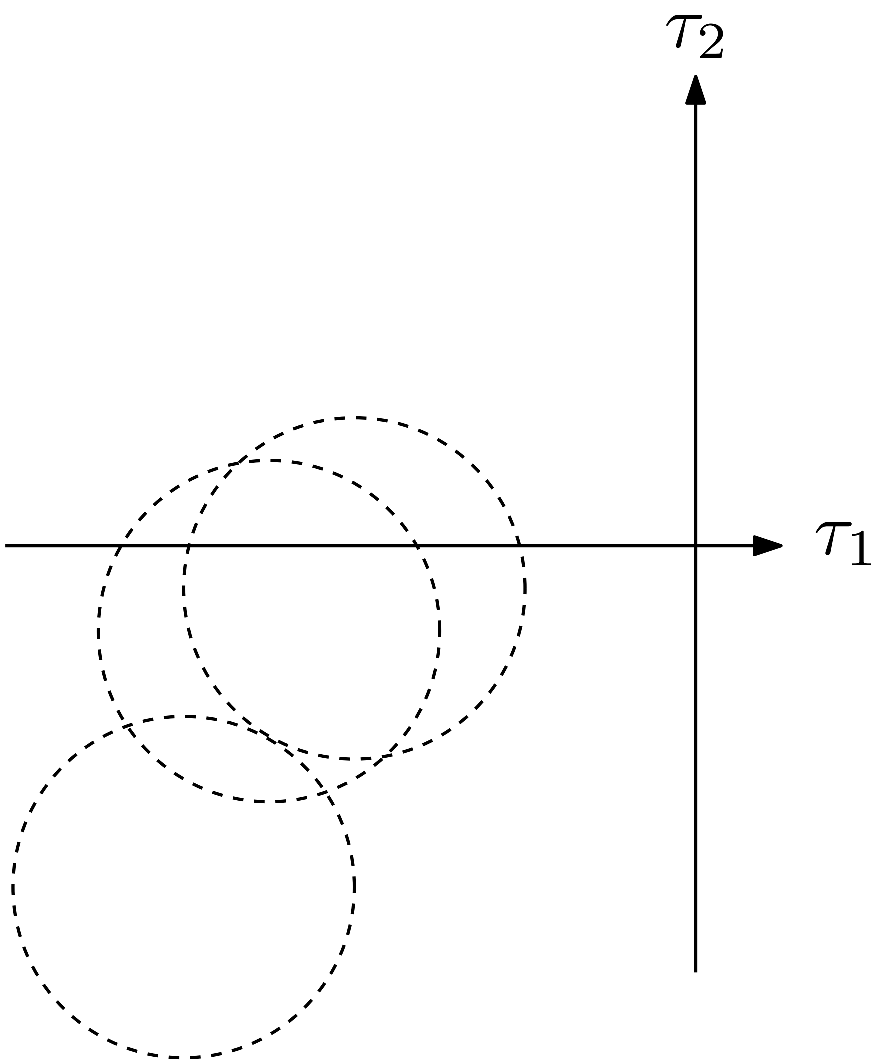

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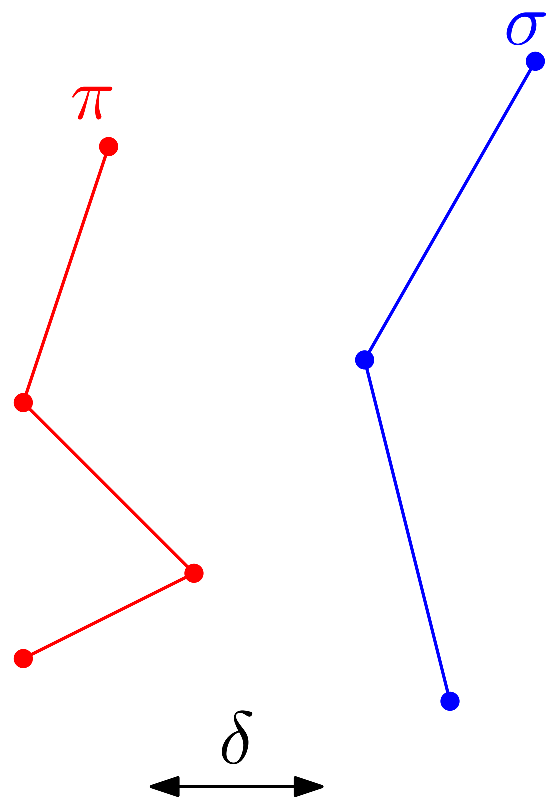

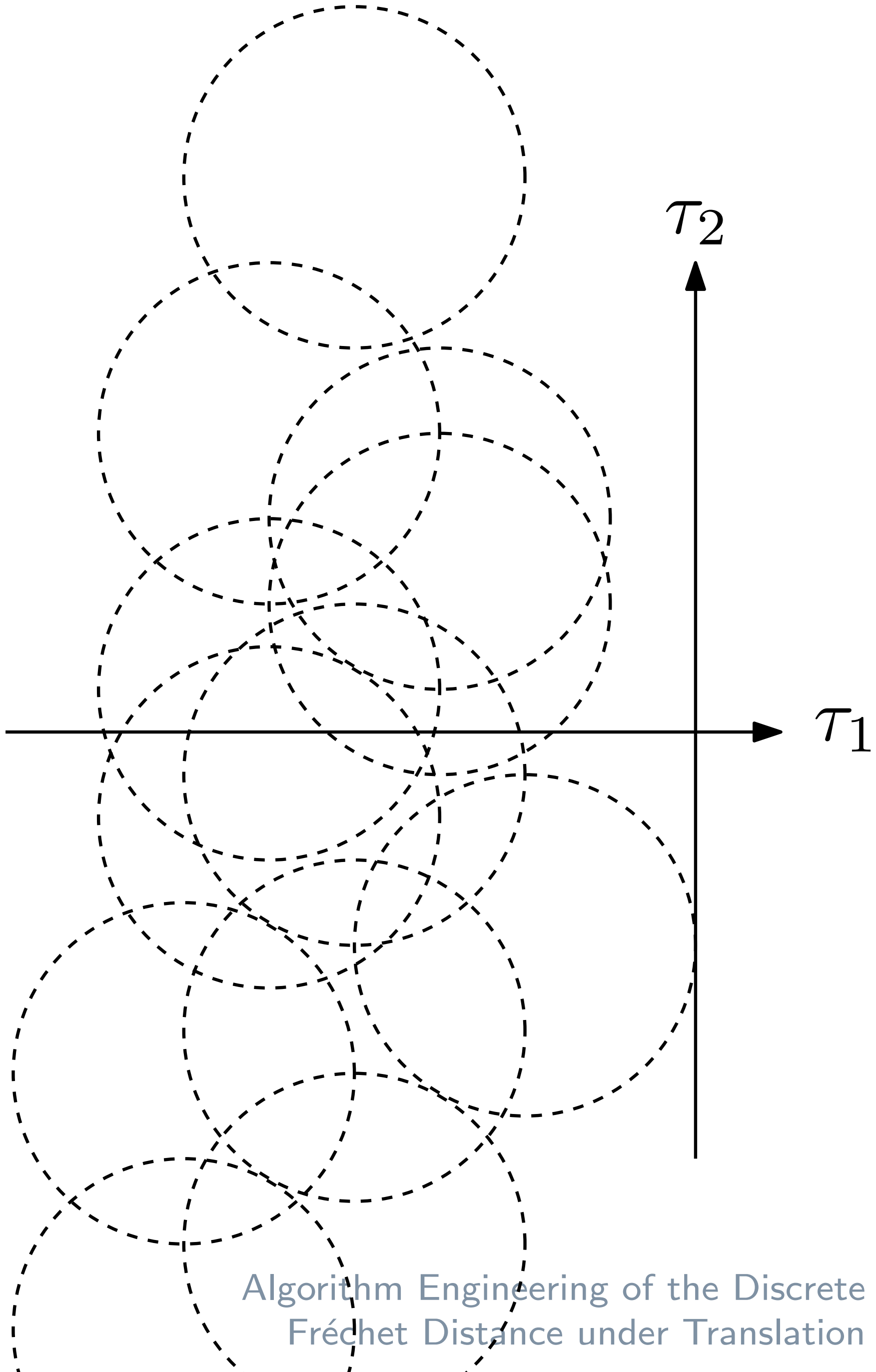

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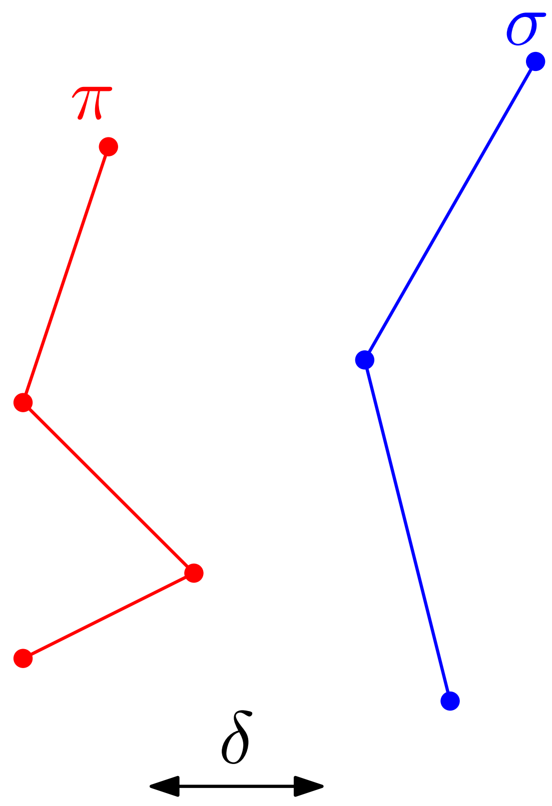
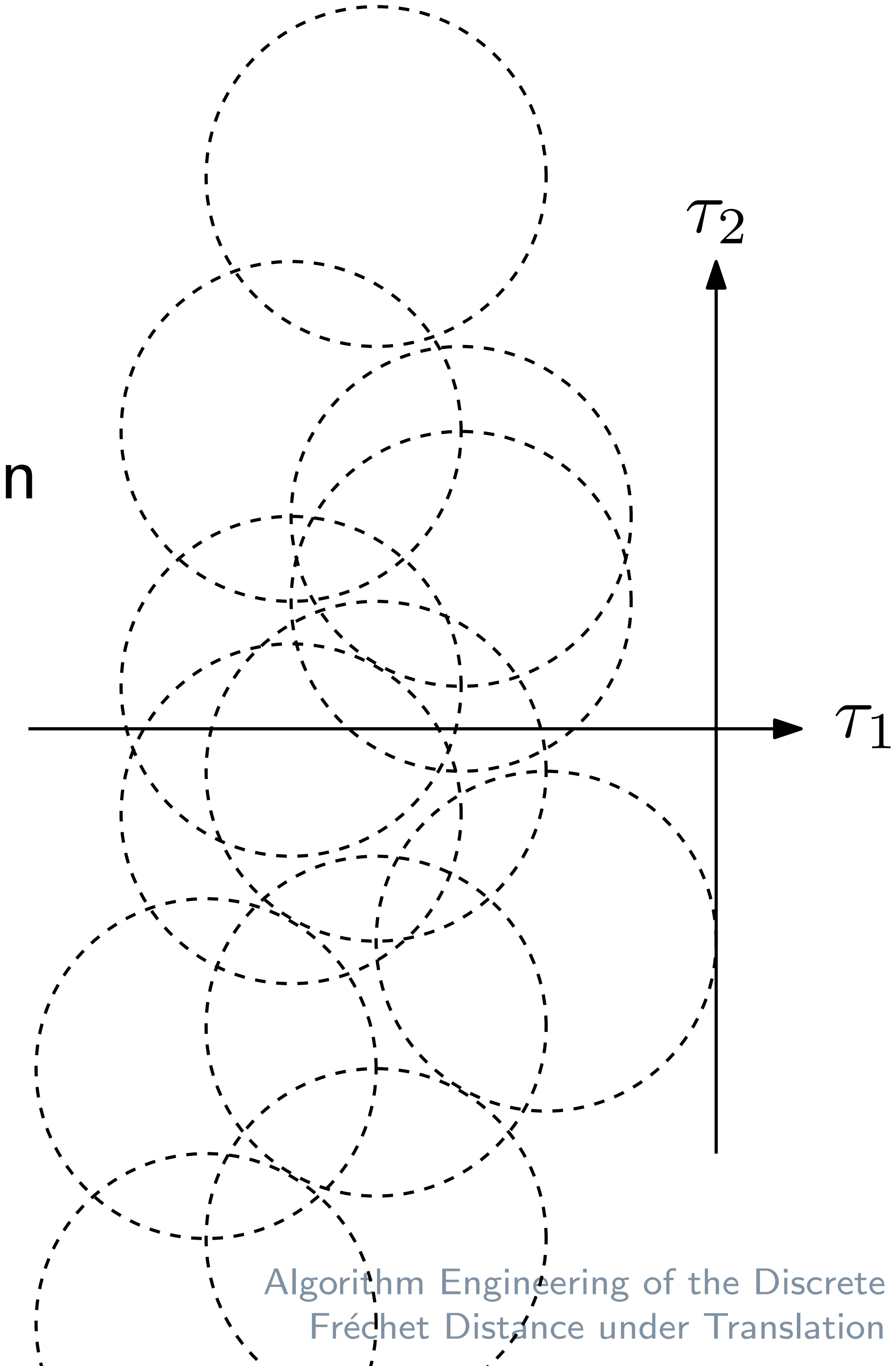


Observation: All translations in a cell of the arrangement have the same closeness relation.



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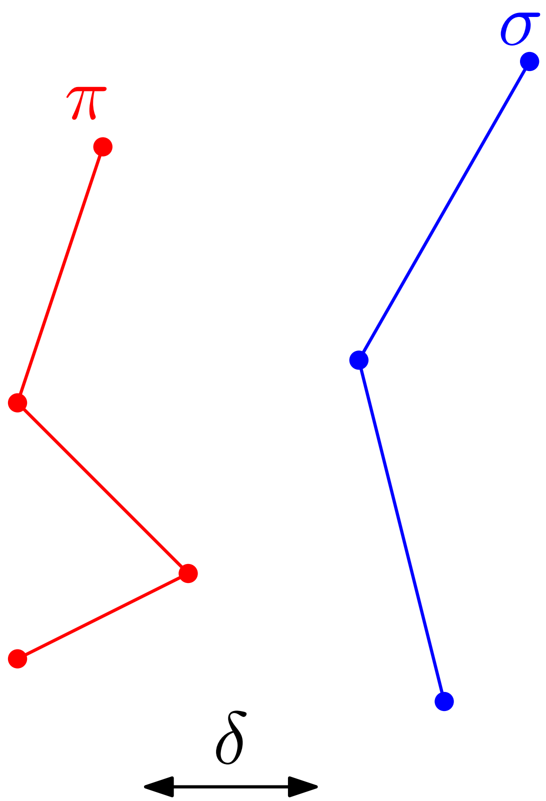

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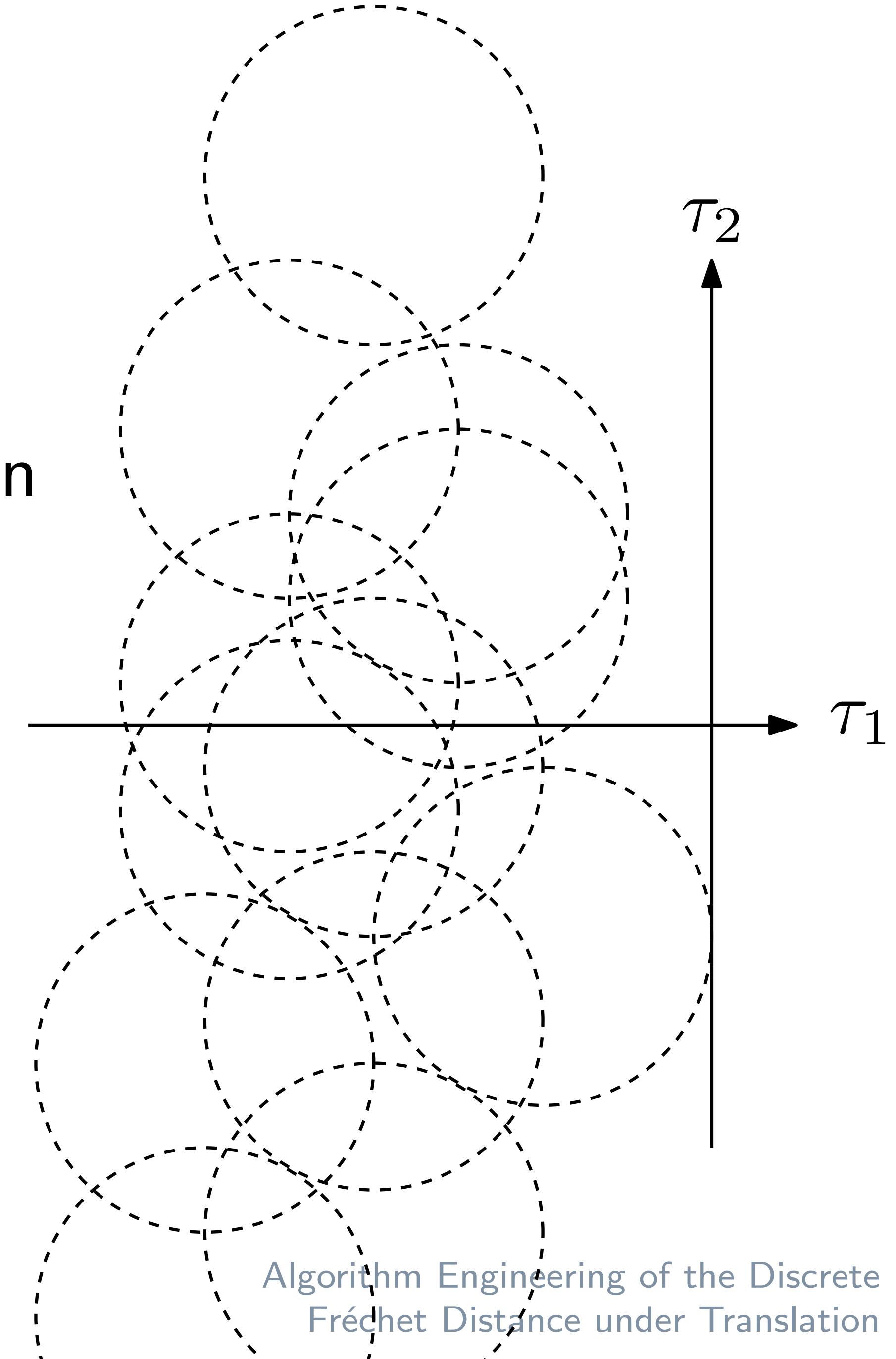
→ for each cell, pick some τ and check $d_F(\pi, \sigma + \tau)$



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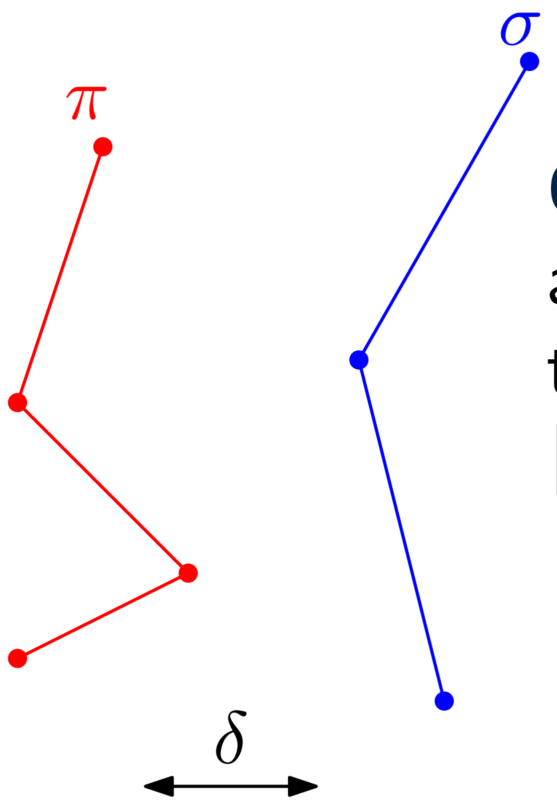

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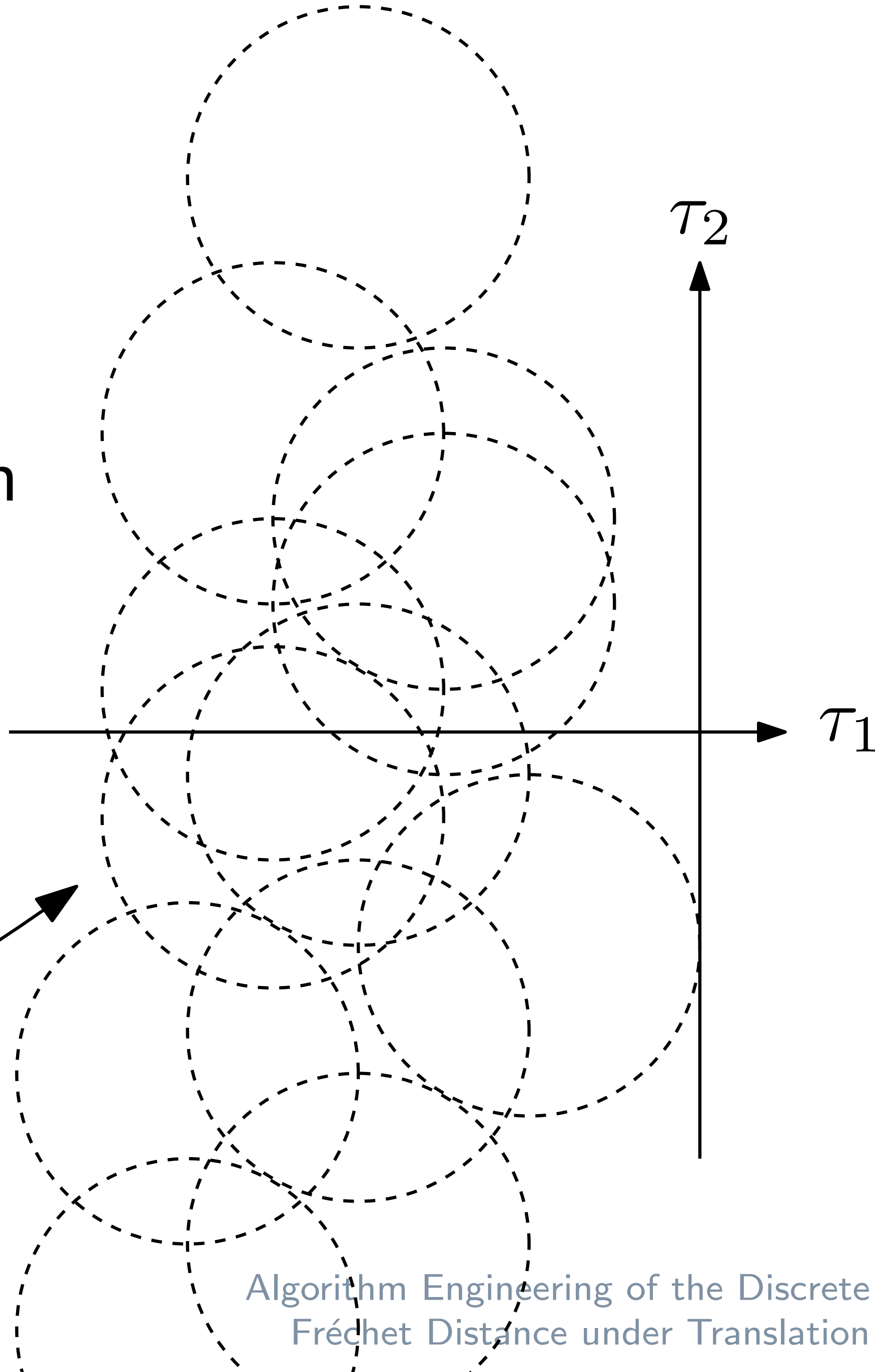
$\mathcal{O}(n^4)$ complexity



Approach I: Discrete Algorithms

- All known algorithms build an $\mathcal{O}(n^4)$ arrangement.

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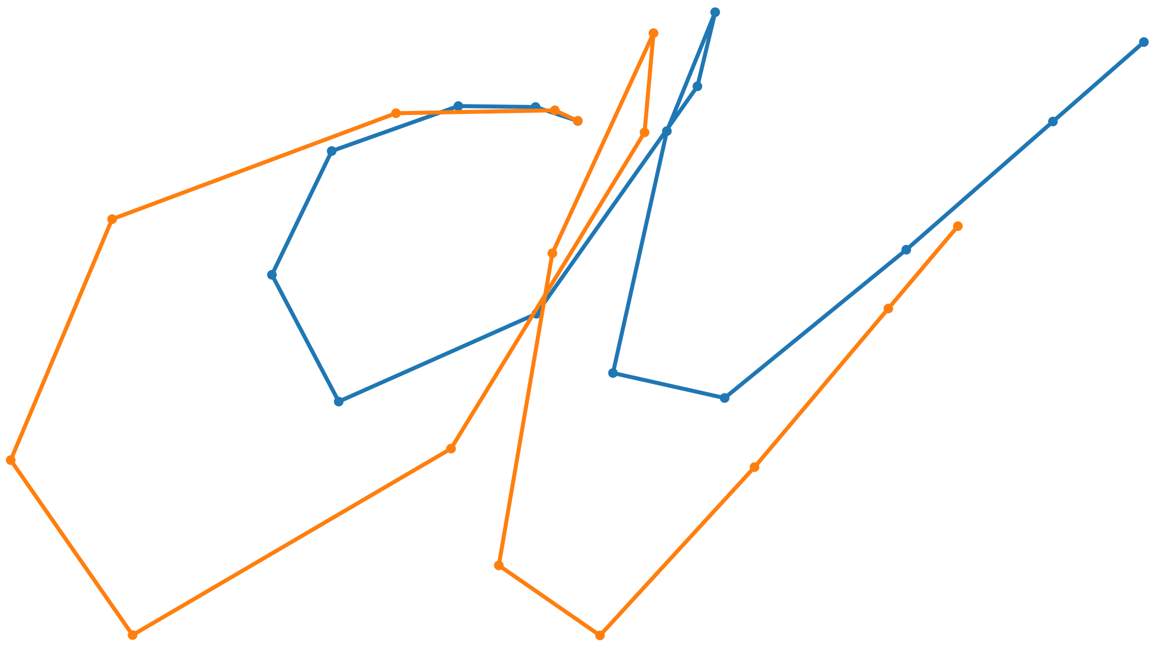

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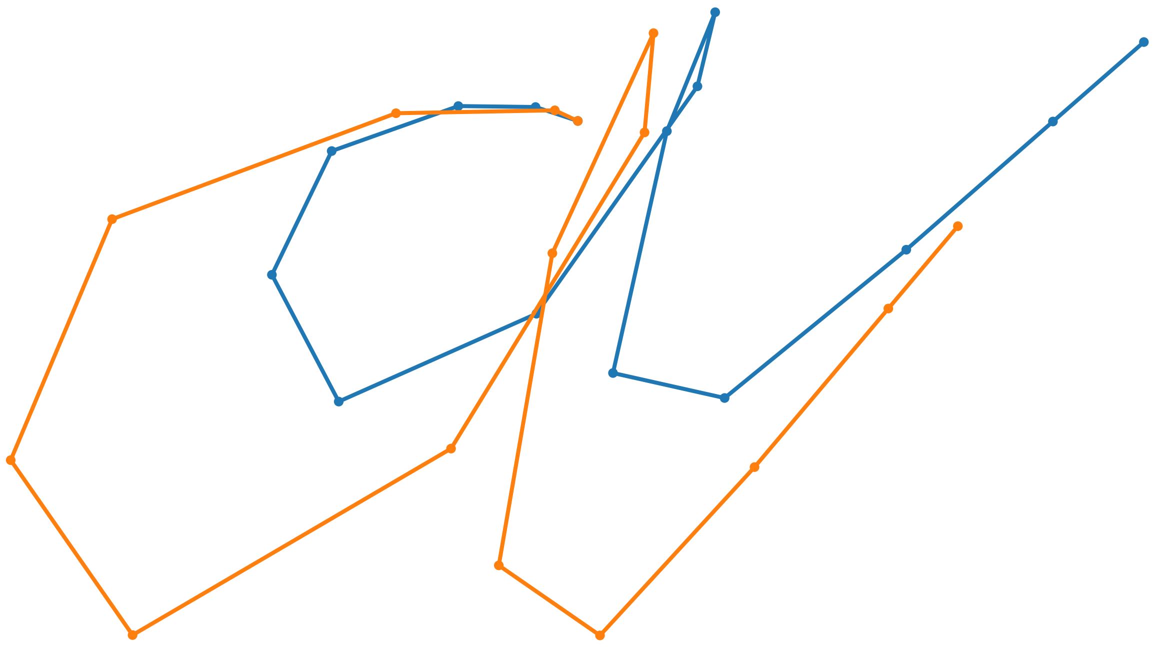

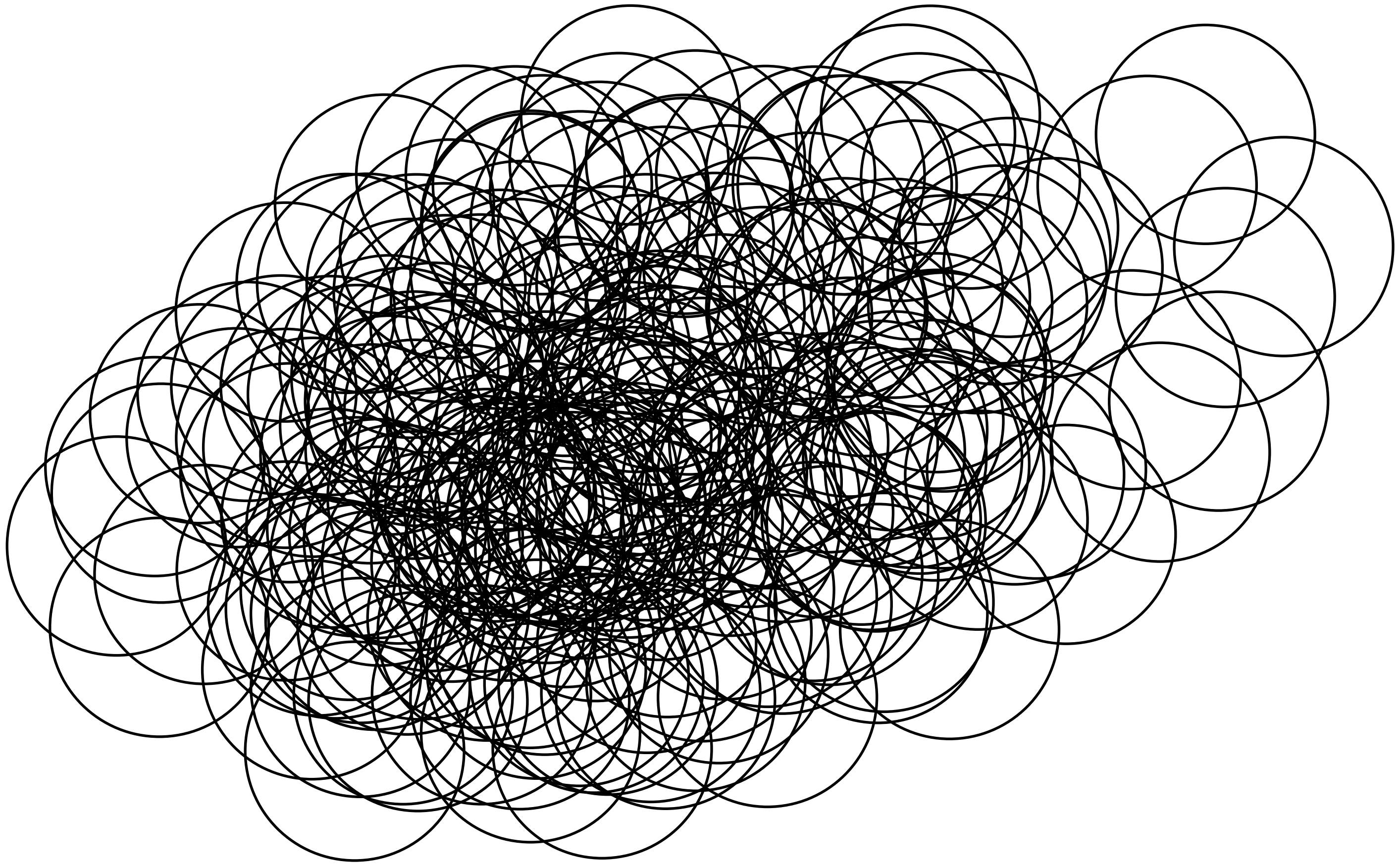

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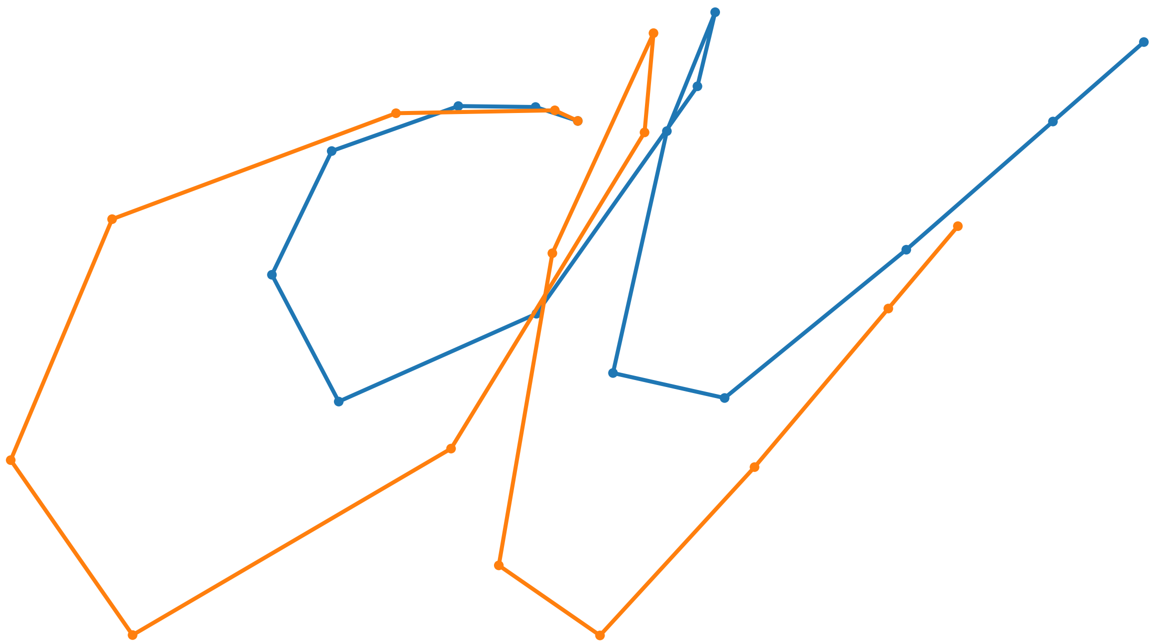
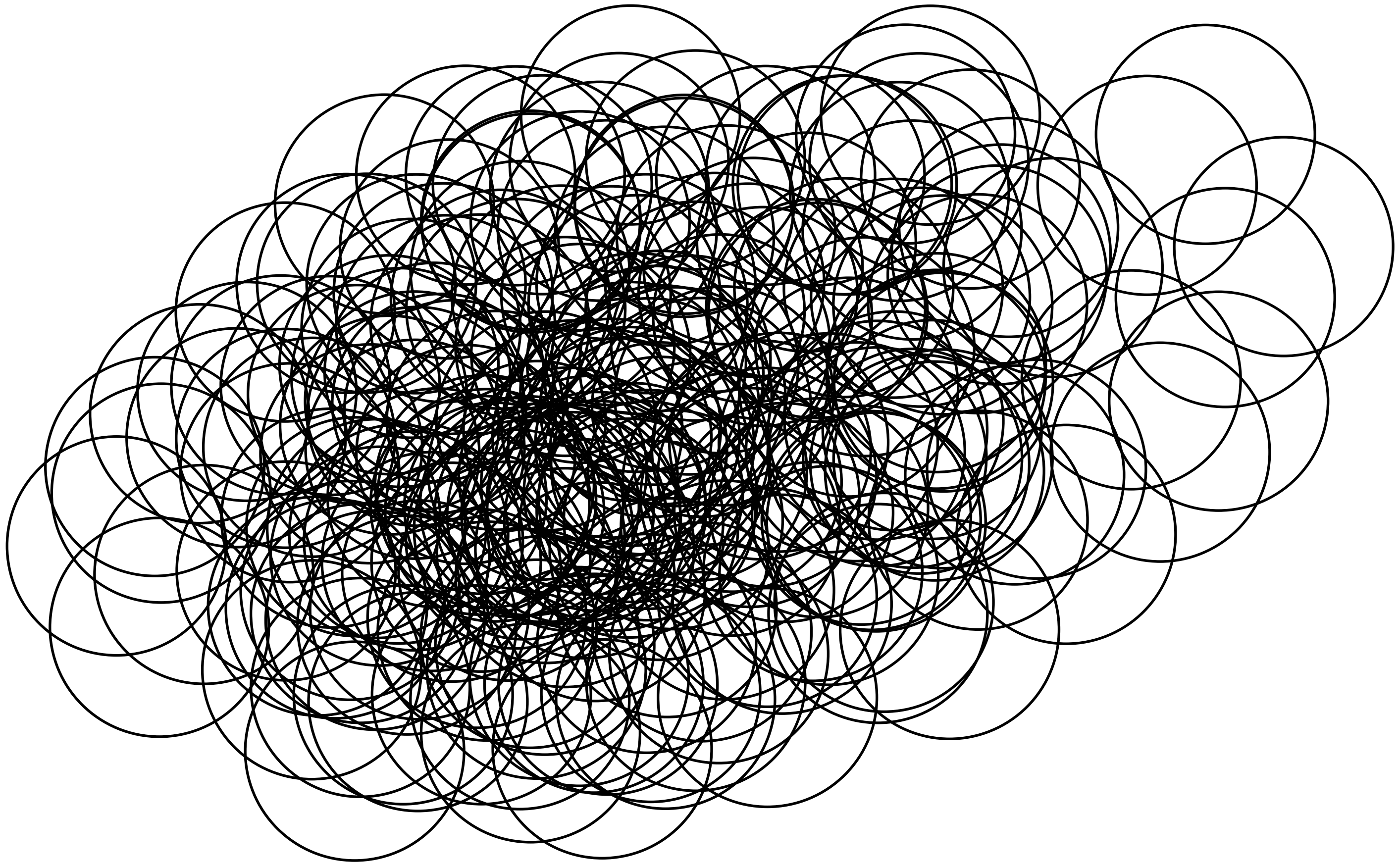
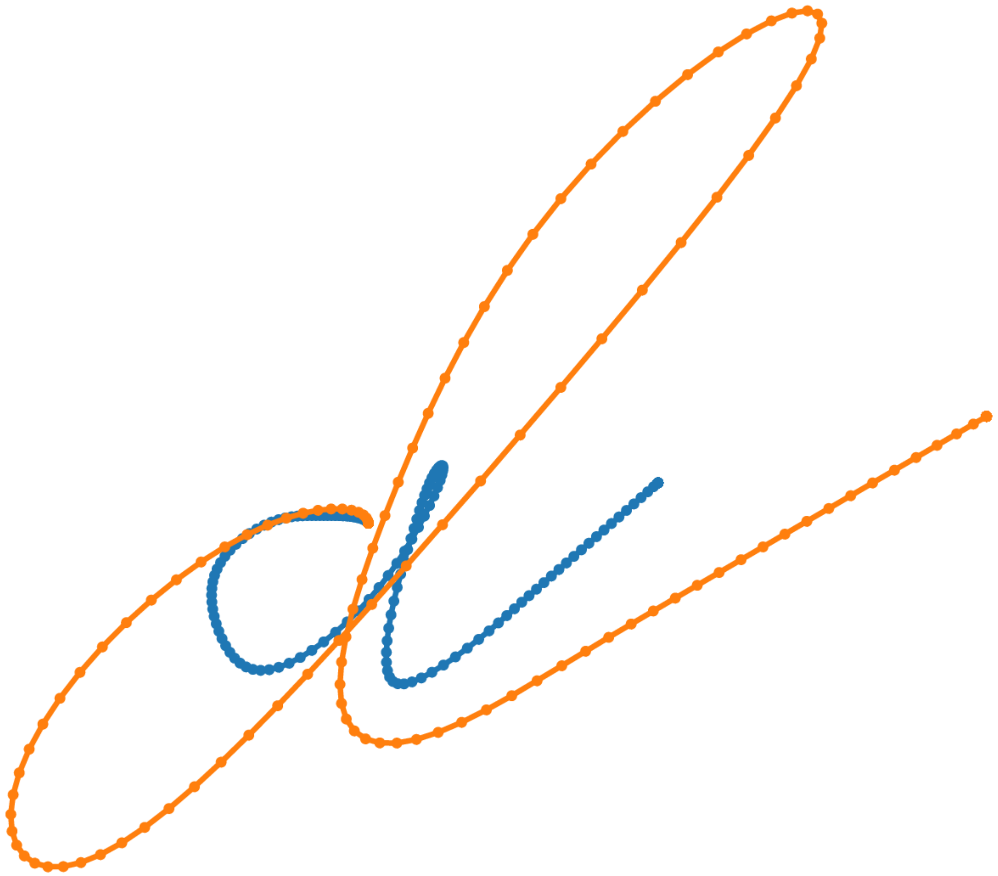




Ewwww.....

Approach II: Continuous Optimization



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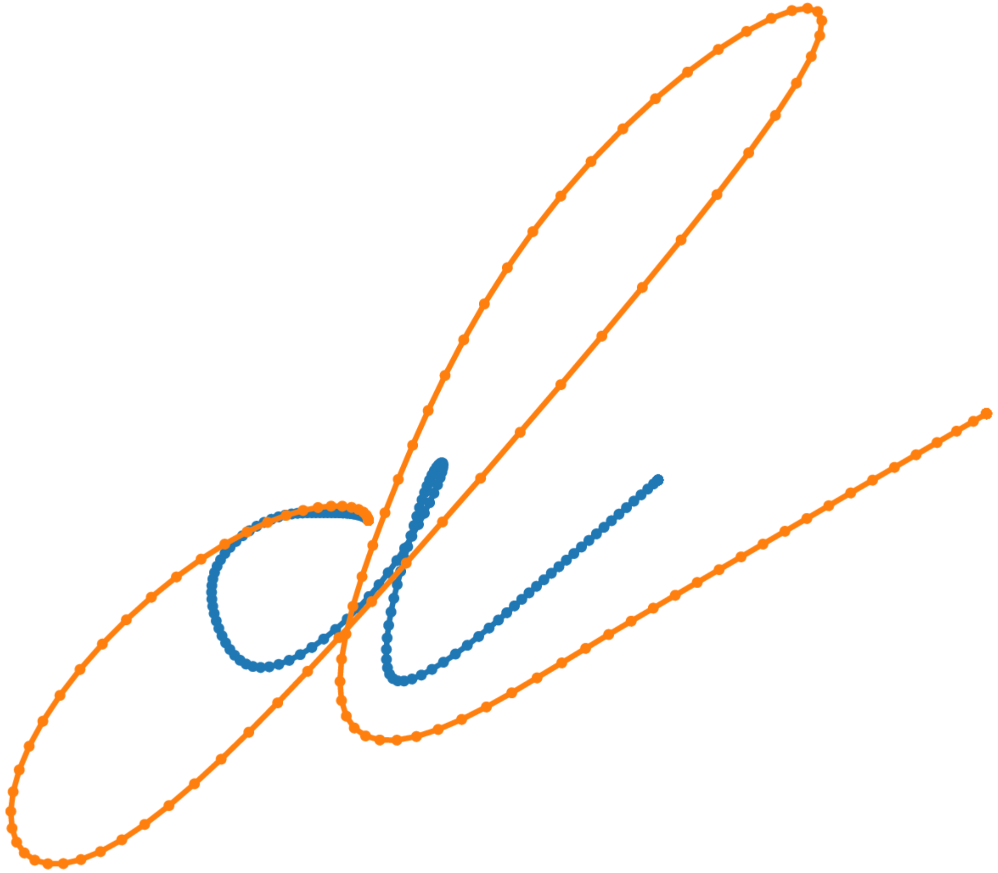
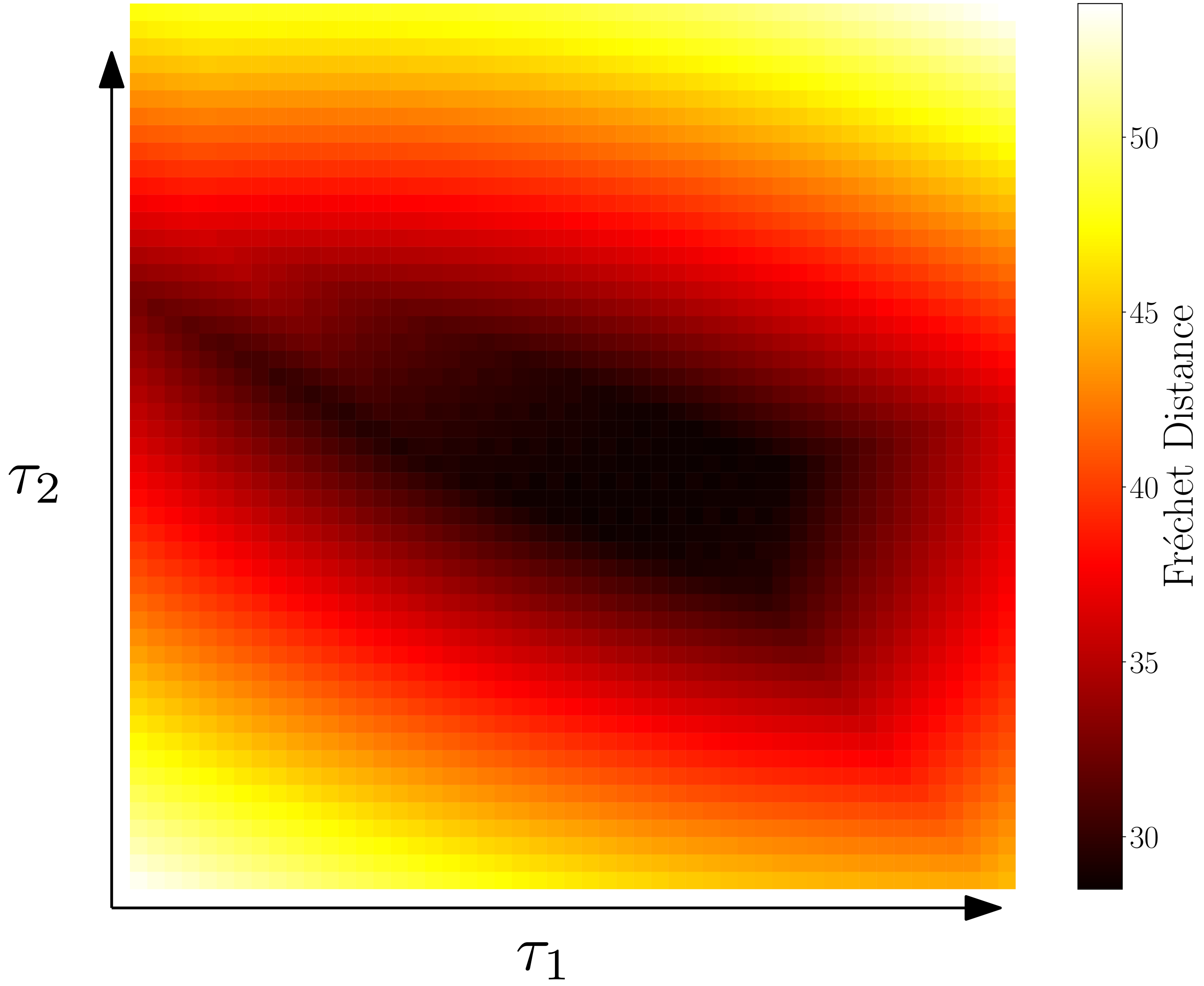




Approach II: Continuous Optimization

Observation: Fréchet under Translation is 1-Lipschitz, i.e.,

$$|d_F(\pi, \sigma + \tau) - d_F(\pi, \sigma + \tau')| \leq \|\tau - \tau'\|$$

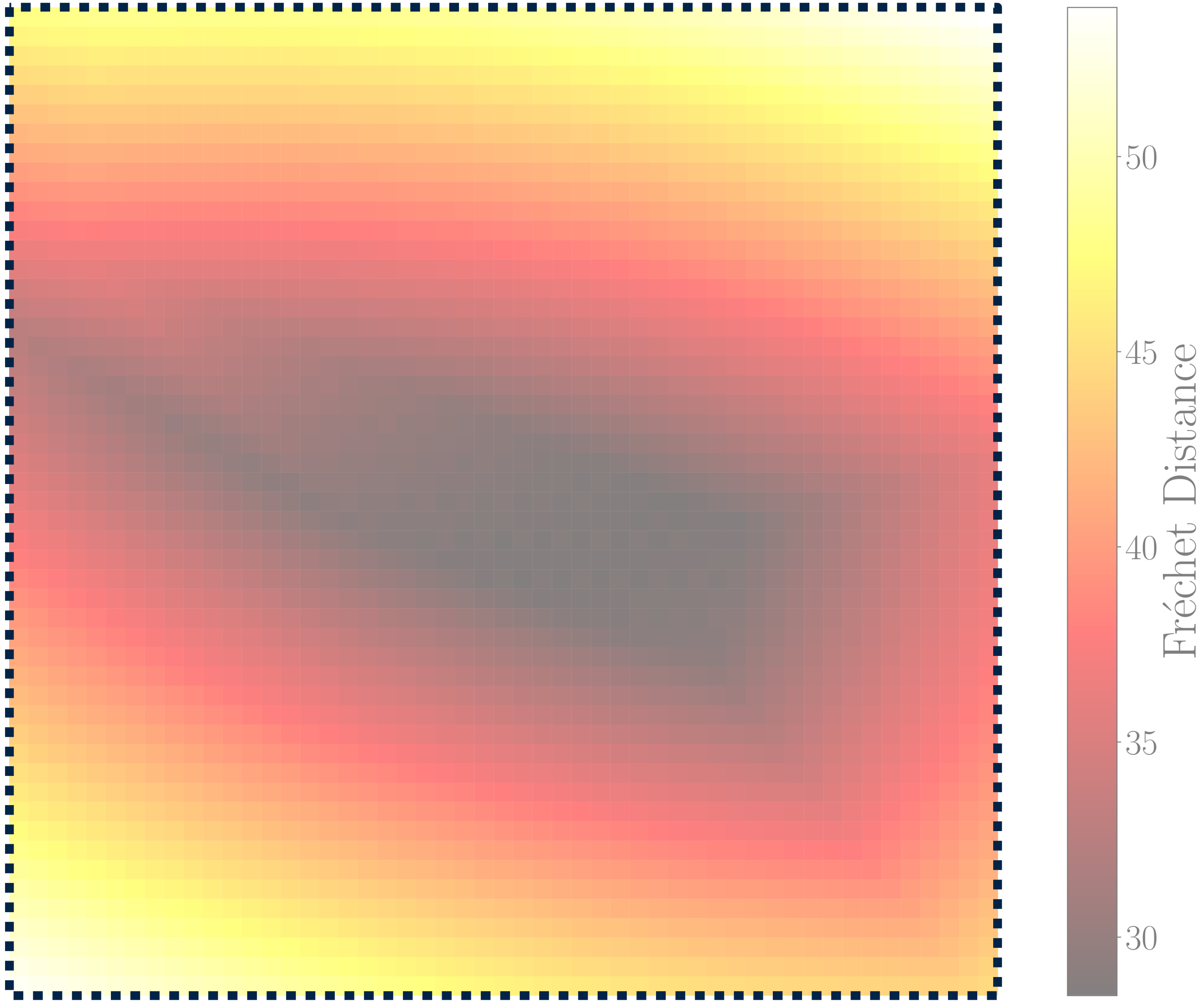
Approach II: Continuous Optimization

Lipschitz Optimization

Approach:

- branch & bound

For each box:



Approach II: Continuous Optimization

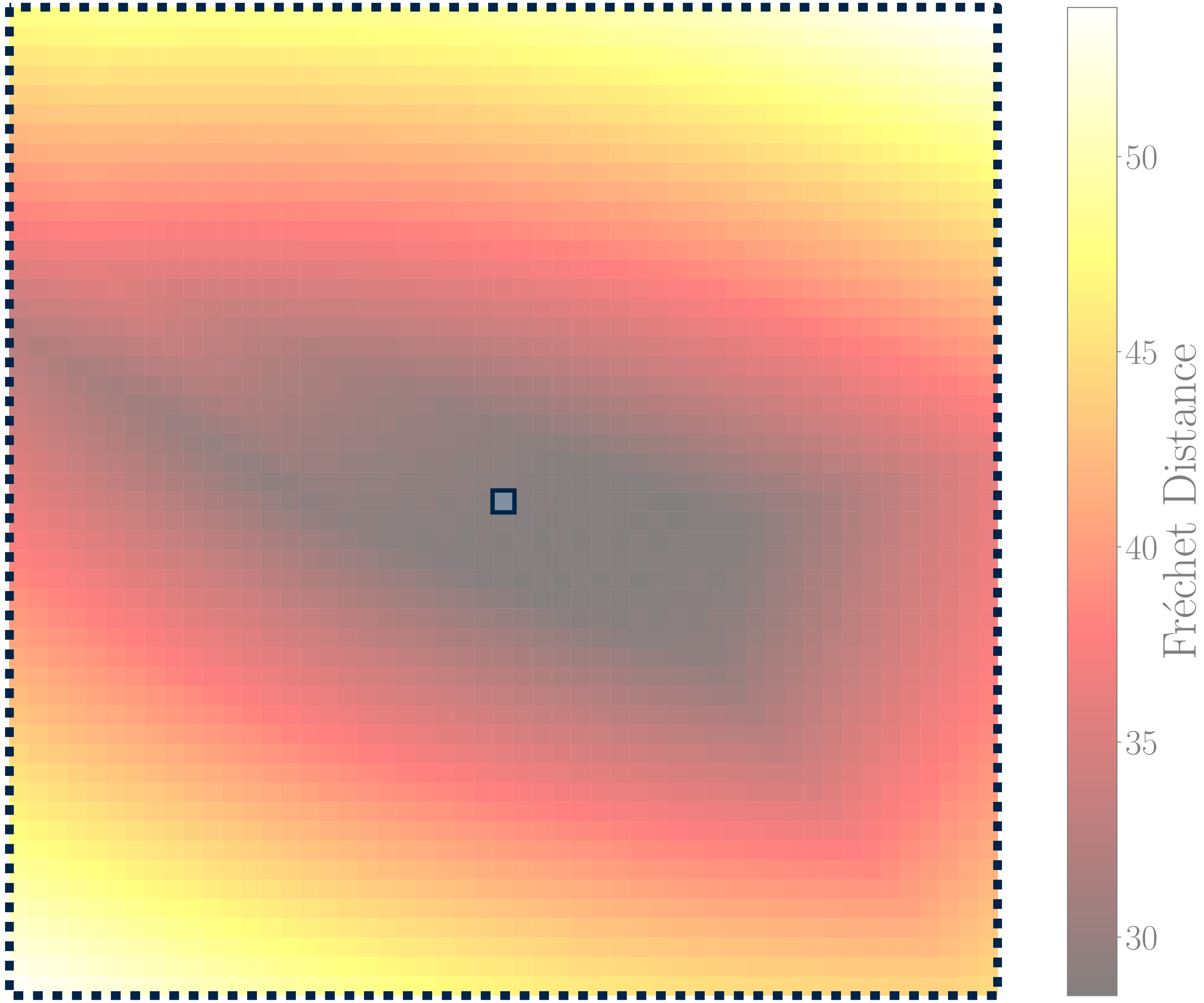
Lipschitz Optimization

Approach:

- branch & bound

For each box:

- if $d_F(\pi, \sigma + \blacksquare) \leq \delta$
  - return LESS



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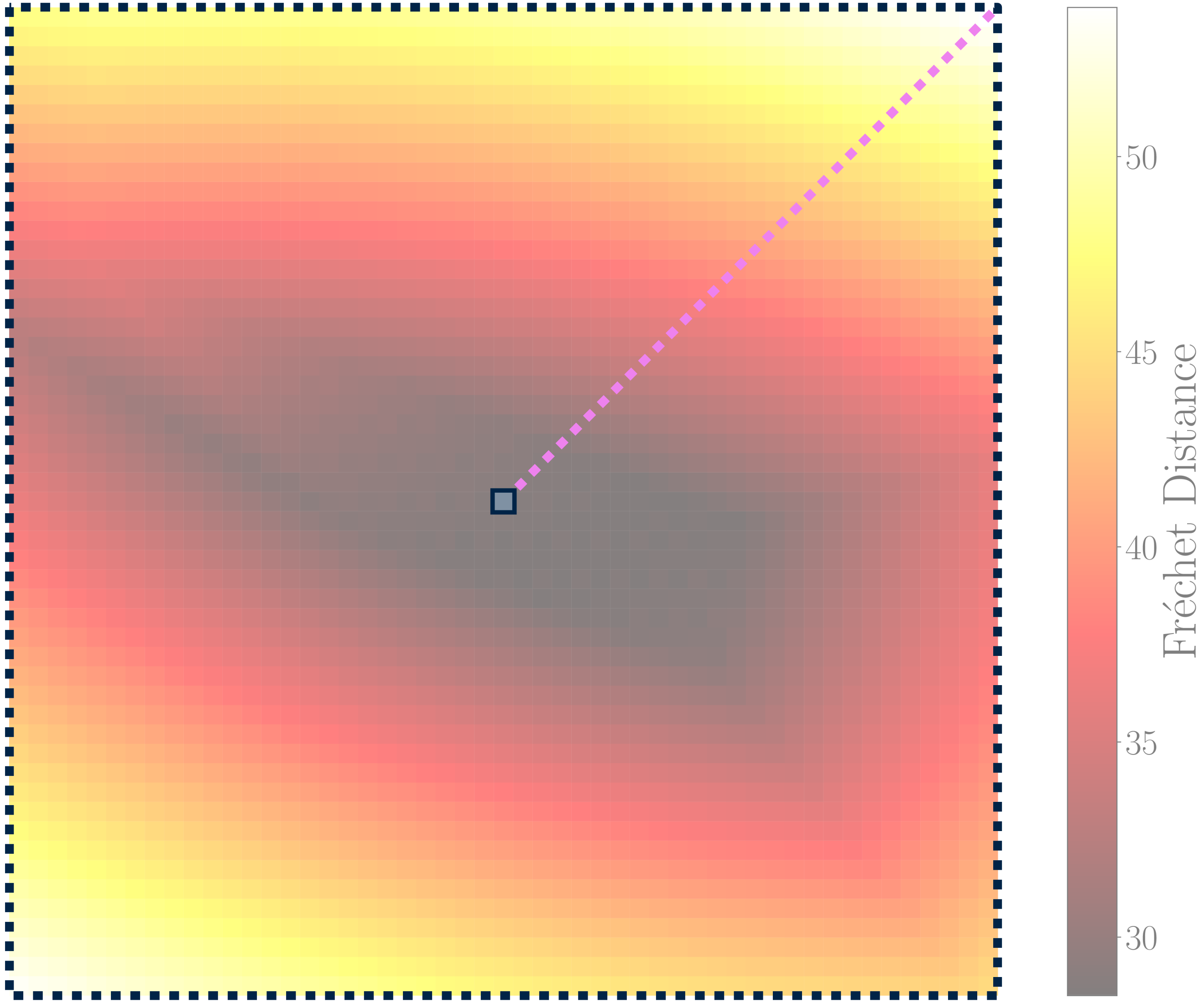
- if $d_F(\pi, \sigma + \blacksquare) > \delta +$
  - skip box



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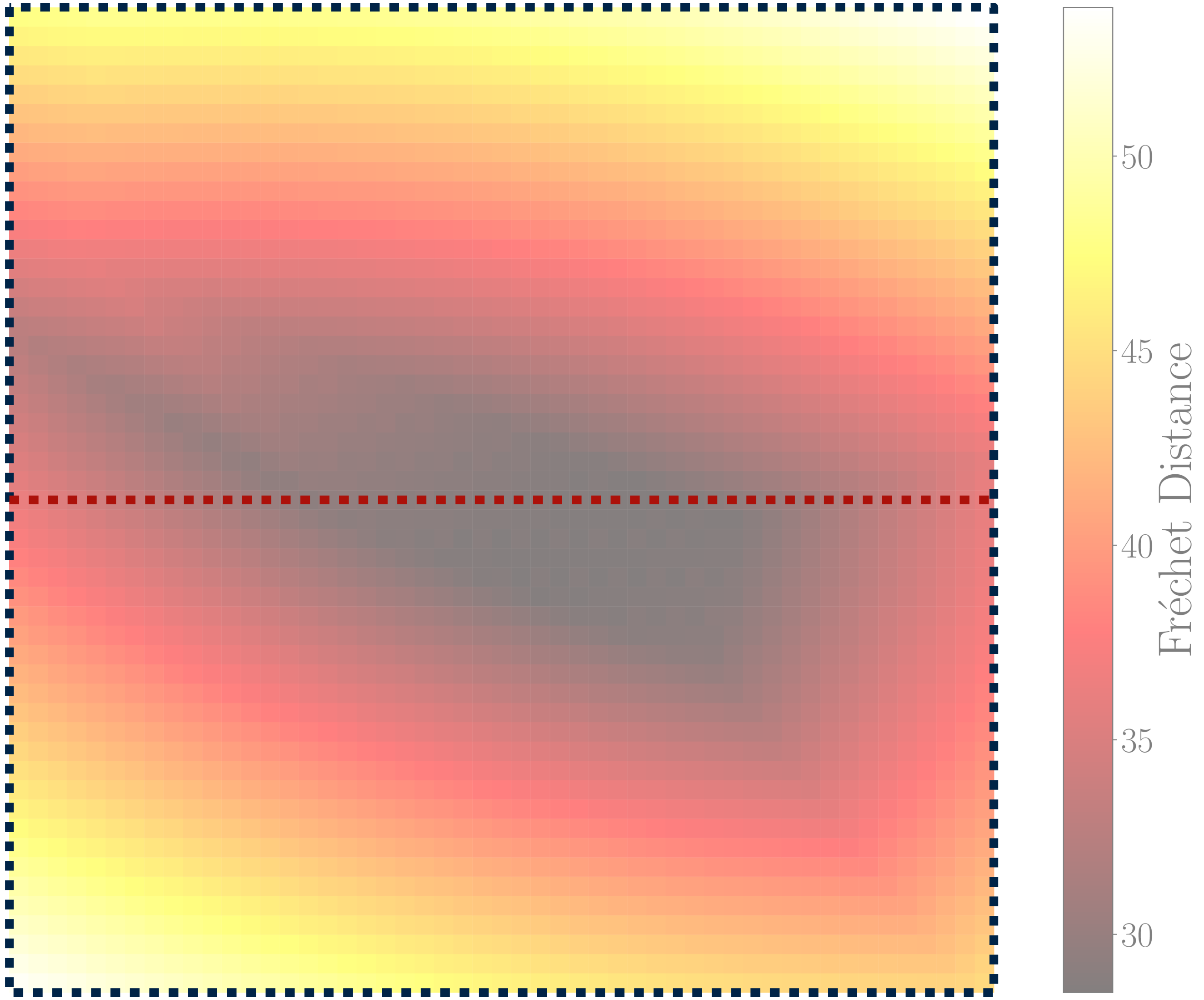
- if $d_F(\pi, \sigma + \blacksquare) > \delta +$
  - skip box
- if both fail: split



Approach II: Continuous Optimization

Issues

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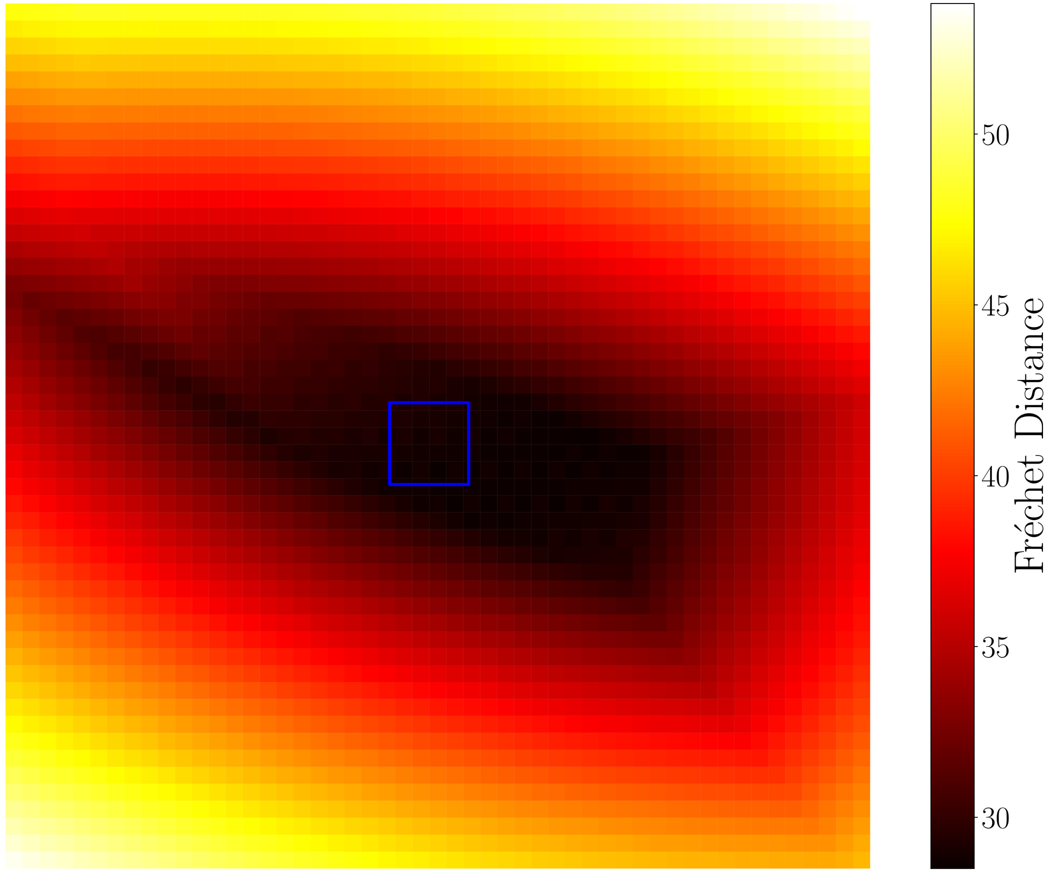

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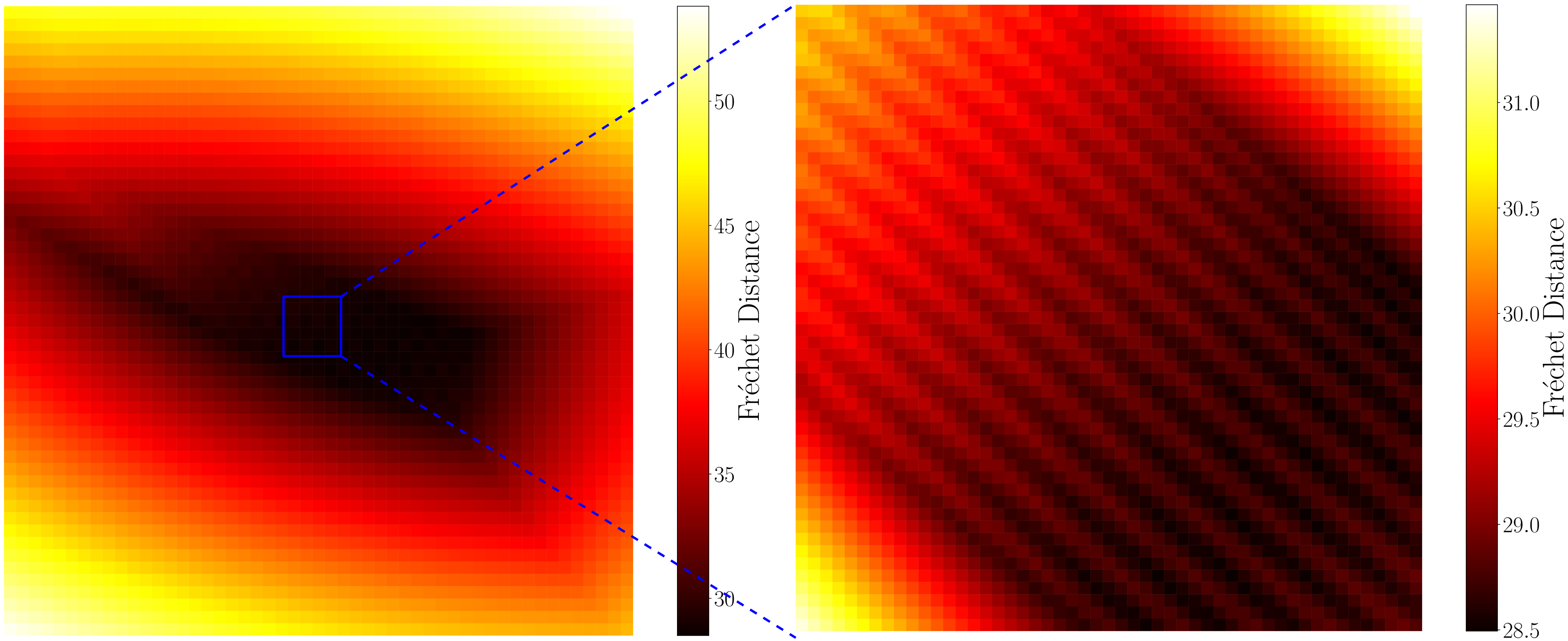

Combine Both Approaches!

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1) Use Lipschitz optimization to identify important regions

Core Idea

Combine Both Approaches!

1) Use Lipschitz optimization to identify important regions
2) Use arrangement algorithm inside these regions

Contribution I: Exact Decider

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Approach:

- augment branch & bound approach
- for each box:
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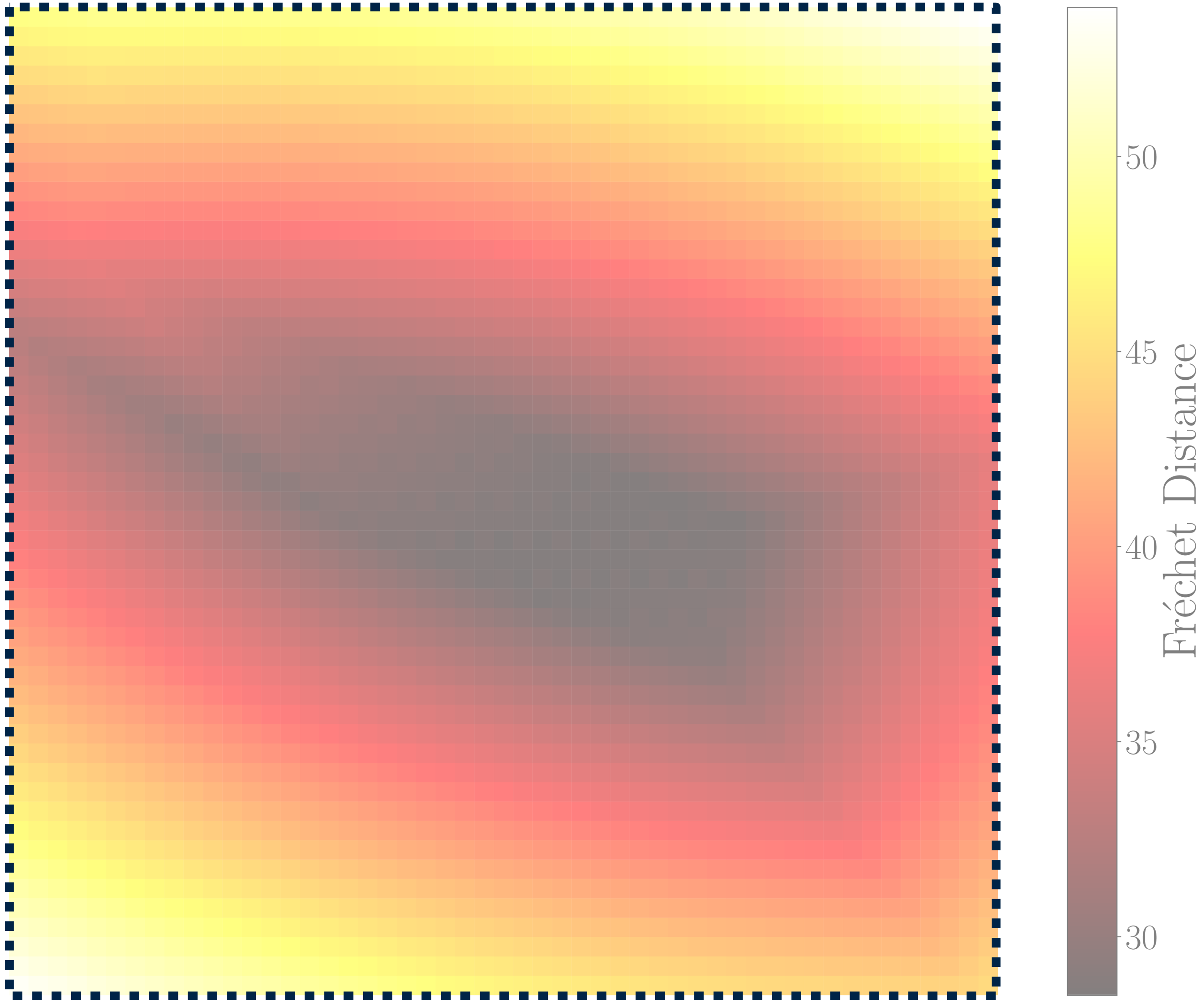
exact decision!



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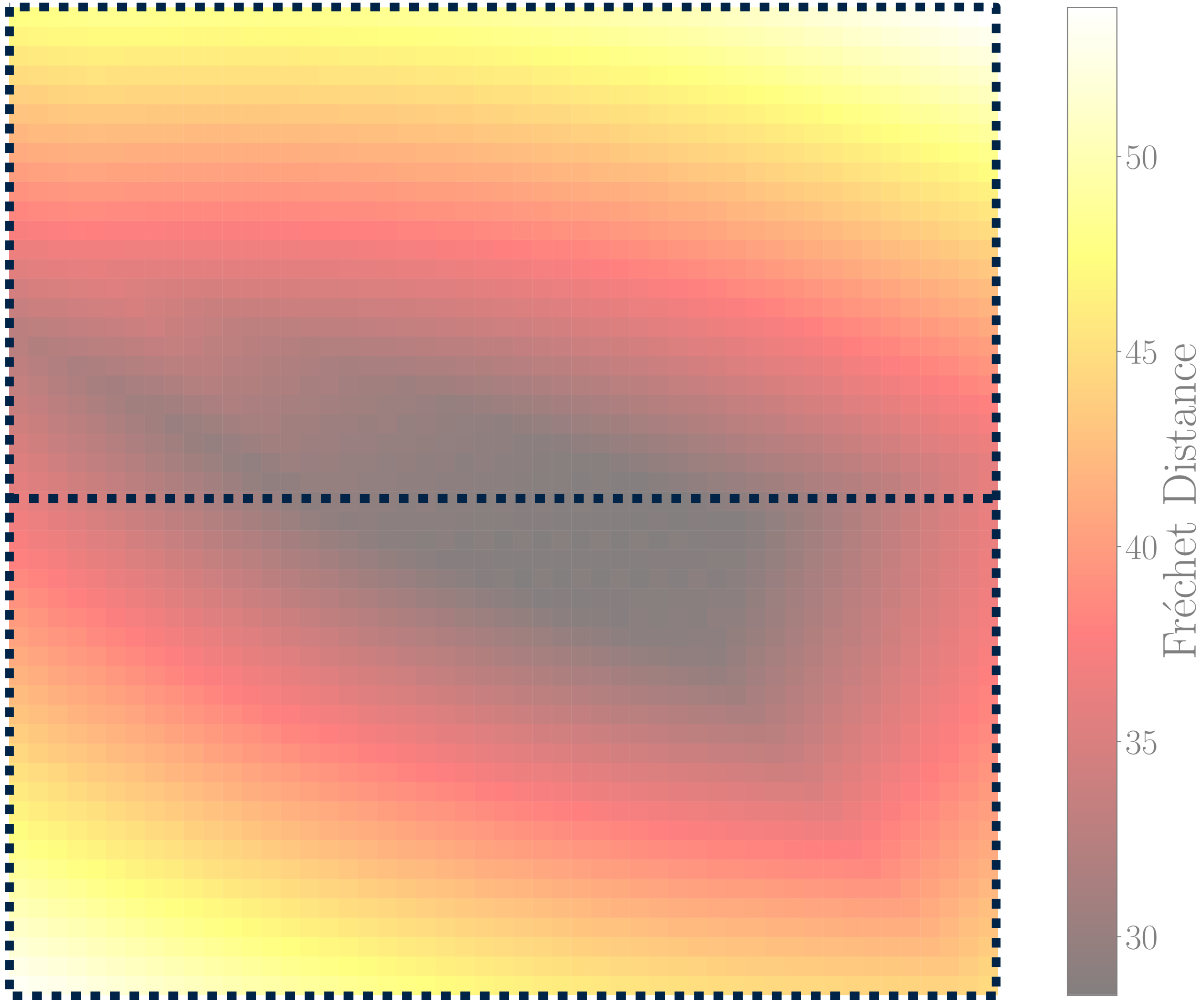

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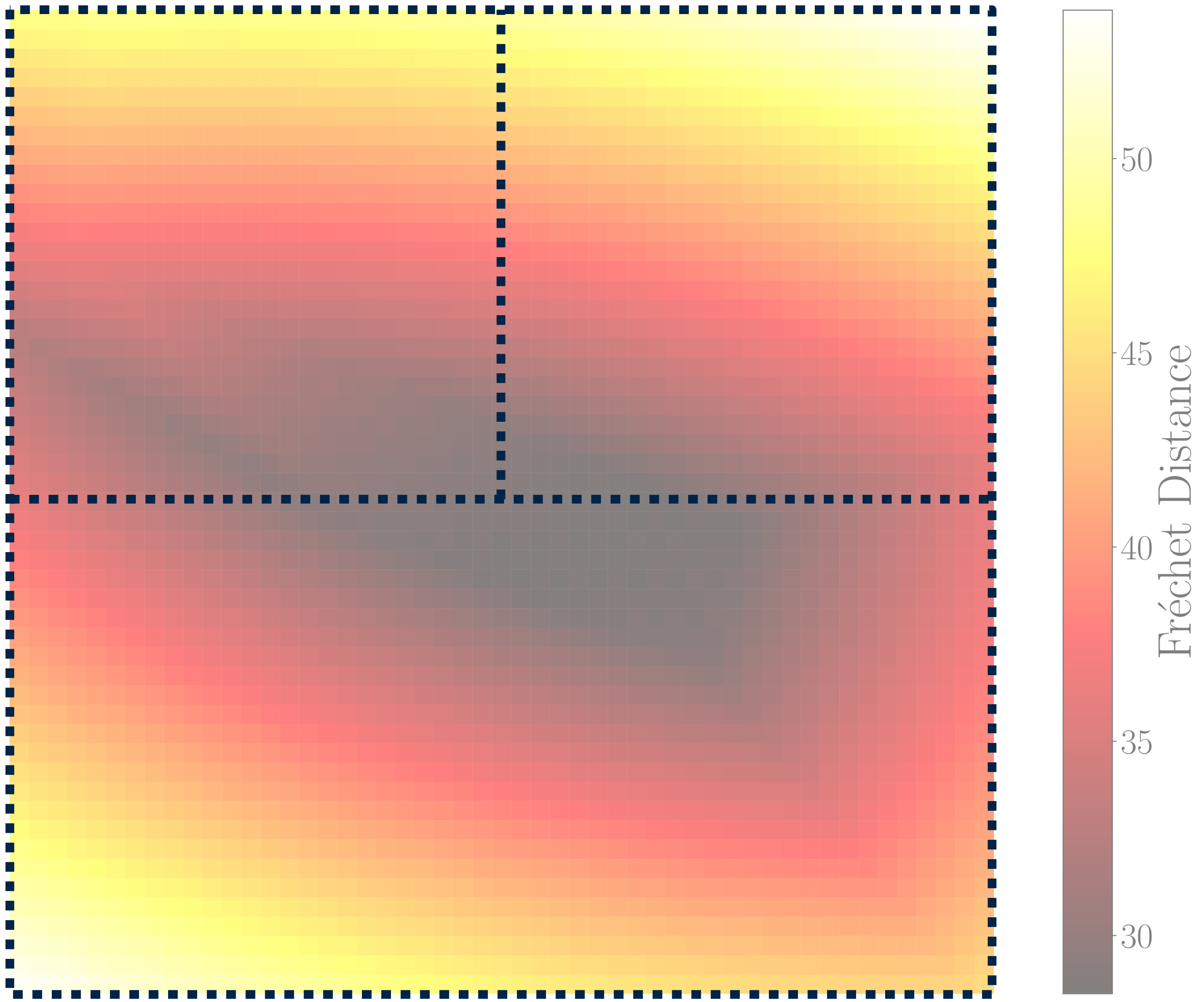

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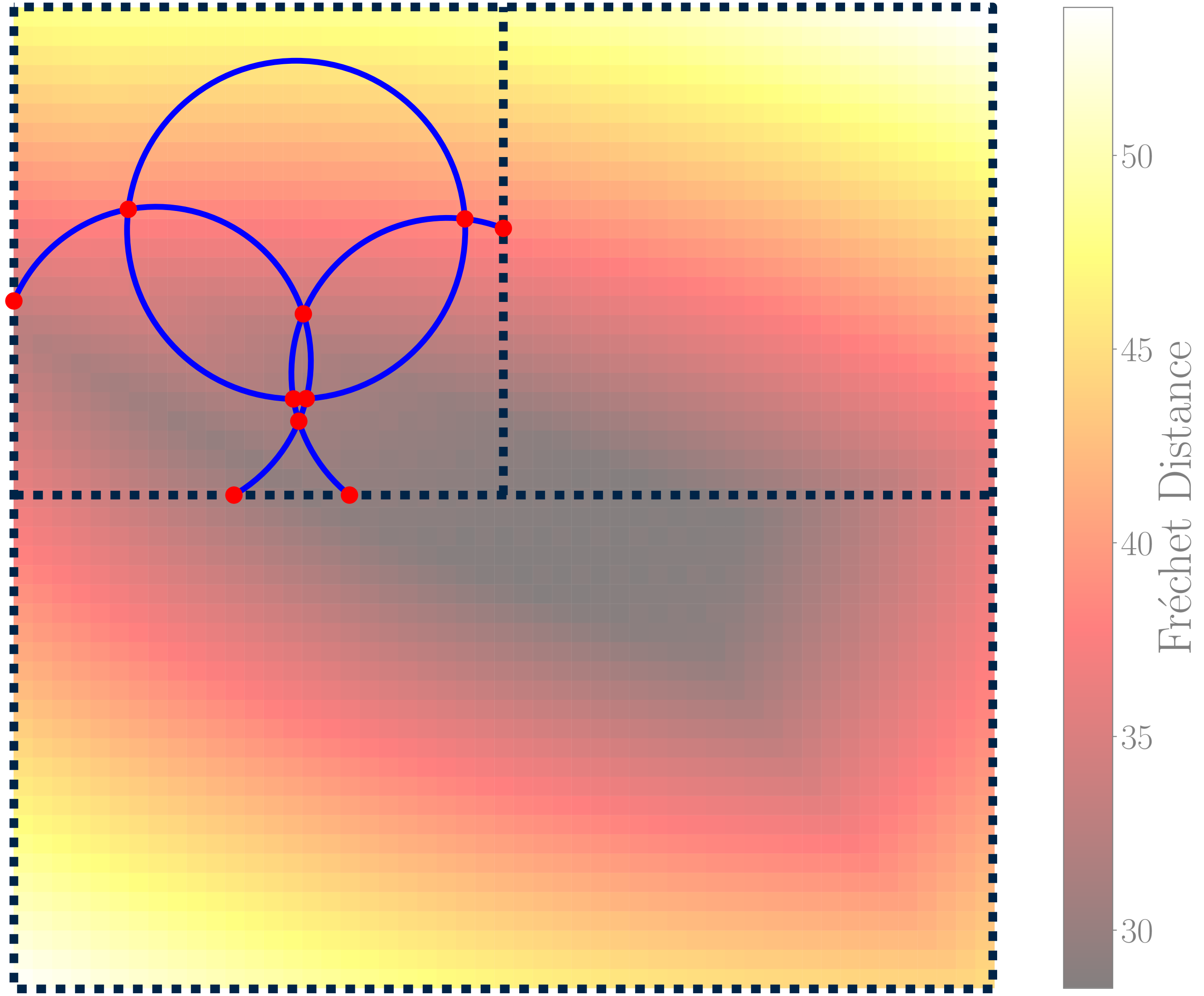

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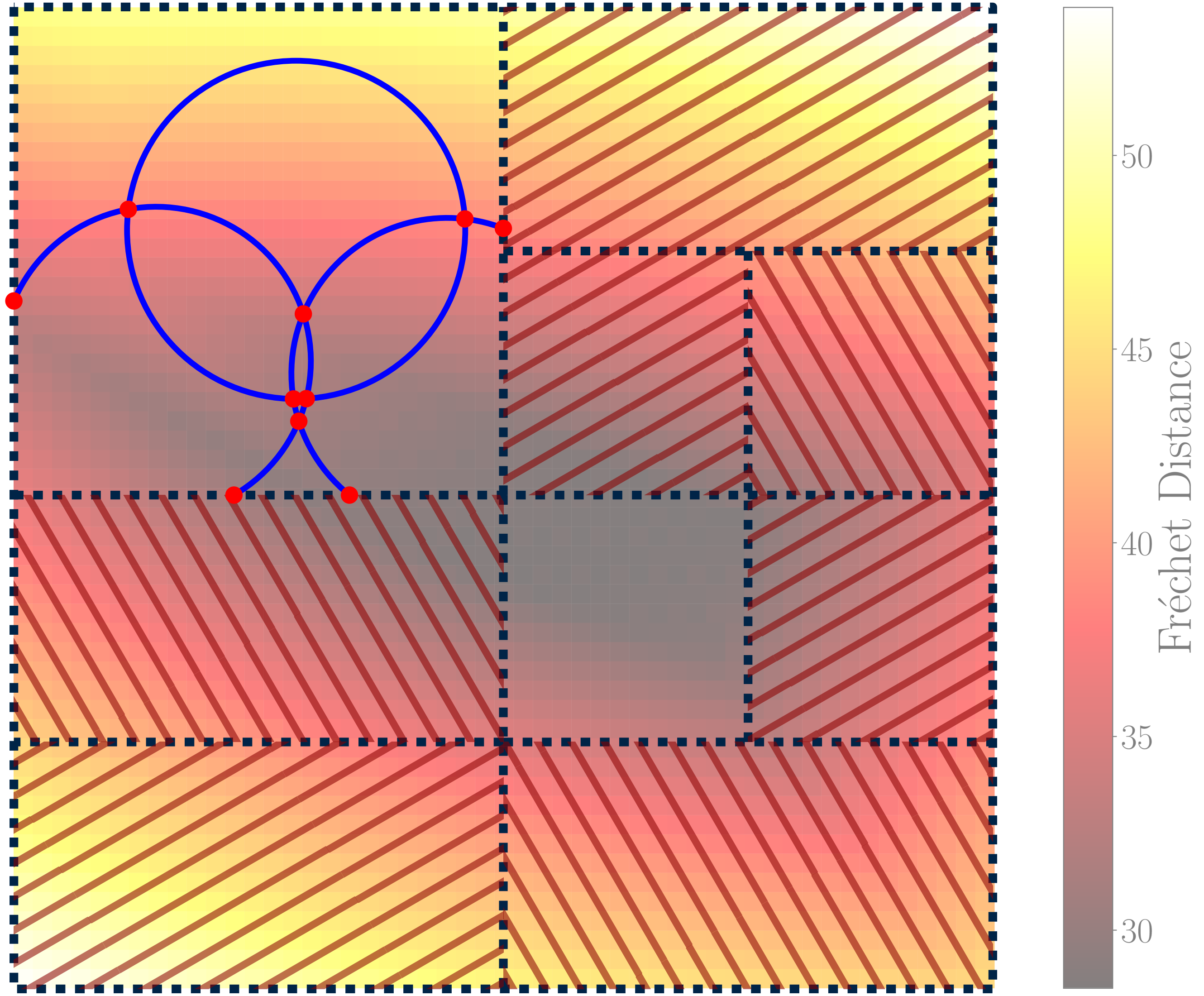

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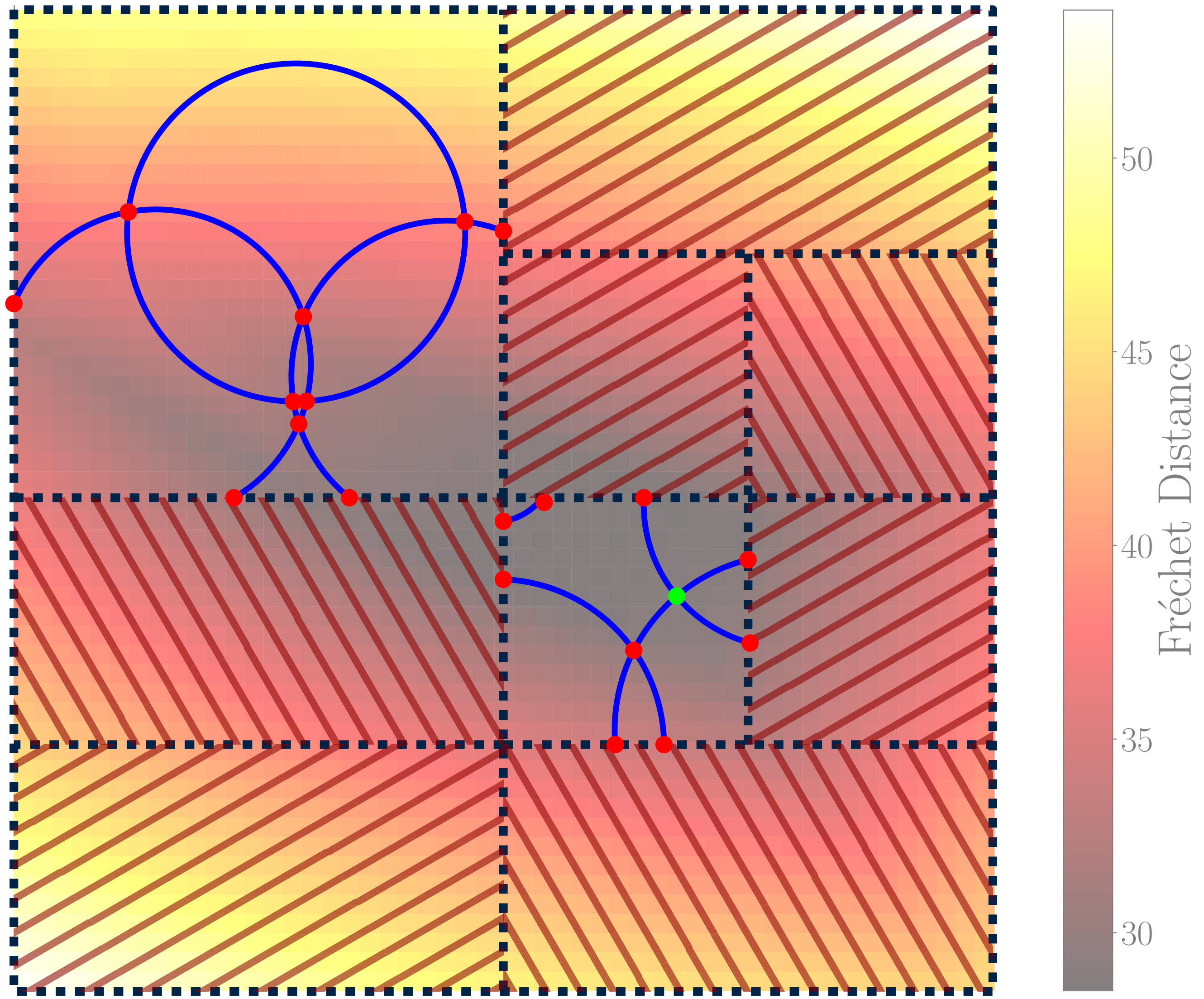

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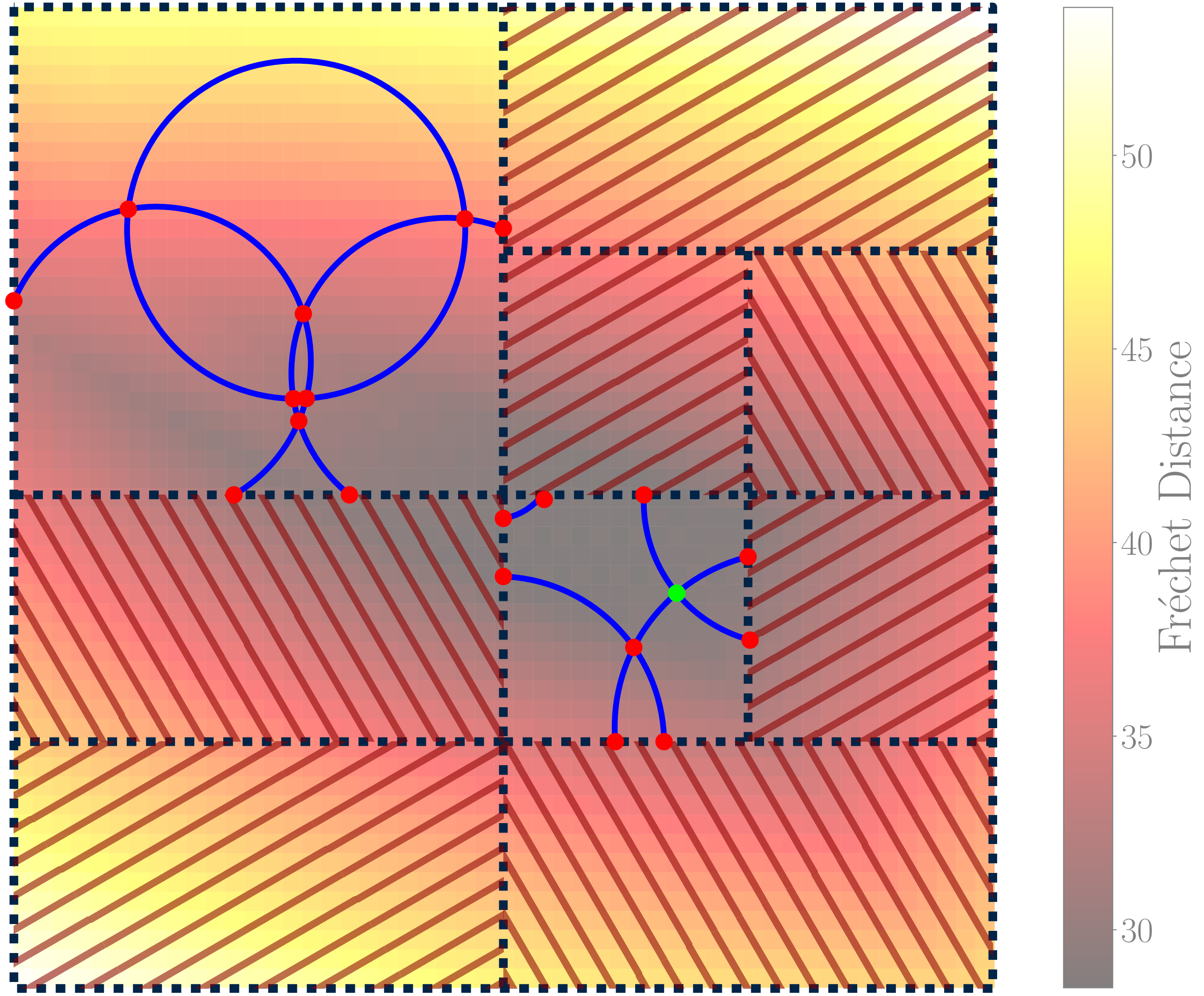

Issue: When to build the arrangement?

Main Ingredients:

1. Arrangement size estimation
2. Threshold parameter

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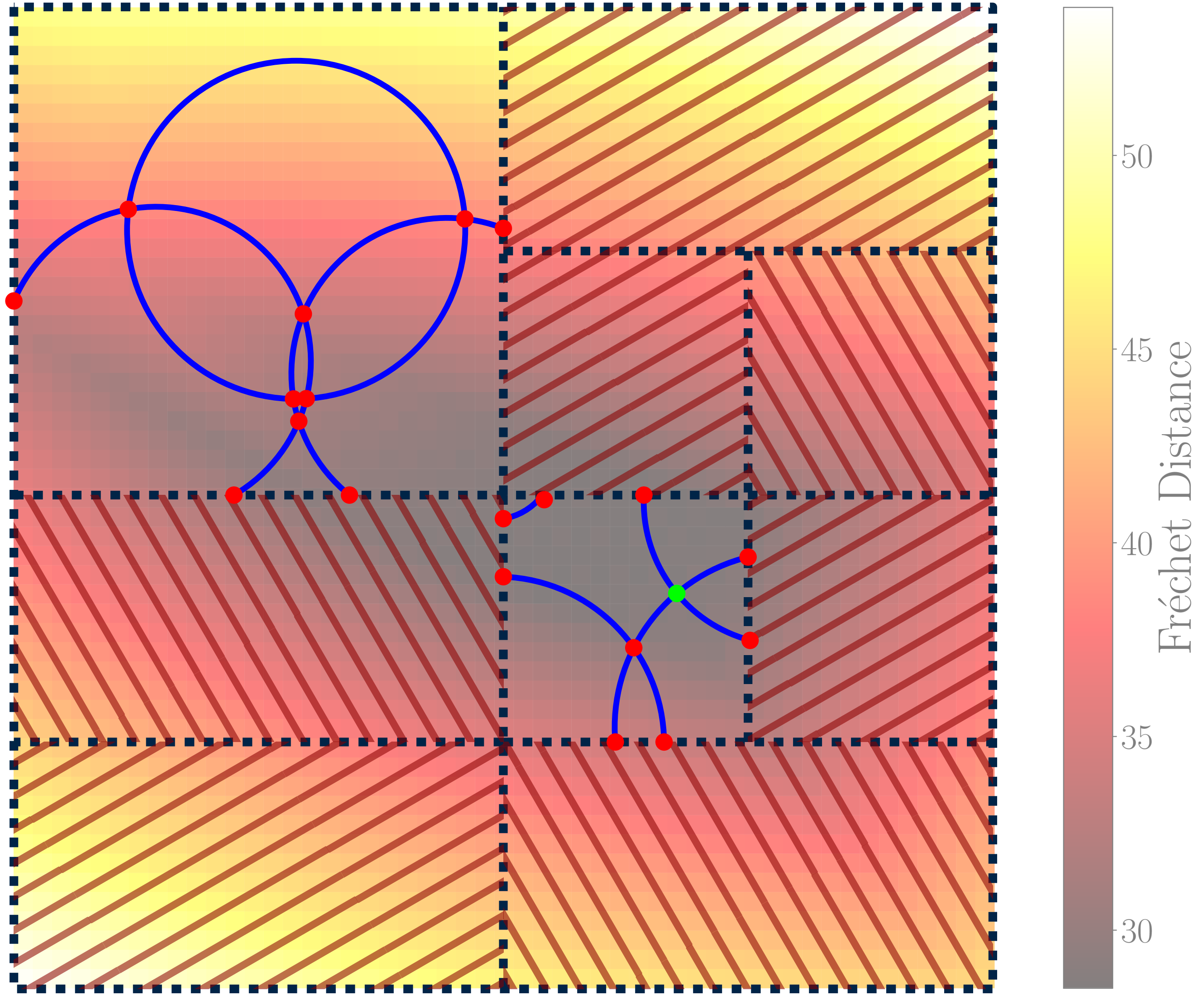

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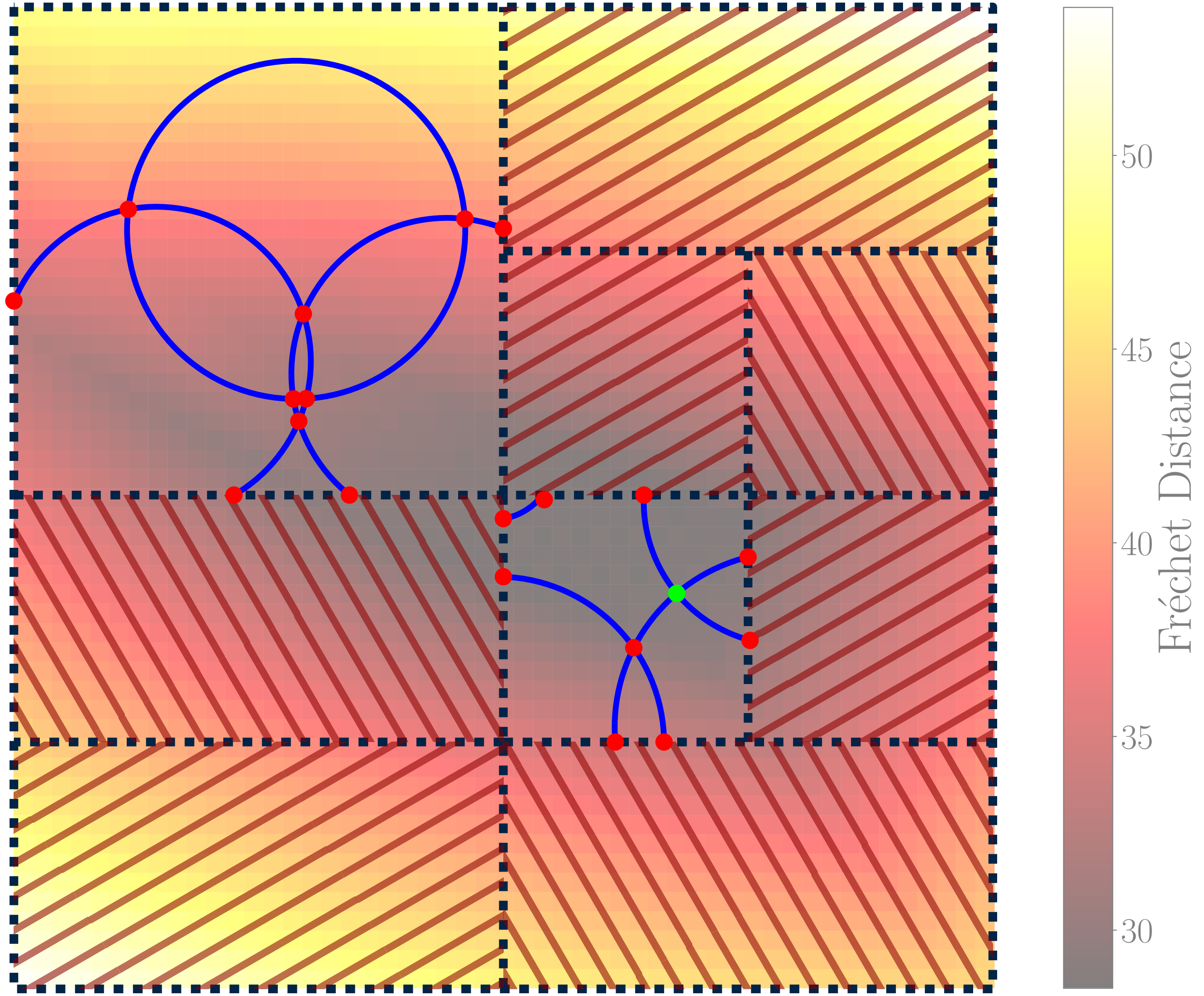

Issue: When to build the arrangement?

Main Ingredients:

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empirically choose

modified kd-tree

Contribution I: Exact Decider

Implementation Details

- Adaption of (fixed-translation) [SoCG'19] implementation to discrete case
- Lazy translation
- Parameter choice for arrangement size estimation

Contribution II: From Decider to Value Computation

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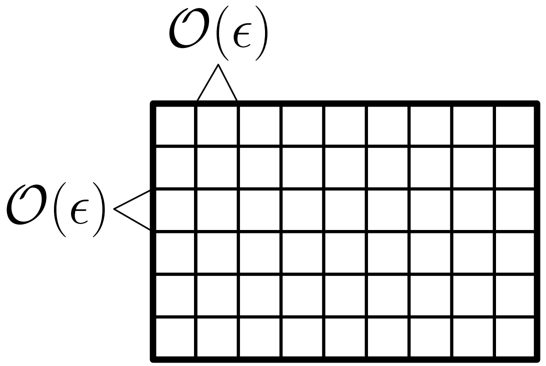
Approaches

Epsilon-approximate Set:



Contribution II: From Decider to Value Computation

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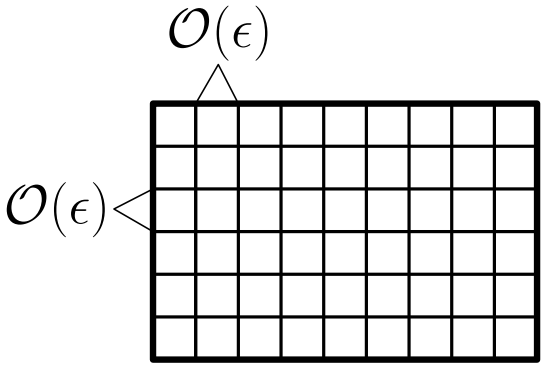

Binary Search via Decision Problem:

- Binary search over δ using decider

Contribution II: From Decider to Value Computation

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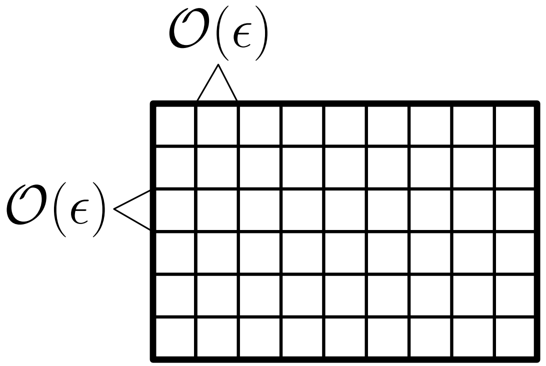

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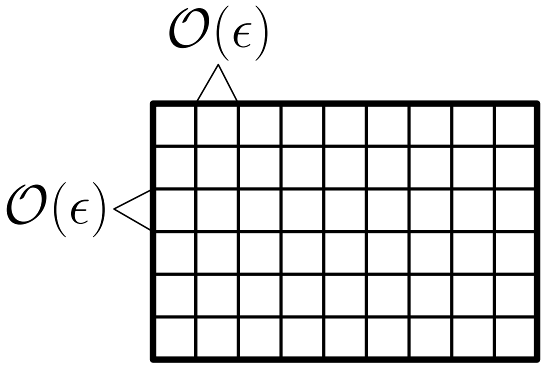

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Lipschitz-meets-Fréchet:

next slide

Contribution II: From Decider to Value Computation

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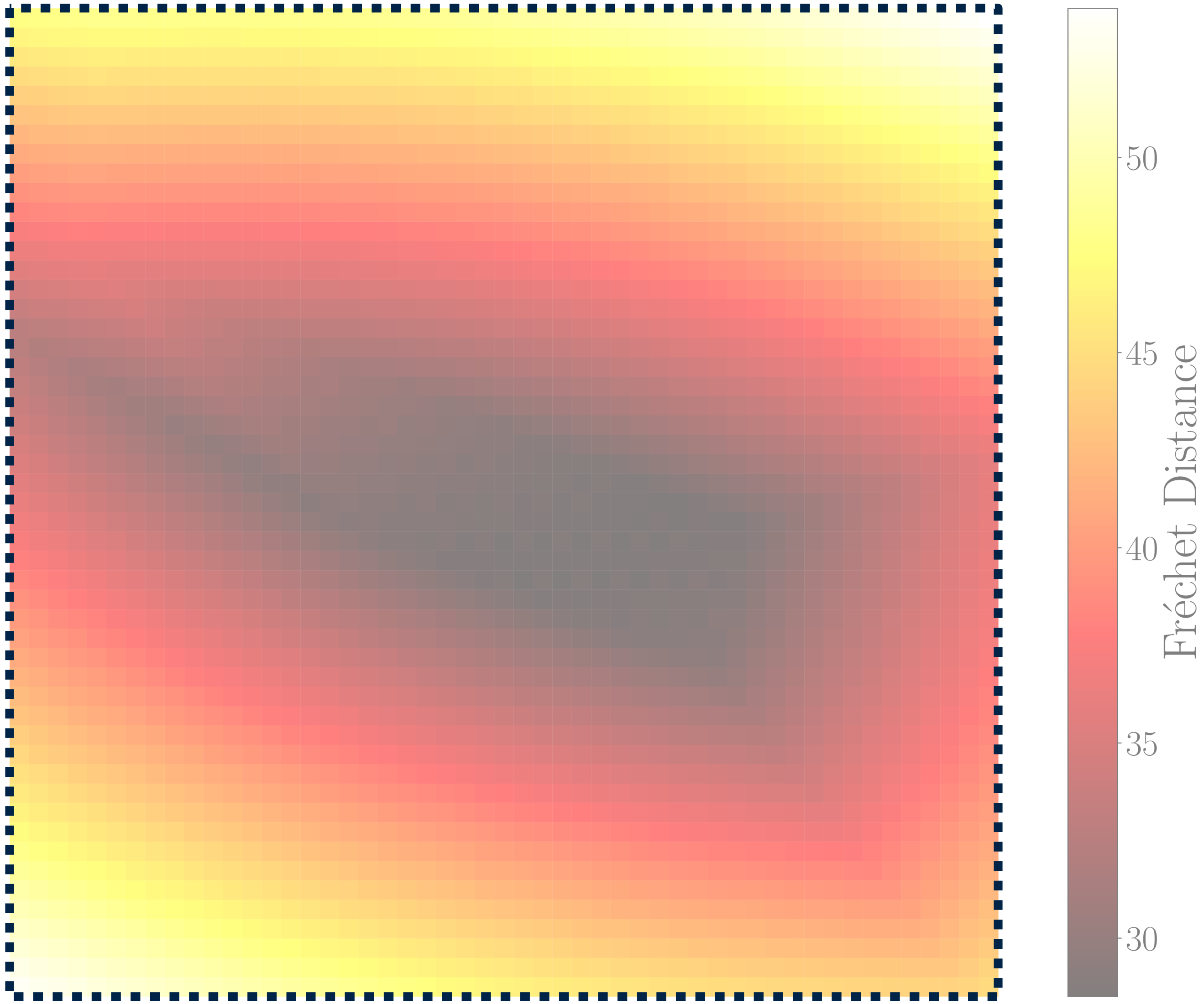
2. Maintain global upper bound
3. Arrangement size estimation

adapt!

For each box:

Lipschitz-meets-Fréchet



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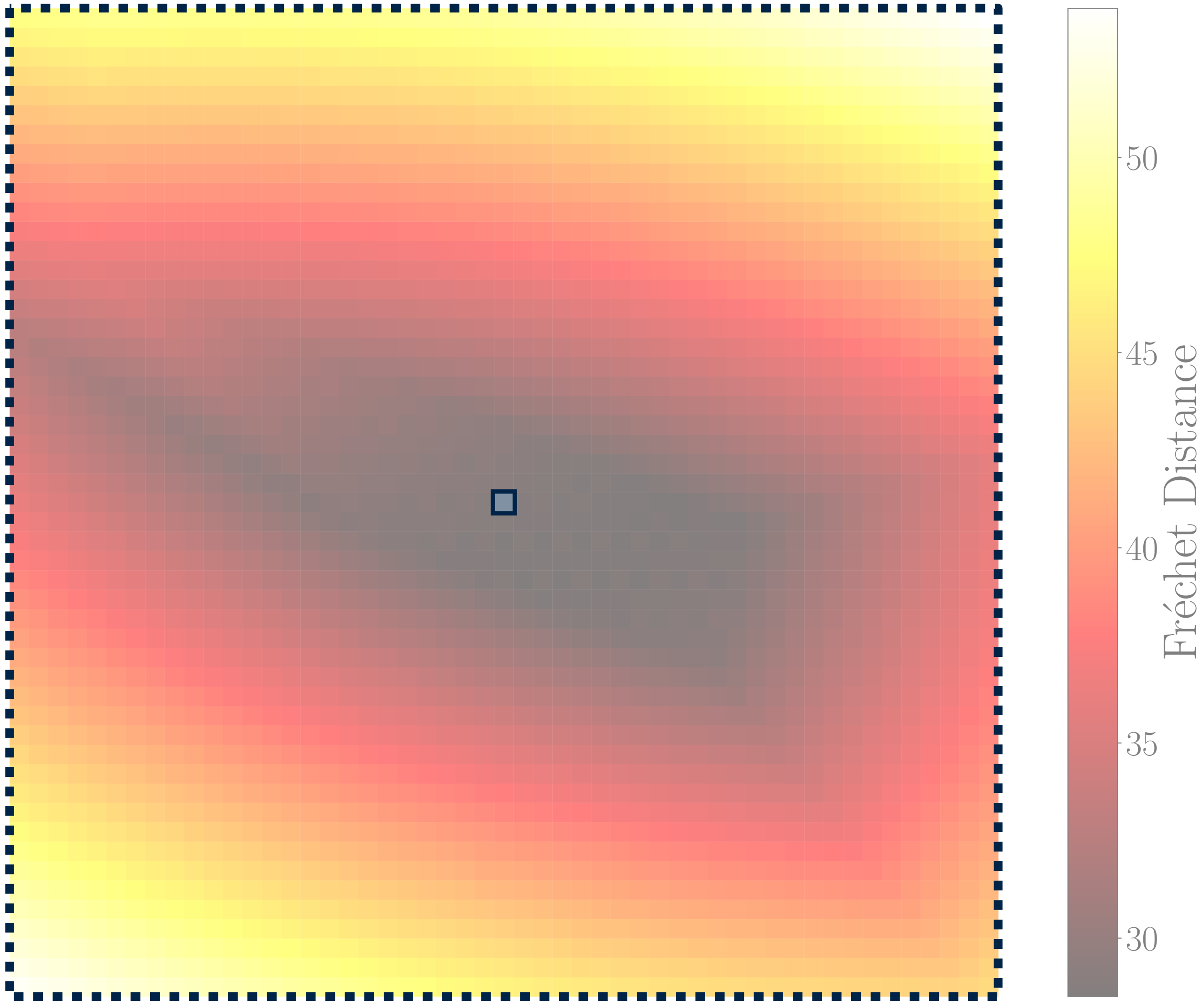

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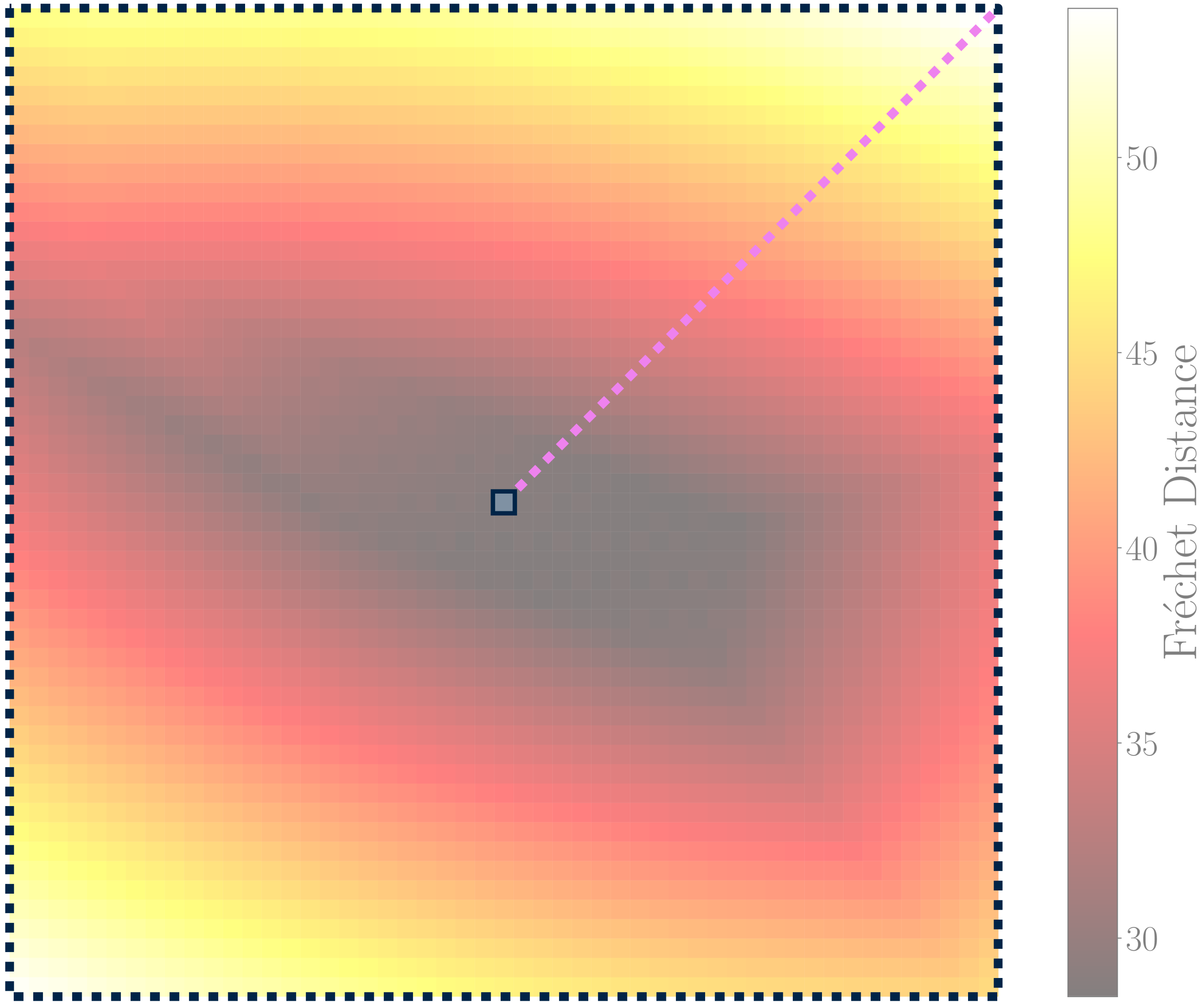

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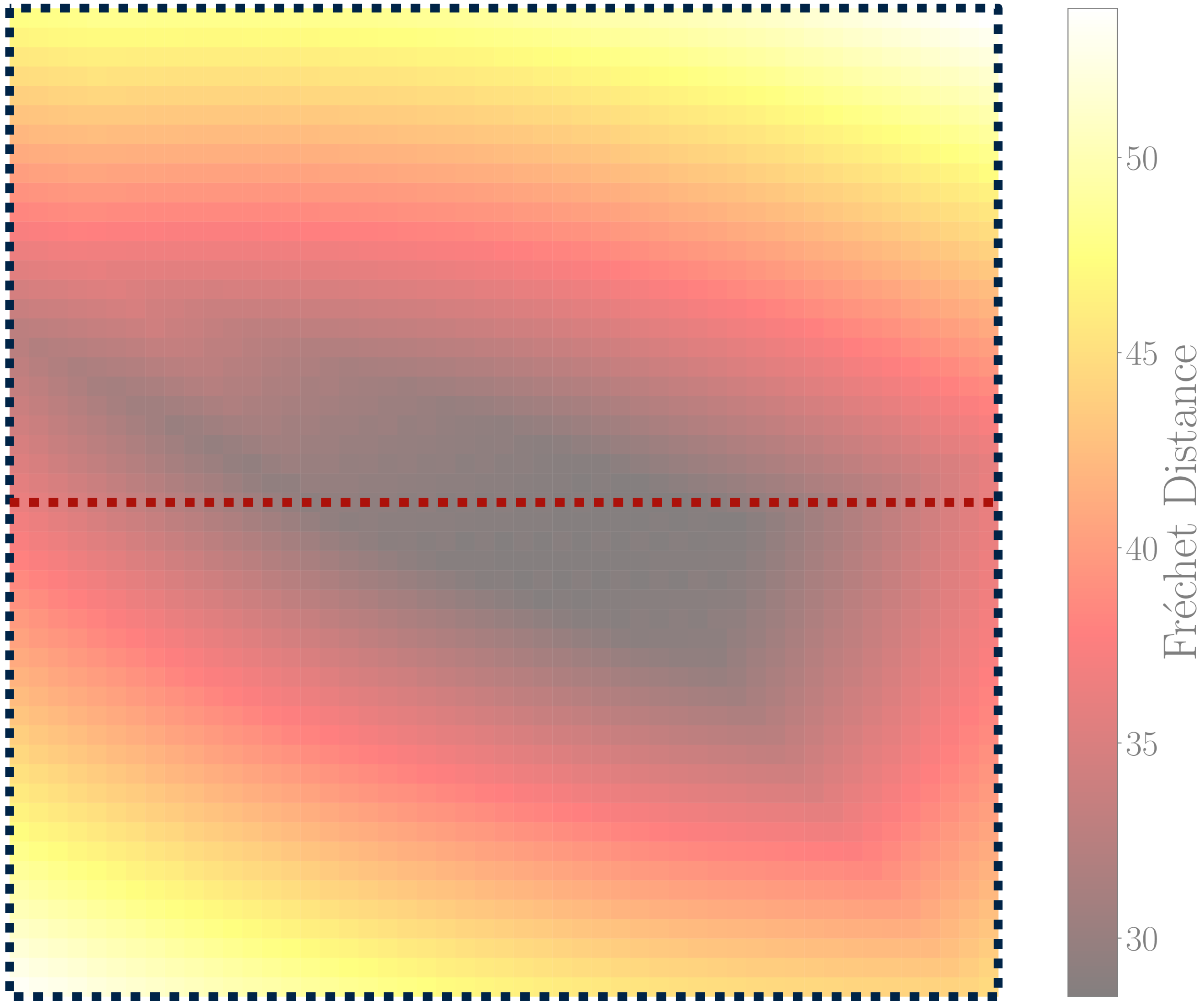

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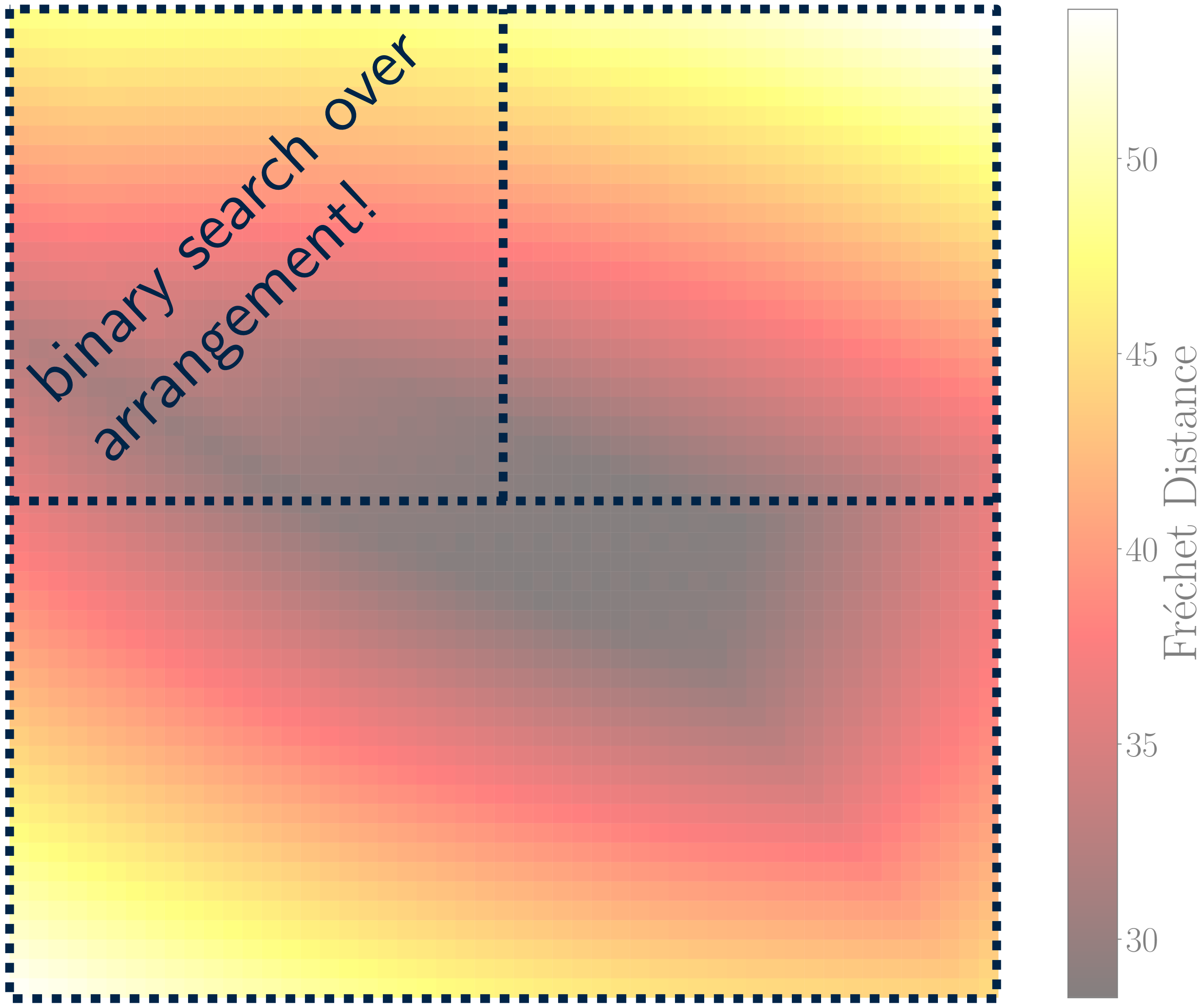


Contribution II: From Decider to Value Computation

Lipschitz-meets-Fréchet: Implementation Details

- Arrangement size estimation
- Initial estimates
- Priority queue on lower bound $\rightarrow$ no regret strategy!

Experiments

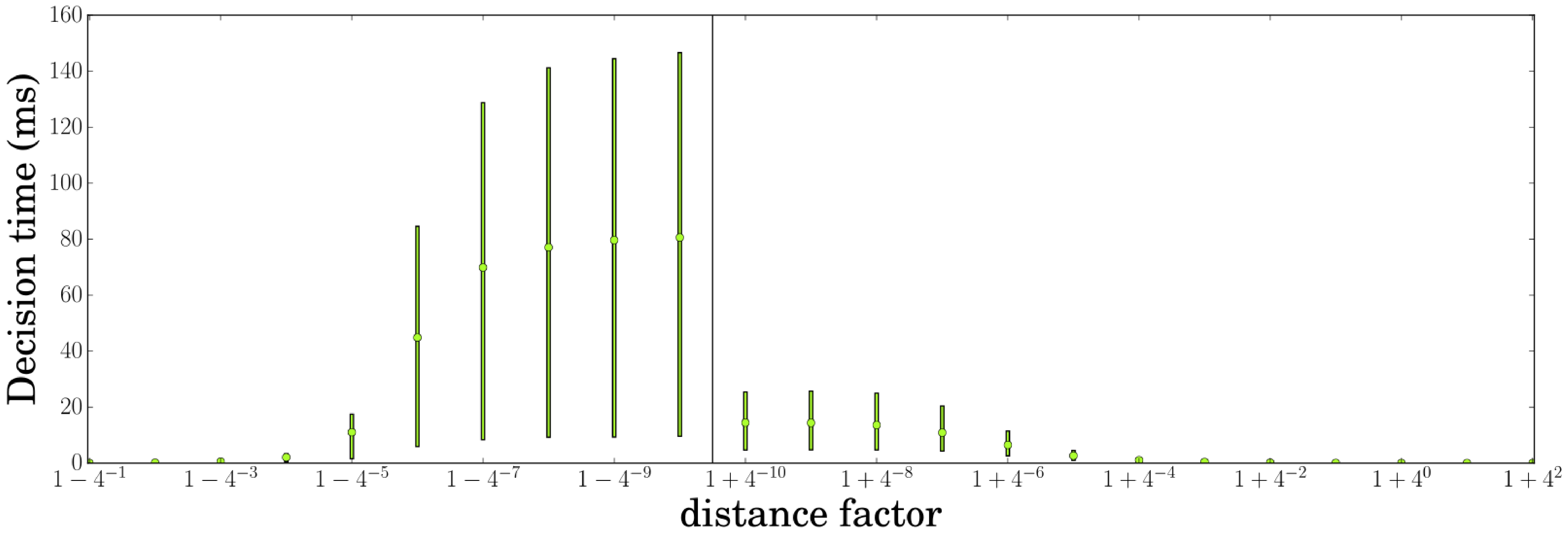
Data Sets

Data set	Type	#Curves	Mean #vertices
SIGSPATIAL	synthetic GPS-like	20199	247.8
CHARACTERS	20 handwritten chars	2858 (142.9 per character)	120.9

Experiments

Running Times

Same-characters:



Experiments

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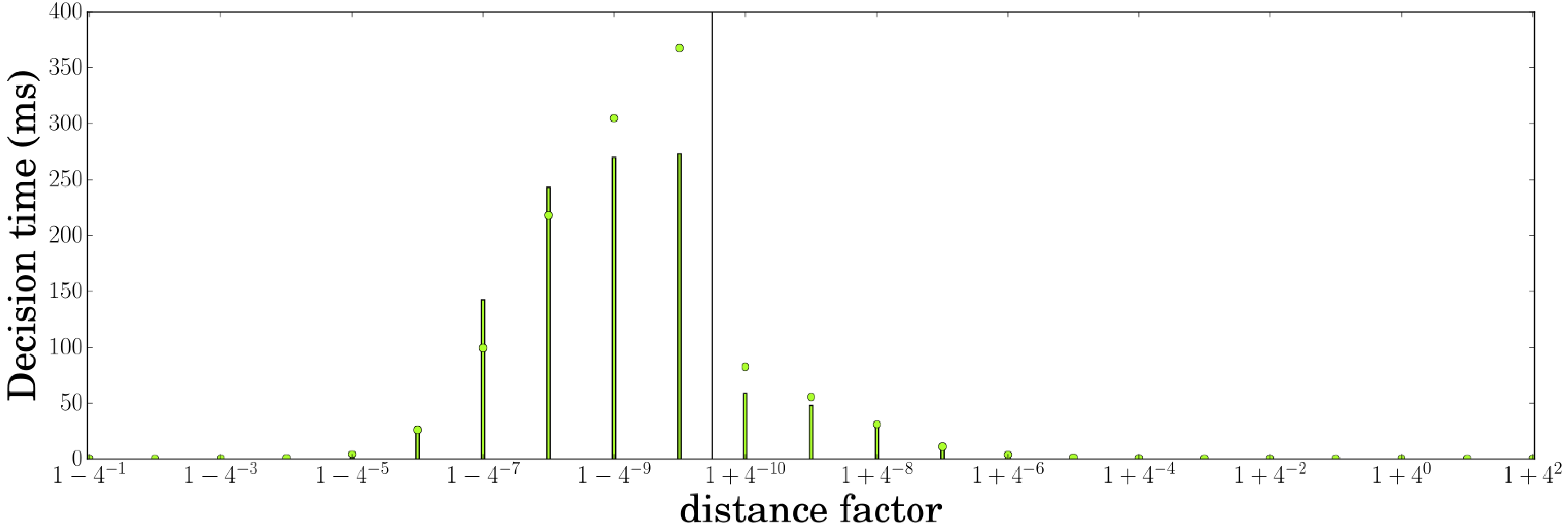
SIGSPATIAL:



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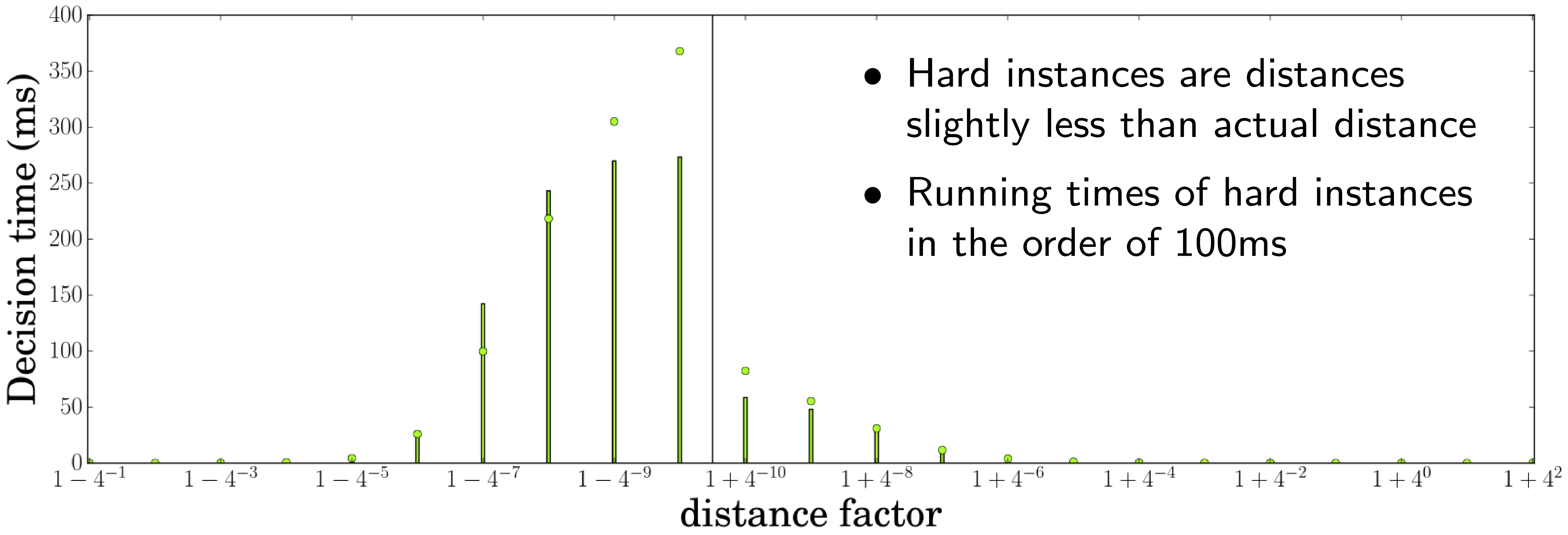
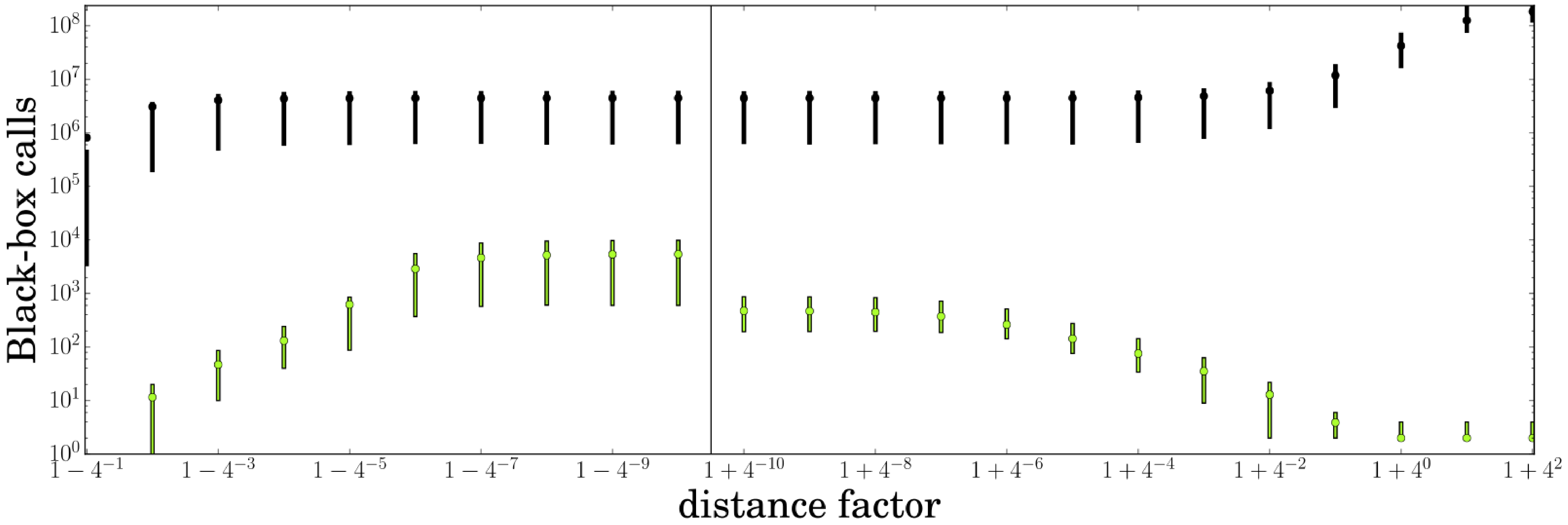


- Hard instances are distances slightly less than actual distance
- Running times of hard instances in the order of 100ms

Experiments

Black box calls vs. arrangement size

SAME-CHARACTERS:



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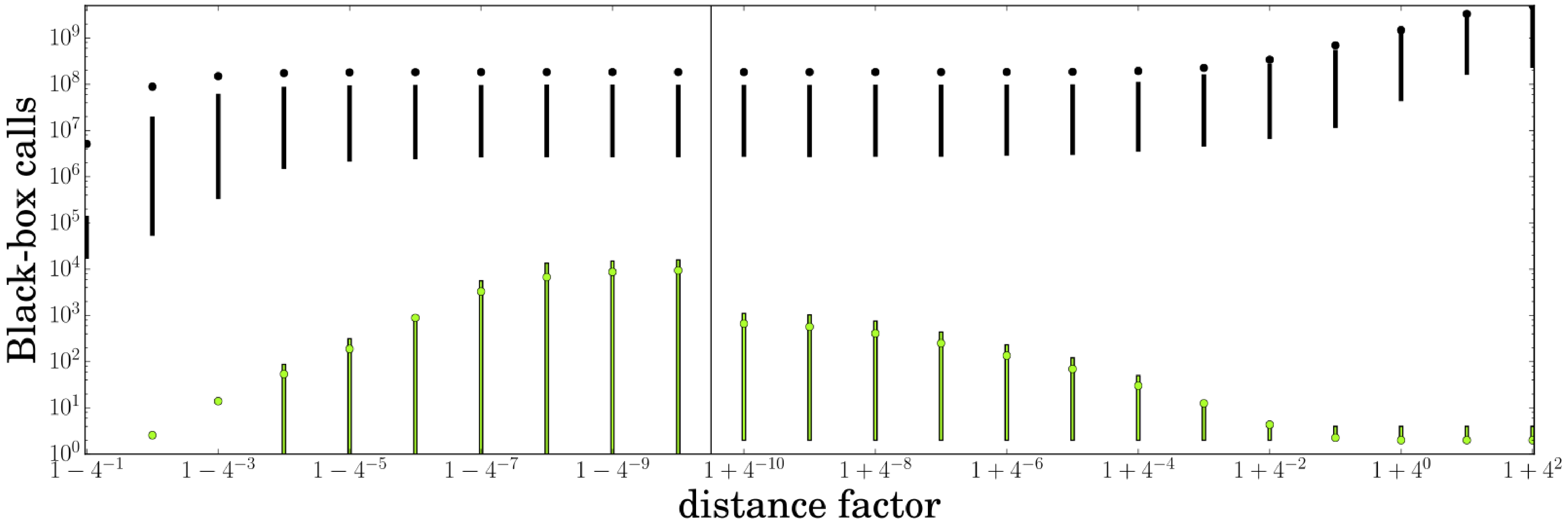

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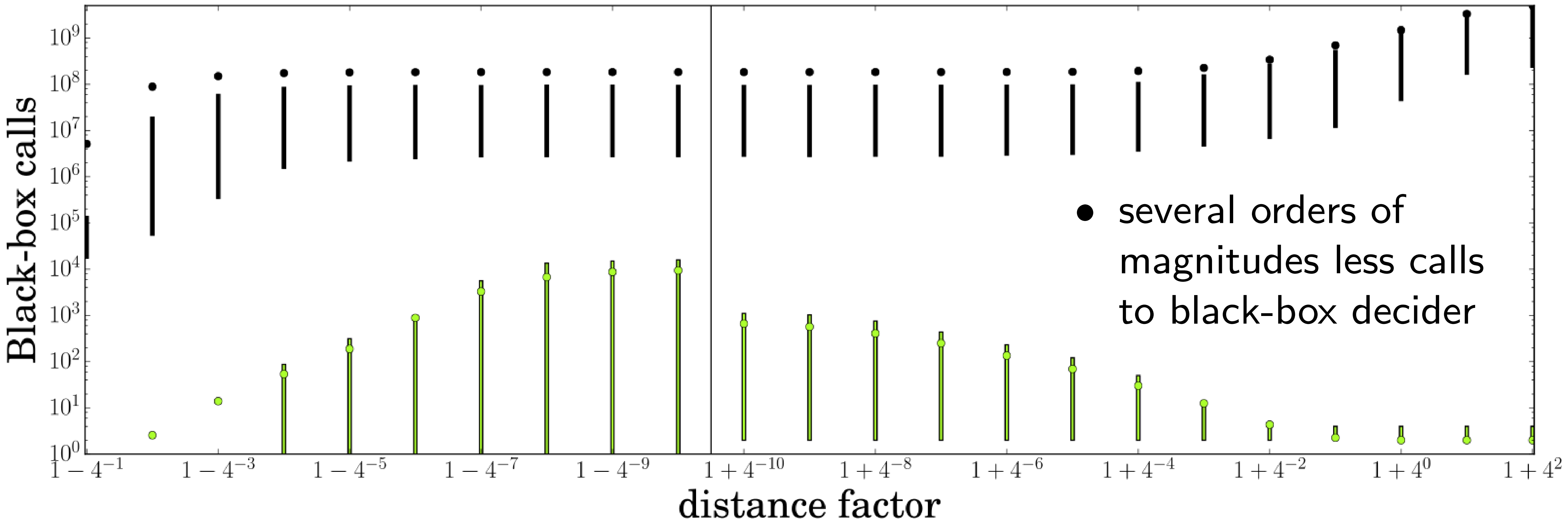


- several orders of magnitudes less calls to black-box decider

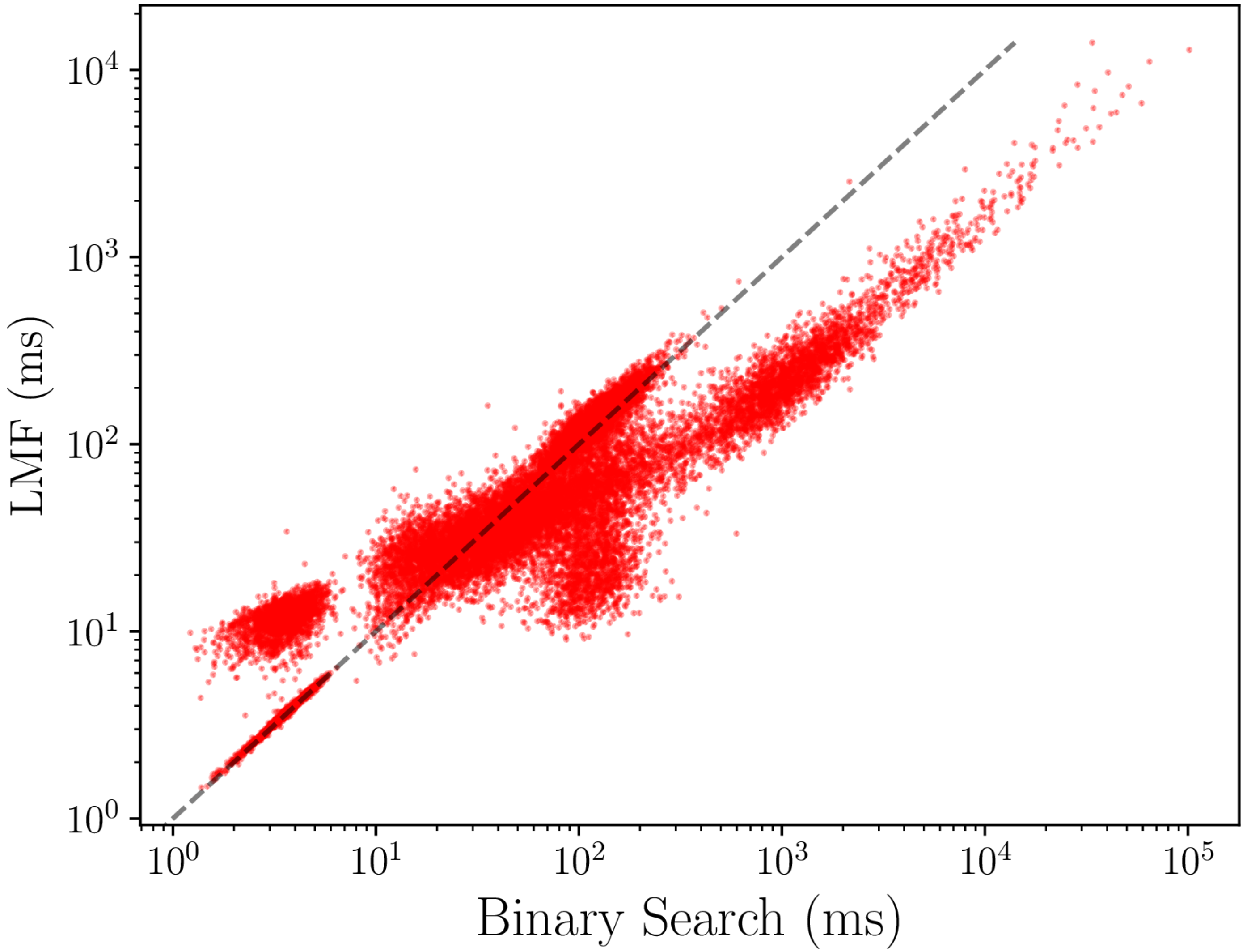
Experiments

Value Computation Times

Approach	Time	Black-Box Calls
LMF	148,032 ms (141.0 ms per instance)	13,323,232 (12,688.8 per instance)
Binary Search	536,853 ms (511.3 ms per instance)	45,909,628 (43,723.5 per instance)
Lipschitz-only	4,204,521 ms (4,004.3 ms per instance)	820,468,224 (781,398.3 per instance)

Experiments

Binary Search vs. LMF



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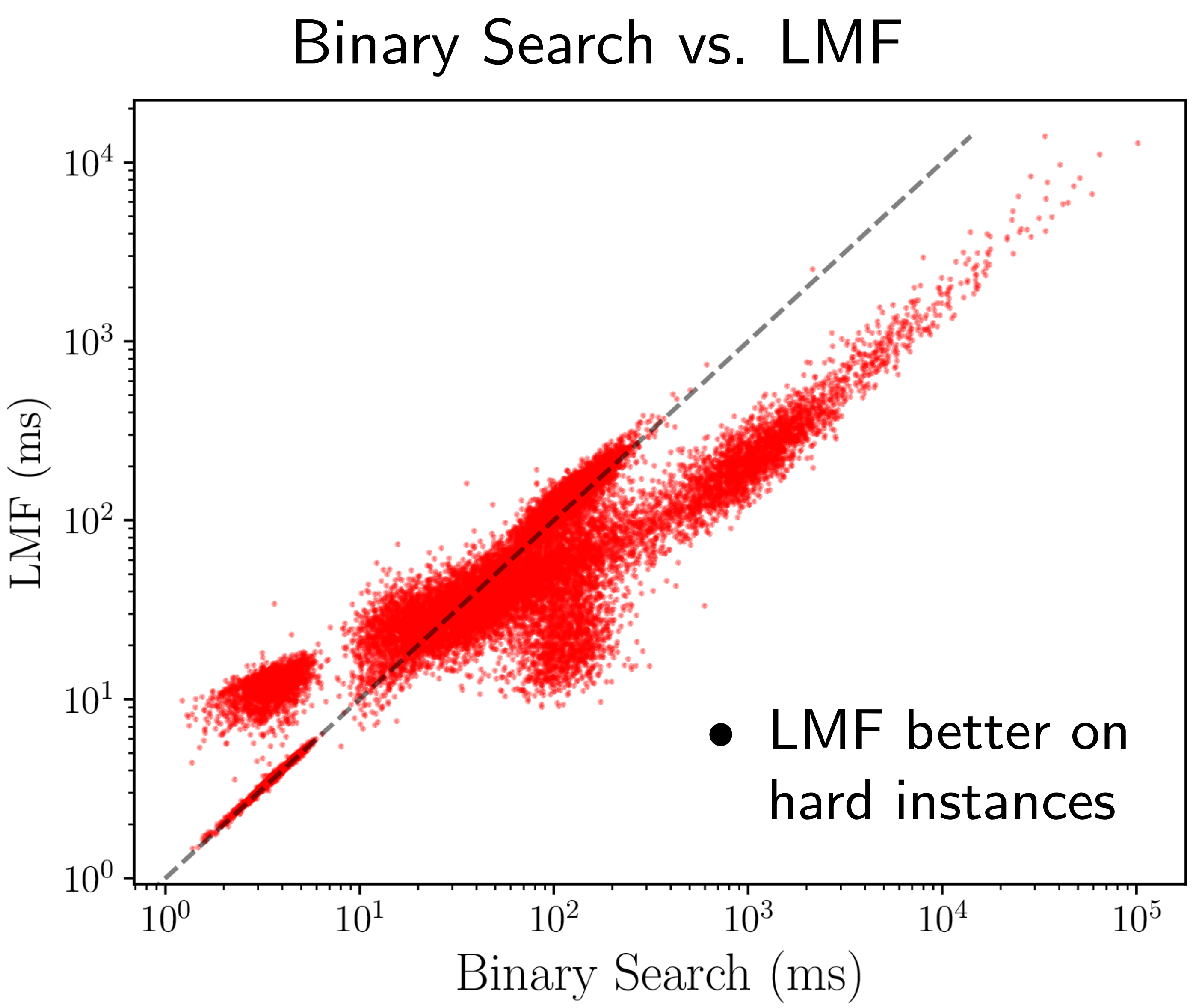
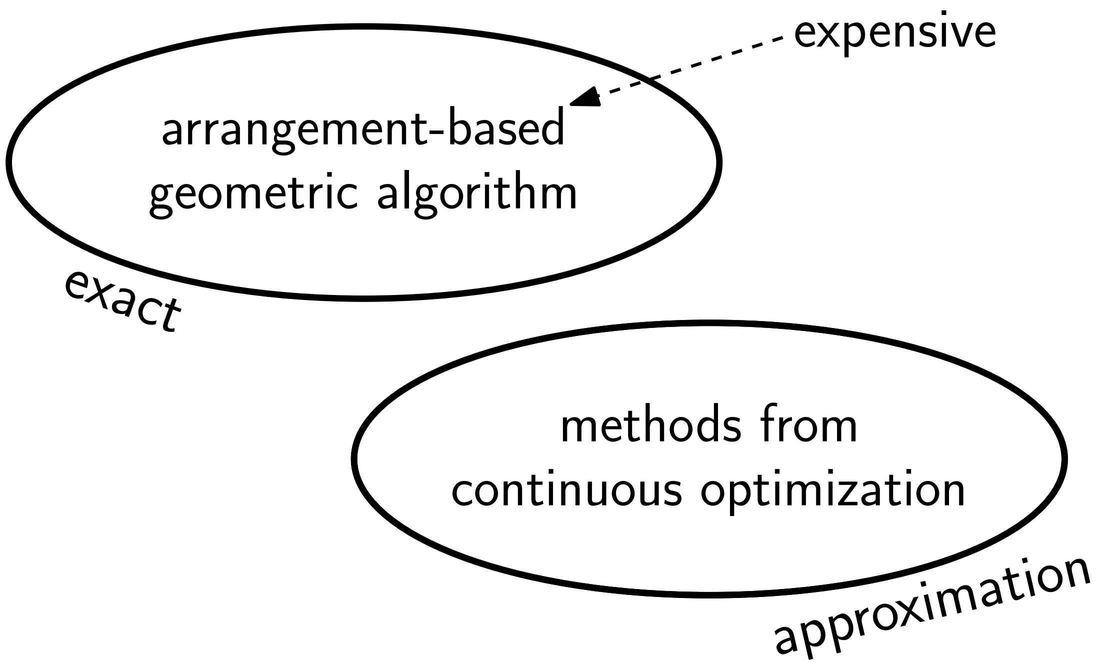


Summary


expensive
arrangement-based
geometric algorithm
exact
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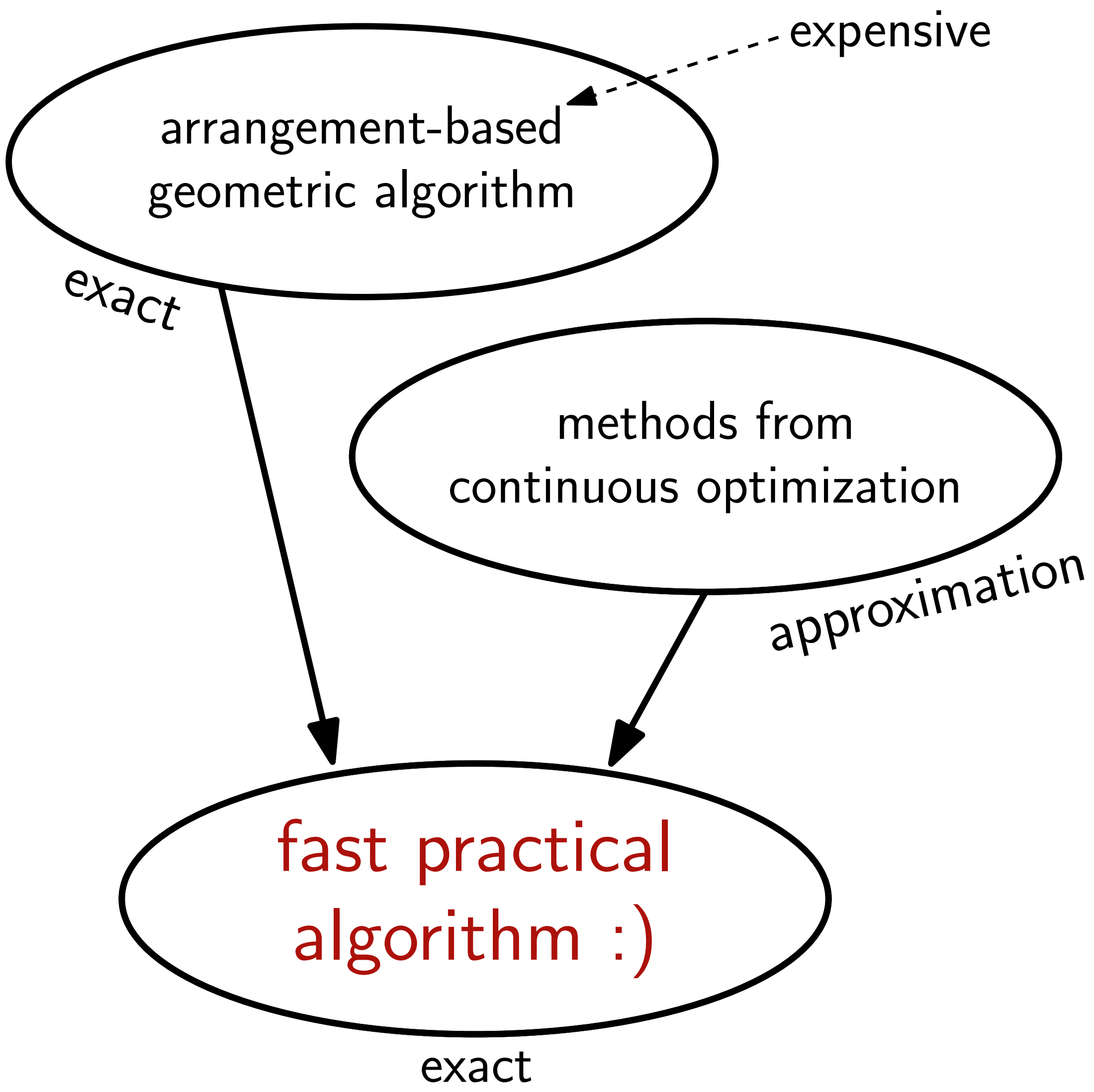

expensive
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fast practical algorithm :)
exact

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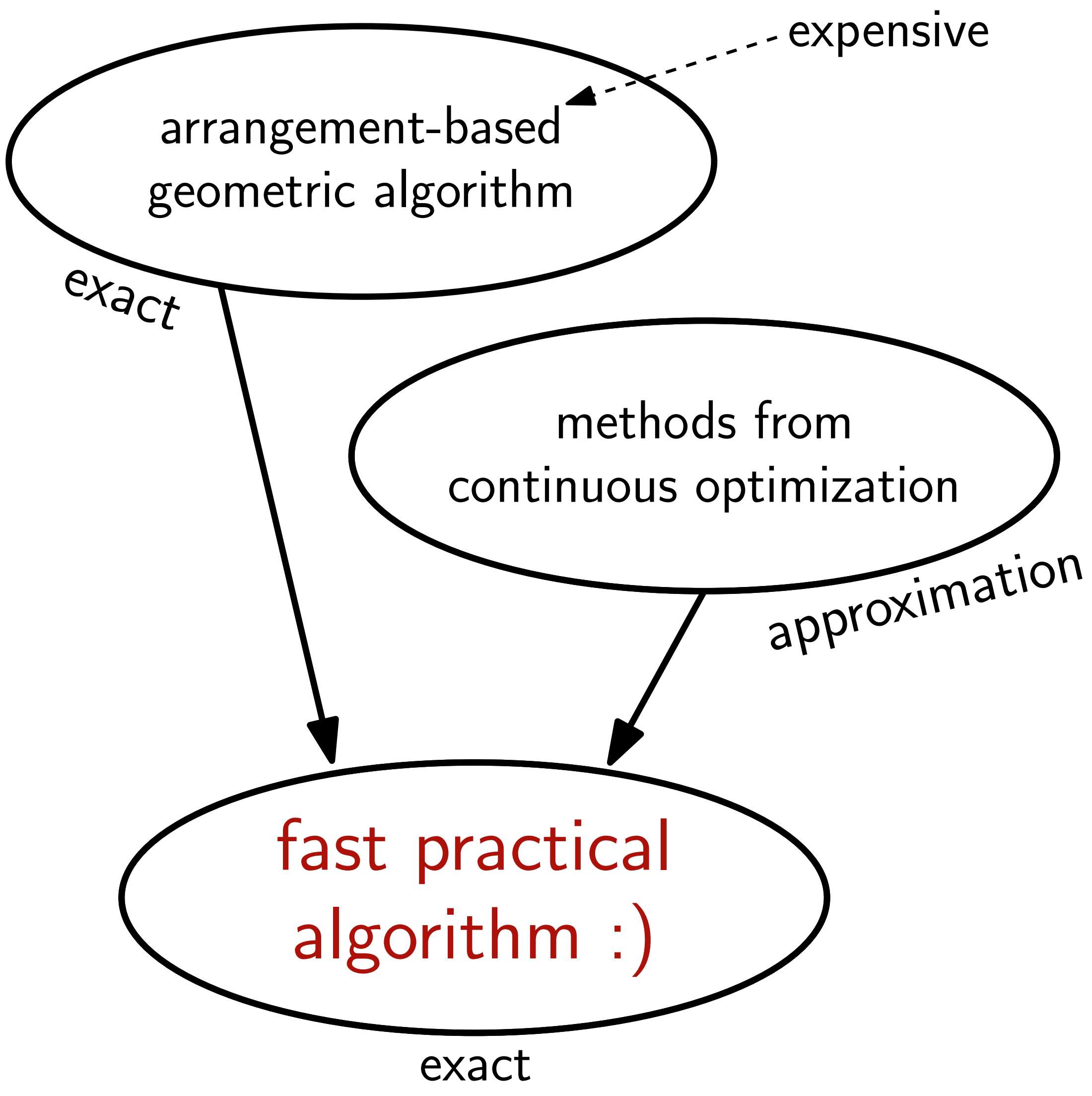

Future Directions:

- Apply approach to other problems
- Find optimal point of building the arrangement

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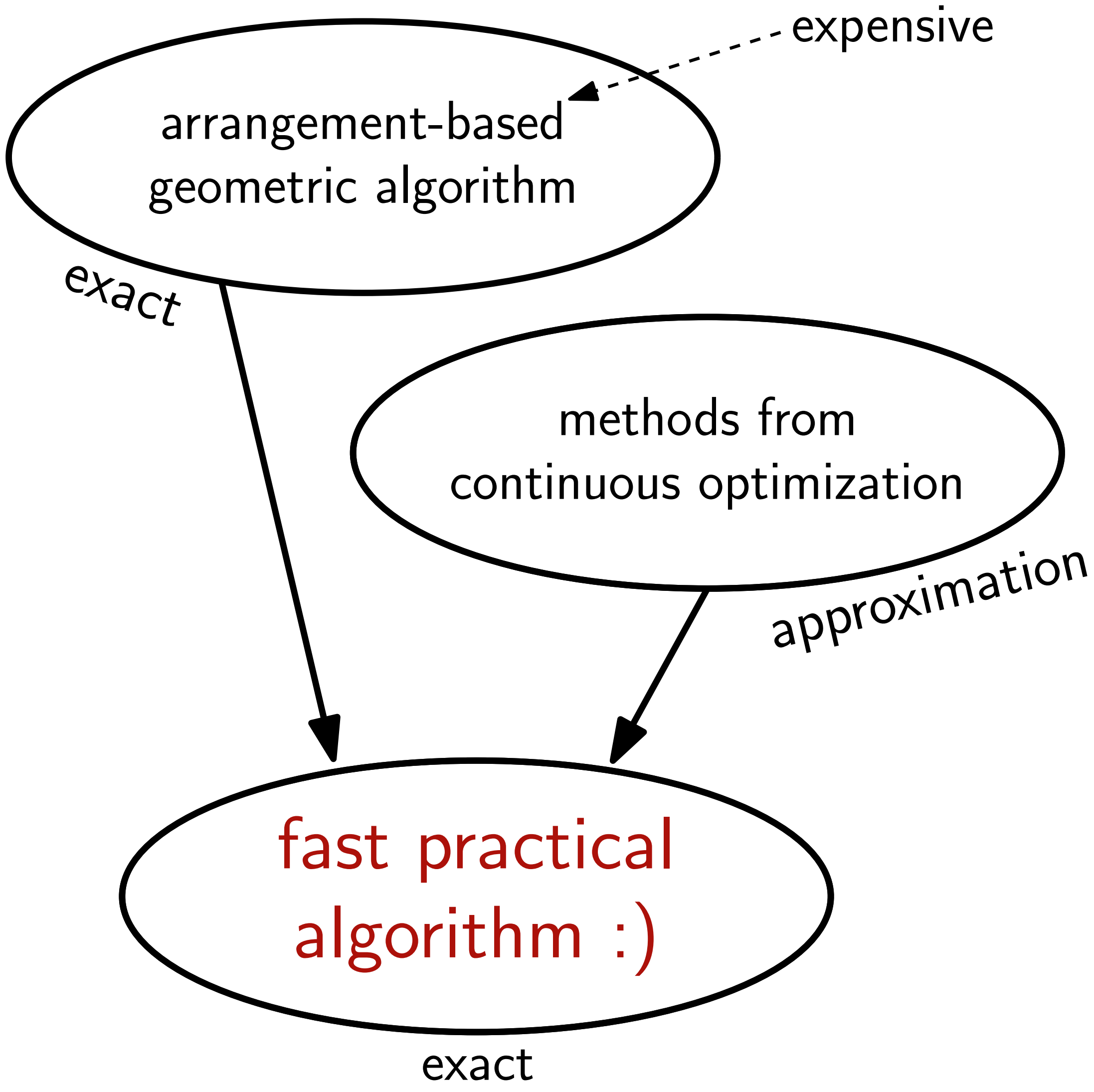

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Code:
https://gitlab.com/anusser/frechet_distance_under_translation