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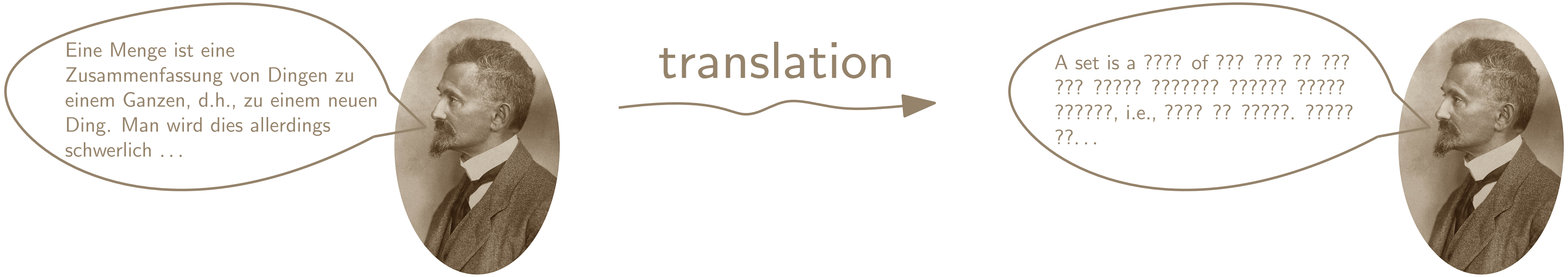
Translating Hausdorff Is Hard: Fine-Grained Lower Bounds for Hausdorff Distance Under Translation

Karl Bringmann and **André Nusser**



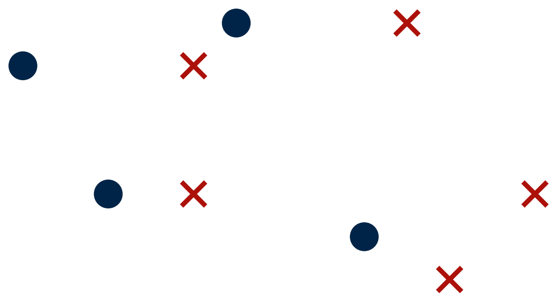
Preamble

Hausdorff Distance

Given: Point sets A (●) and B (×) in $\mathbb{R}^2$

Directed Hausdorff Distance

$$\delta_{\vec{H}}(A, B) := \max_{a \in A} \min_{b \in B} \|a - b\|_p$$



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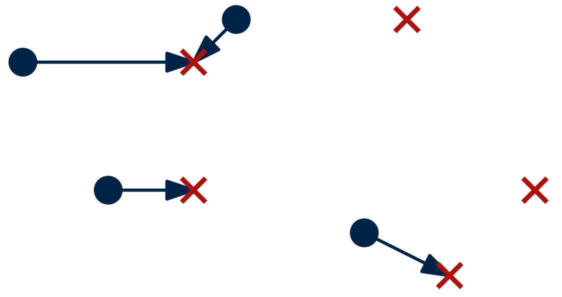

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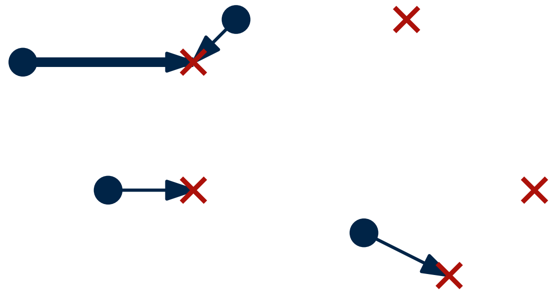

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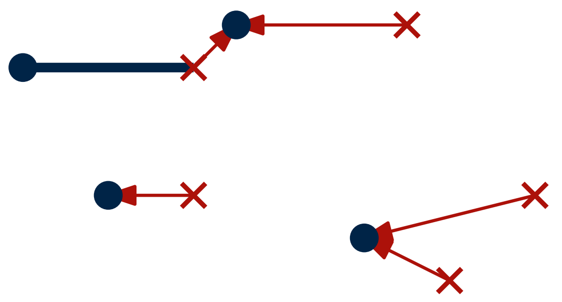
(Undirected) Hausdorff Distance

$$\delta_H(A, B) := \max\{\delta_{\vec{H}}(A, B), \delta_{\vec{H}}(B, A)\}$$



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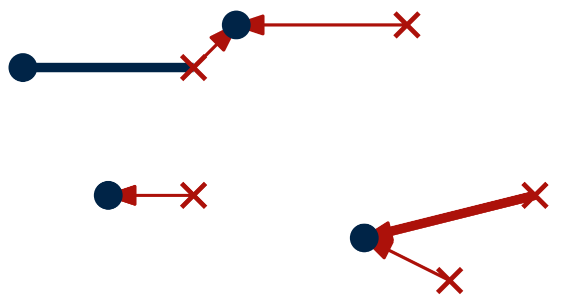

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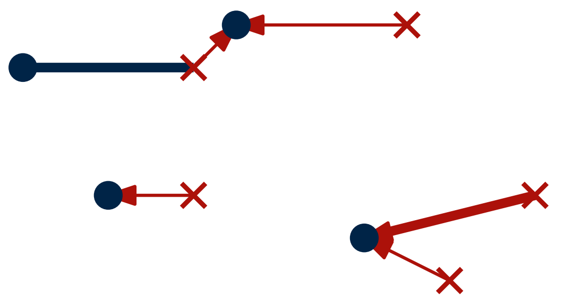

→ Computable in time $\tilde{\mathcal{O}}(n + m)$

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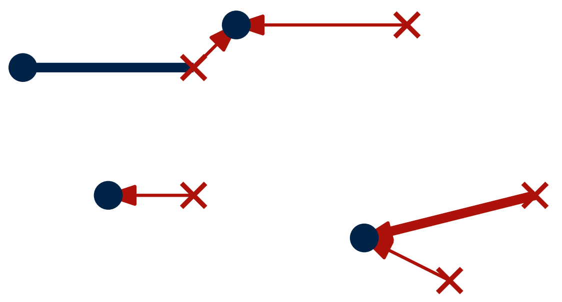

→ Computable in time $\tilde{O}(n+m)$

Hausdorff Distance Under Translation

Given: Point sets A (●) and B (×) in $\mathbb{R}^2$

Directed Hausdorff Distance U. Translation

$$\delta^T_{\vec{H}}(A, B) := \min_{\tau \in \mathbb{R}^2} \delta_{\vec{H}}(A, B + \tau)$$

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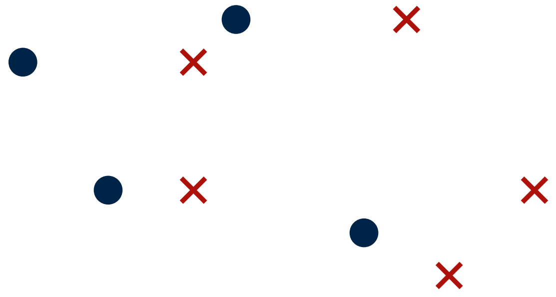

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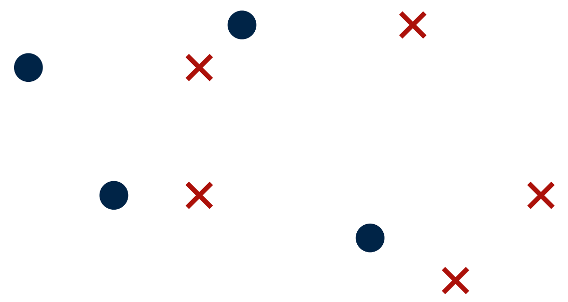

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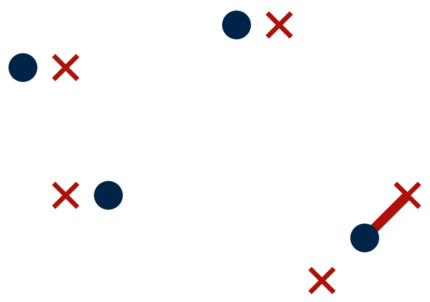

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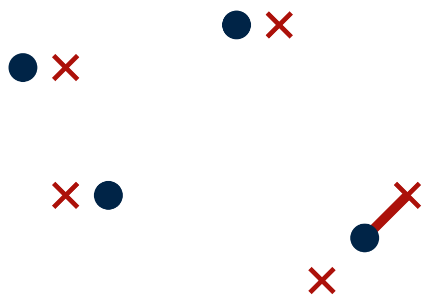


→ **What's the "correct" running time?**

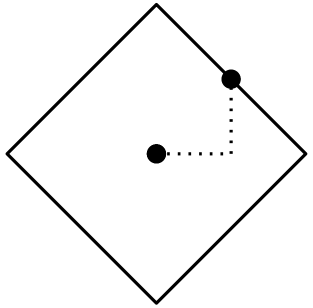
Motivation

Most basic translation invariant point set similarity measure

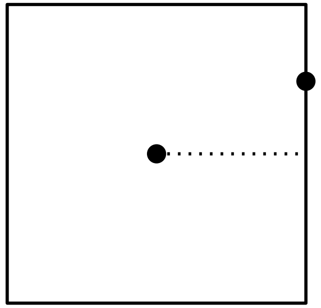
Results

Results

L_1 :



L_∞ :



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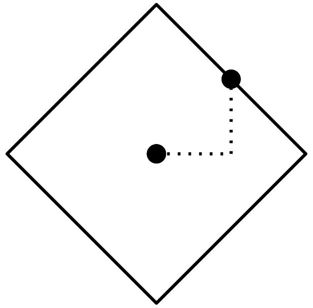

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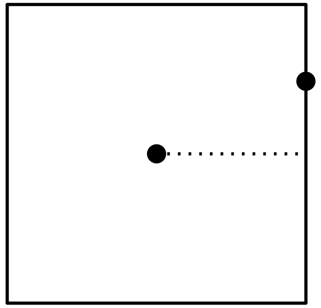

Theorem [Chew, Kedem '92]

Hausdorff Distance Under Translation in L_1/L_∞ can be computed in time $\tilde{O}(nm)$.

Results

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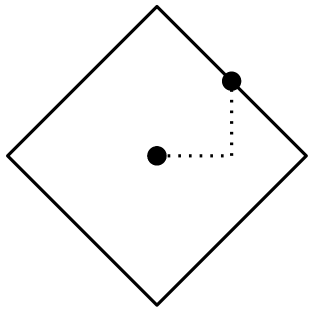

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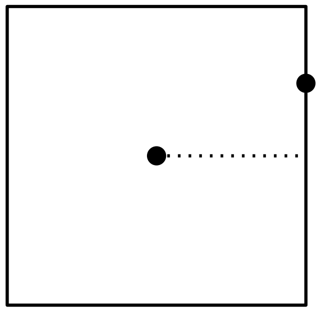

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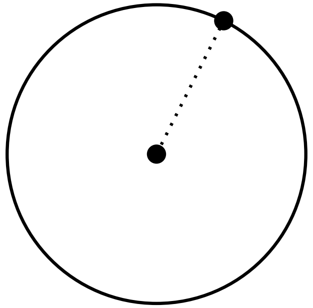
Hausdorff Distance Under Translation in L_1/L_∞ cannot be computed in time $(nm)^{1-\epsilon}$, unless SETH fails.

Results

$L_2:$



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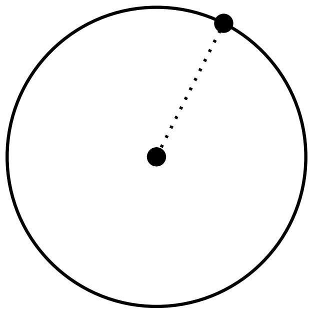

Theorem [Huttenlocher, Kedem, Sharir ’93]

Hausdorff Distance Under Translation in L_2 can be computed in $\tilde{\mathcal{O}}(nm(n+m))$.

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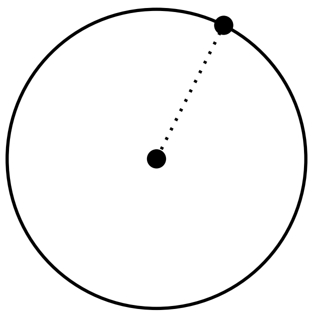

Theorem [Huttenlocher, Kedem, Sharir '93]

Hausdorff Distance Under Translation in L_2 can be computed in $\tilde{\mathcal{O}}(nm(n+m))$.

Theorem

Hausdorff Distance Under Translation in L_2 with $m \in \mathcal{O}(1)$ cannot be computed in time $n^{2-\epsilon}$, unless the 3SUM Hypothesis fails.

First Reduction

Theorem

Hausdorff Distance Under Translation in L_1/L_∞ cannot be computed in time $(nm)^{1-\epsilon}$, unless SETH fails.

Orthogonal Vectors

Orthogonal Vectors (OV)

Given sets $X, Y \subset \{0,1\}^d$, do there exist $x \in X$ and $y \in Y$ such that $\langle x, y \rangle = 0$?

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Orthogonal Vectors cannot be solved in time $\mathcal{O}((|X| \cdot |Y|)^{1-\epsilon} \text{poly}(d))$.

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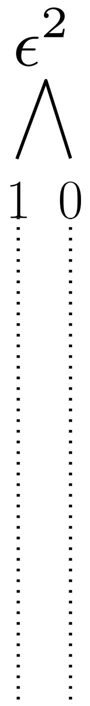
$\rightarrow$ SETH implies OVH [Williams '05]

OV $\leq \delta_H^T$ in L_1

Vector Gadgets



$$x = (0, 1, 1, 0, 1, 0, 0)$$

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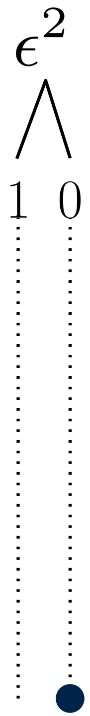

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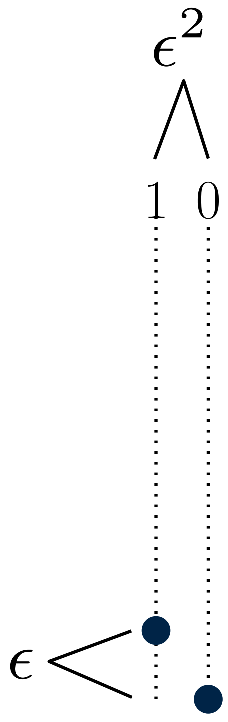

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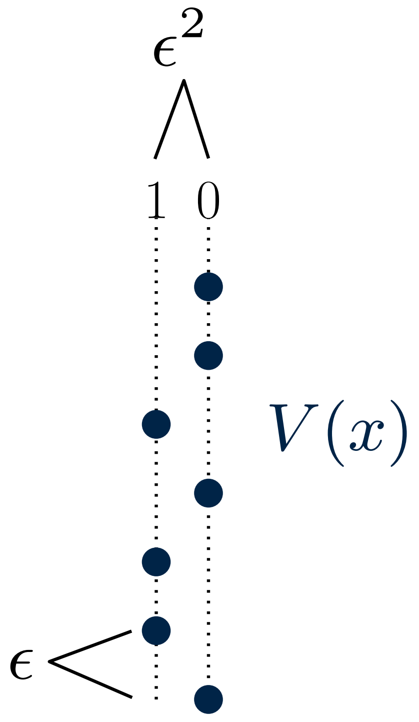

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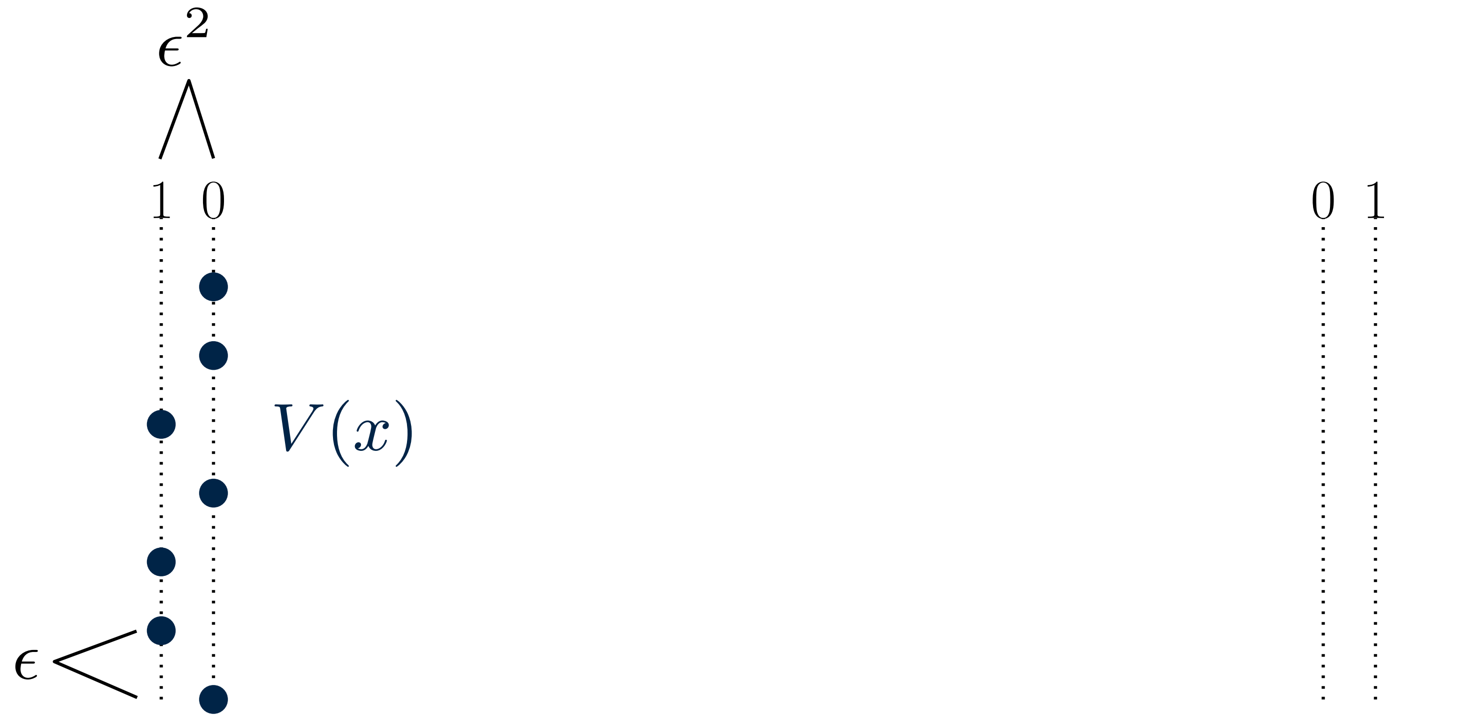

$x = (0, 1, 1, 0, 1, 0, 0)$ $y = (0, 1, 0, 1, 0, 0, 1)$

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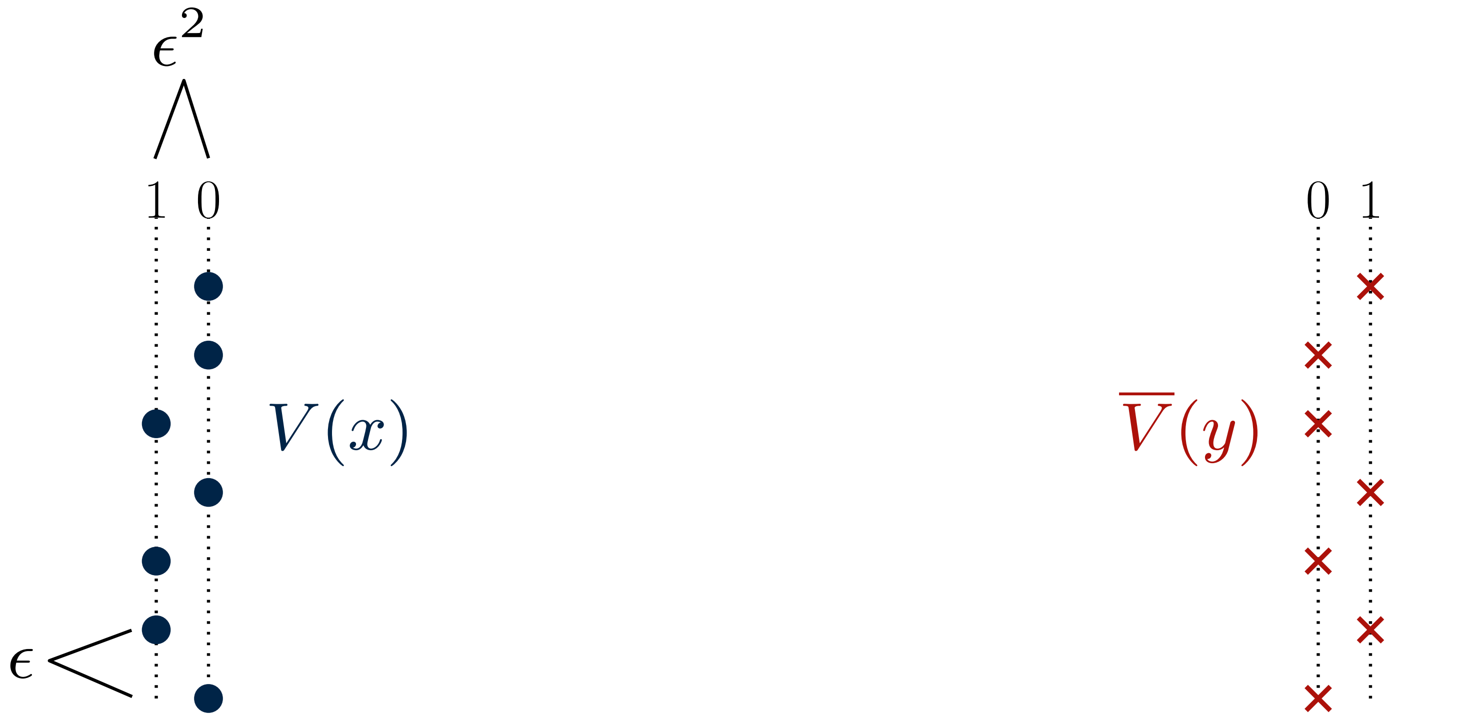

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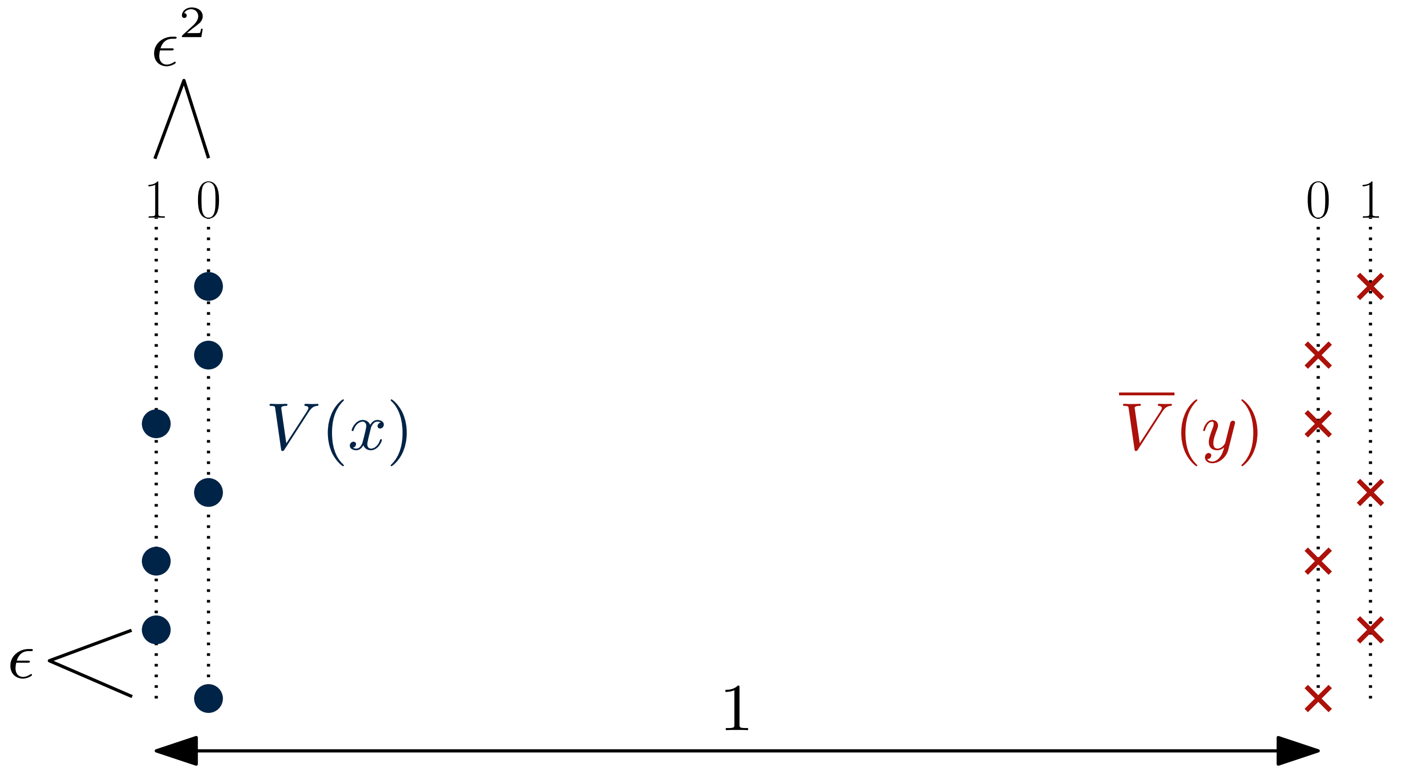

$x = (0, 1, 1, 0, 1, 0, 0)$ $y = (0, 1, 0, 1, 0, 0, 1)$

Hausdorff distance at most $1 \iff \langle x, y \rangle = 0$

OV $\leq \delta_H^T$ in L_1

OV Input:
$X = \{x_1, \ldots, x_4\}$
$Y = \{y_1, \ldots y_3\}$

Choice Gadget

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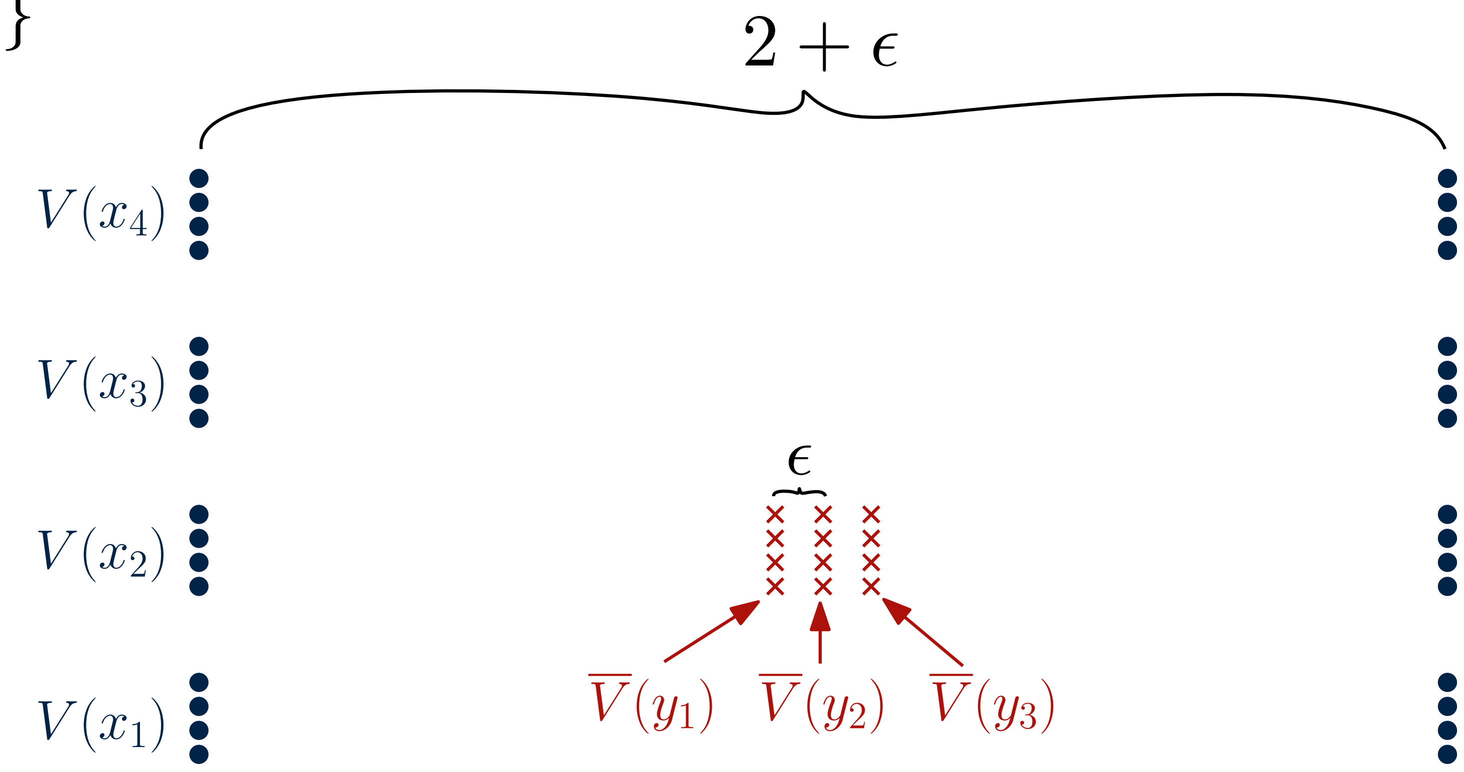

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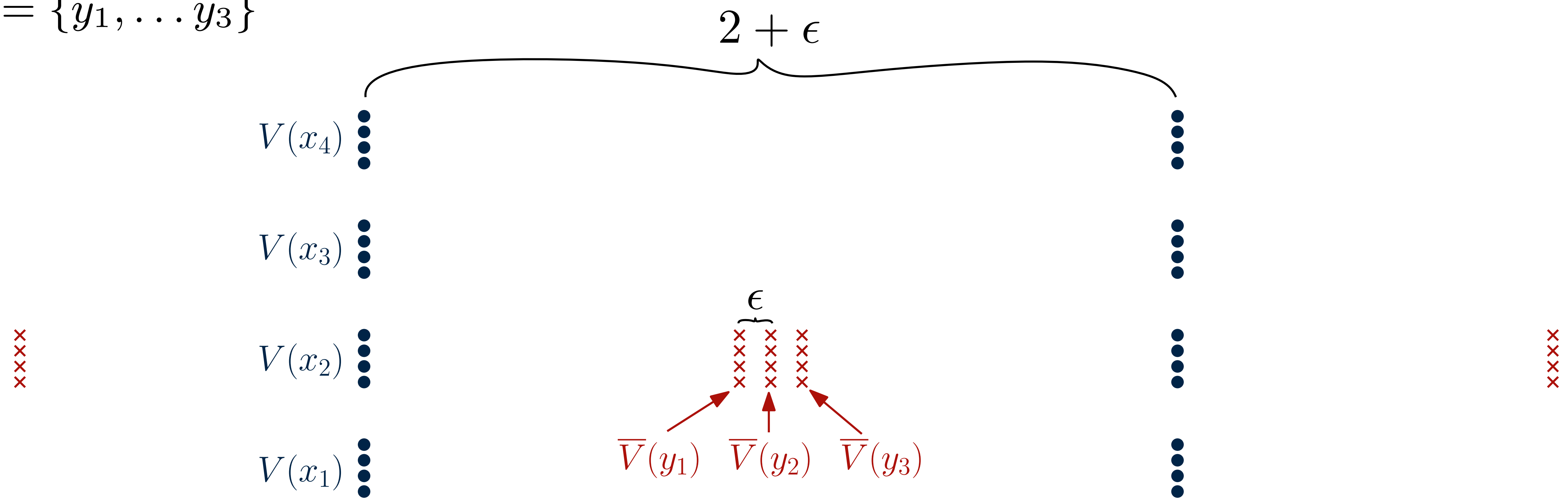

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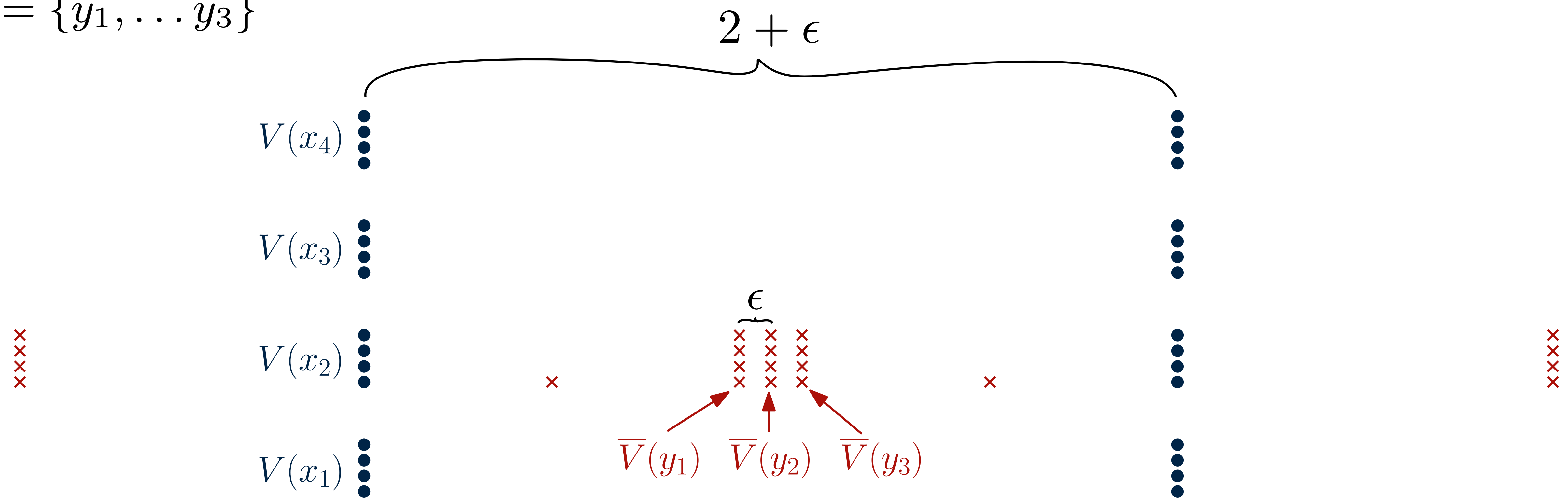

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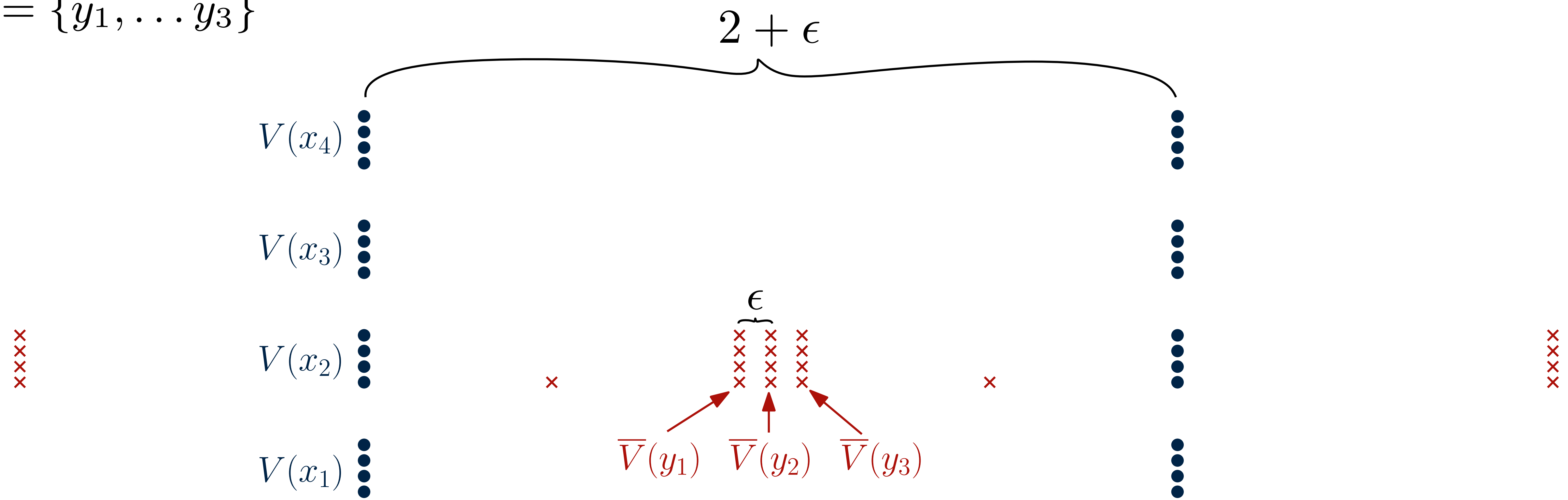

Hausdorff distance under translation at most 1 $\iff \exists i, j : \langle x_i, y_j \rangle = 0$

Second Reduction

Theorem

Hausdorff Distance Under Translation in L_2 with $m \in \mathcal{O}(1)$ cannot be computed in time $n^{2-\epsilon}$, unless the 3SUM Hypothesis fails.

3SUM

3SUM Problem

Given $X = (x_0, \ldots, x_{n-1}) \subset \{1, \ldots, \text{poly}(n)\}$, do there exist $i, j, k \in [n]$ s.t.

$$x_i + x_j = x_k$$

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3SUM Hypothesis

The 3SUM Problem cannot be solved in time $n^{2-\epsilon}$.

3SUM

Convolution

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Given $X = (x_0, \ldots, x_{n-1}) \subset \{1, \ldots, \text{poly}(n)\}$, do there exist $i, j, \cancel{k} \in [n]$ s.t.

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The Convolution 3SUM Problem cannot be solved in time $n^{2-\epsilon}$.

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Convolution

3SUM Hypothesis

The Convolution 3SUM Problem cannot be solved in time $n^{2-\epsilon}$.

→ 3SUM and Convolution 3SUM Hypotheses are equivalent [Pătrașcu '10]

Conv3SUM $\leq \delta_H^T$ in L_2

Encode the sequence $X = (x_0, \ldots, x_{n-1})$:

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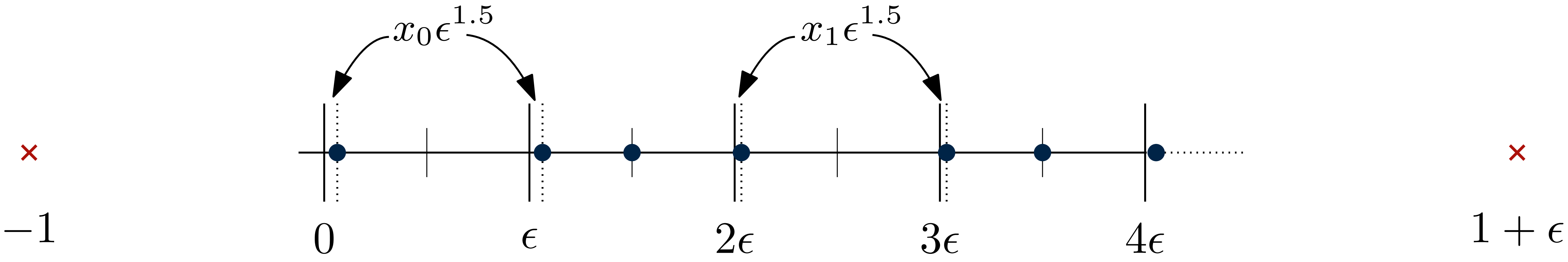

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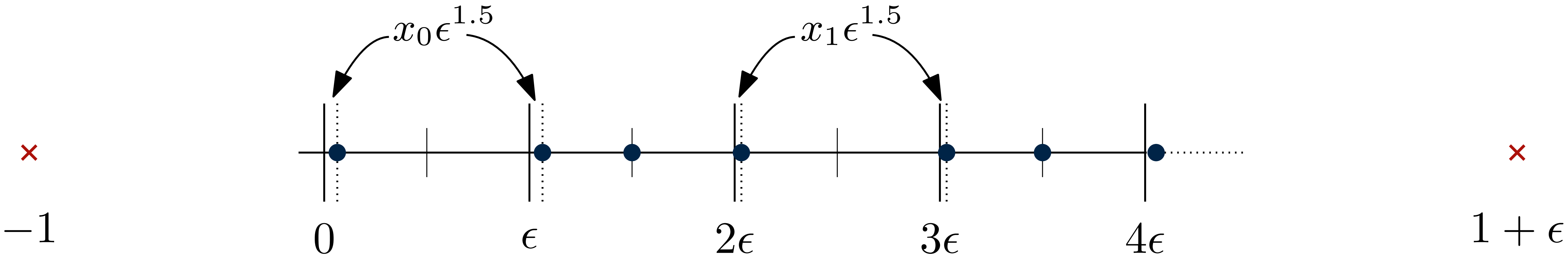
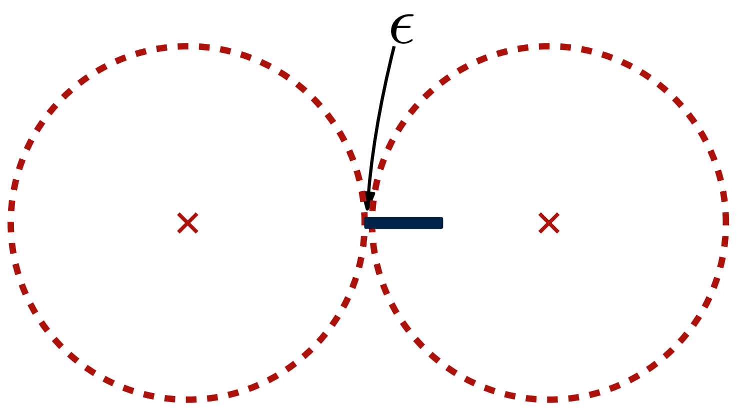


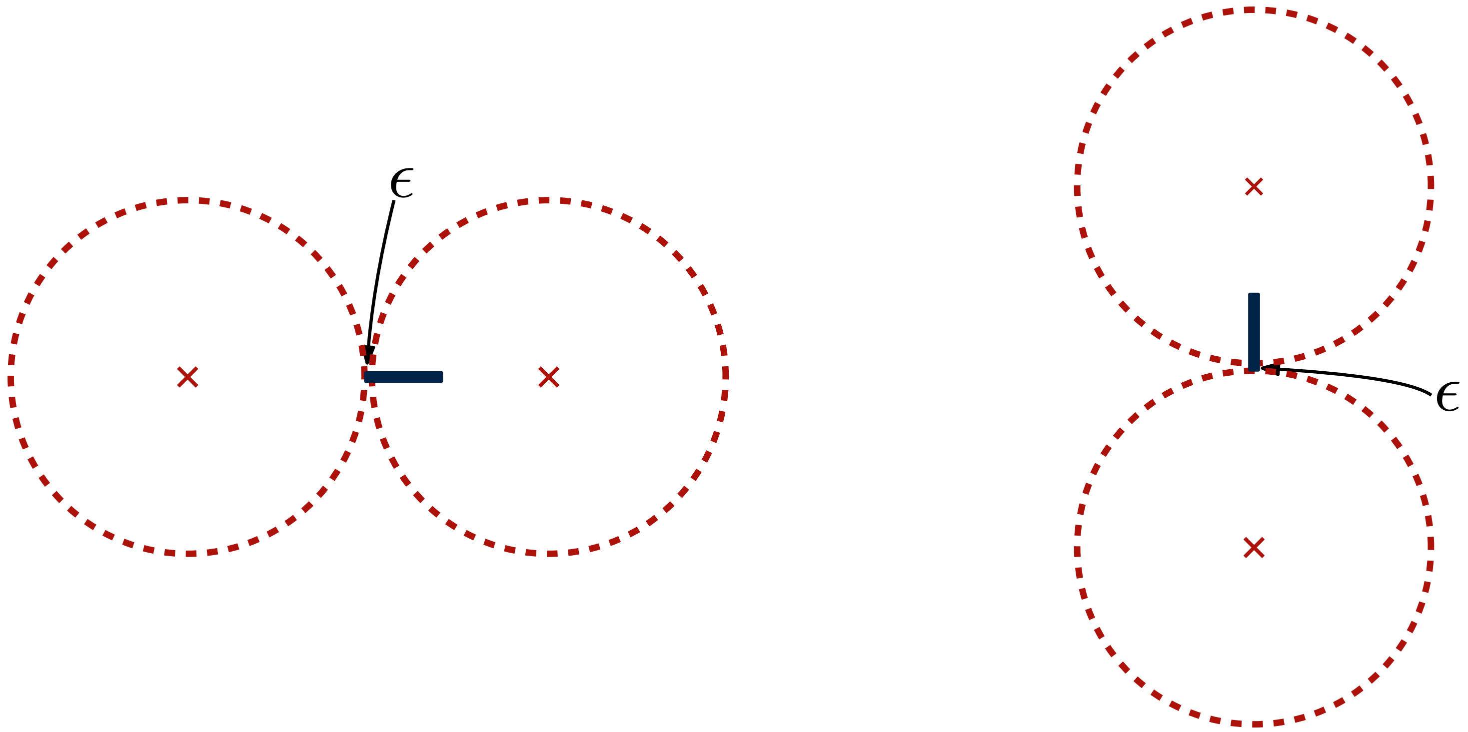
Feasible translations: $\tau_x \approx 2i\epsilon + x_i\epsilon^{1.5}$ for $i \in [n]$

Conv3SUM $\leq \delta_H^T$ in L_2



$$\tau_x \approx 2i\epsilon + x_i\epsilon^{1.5}$$

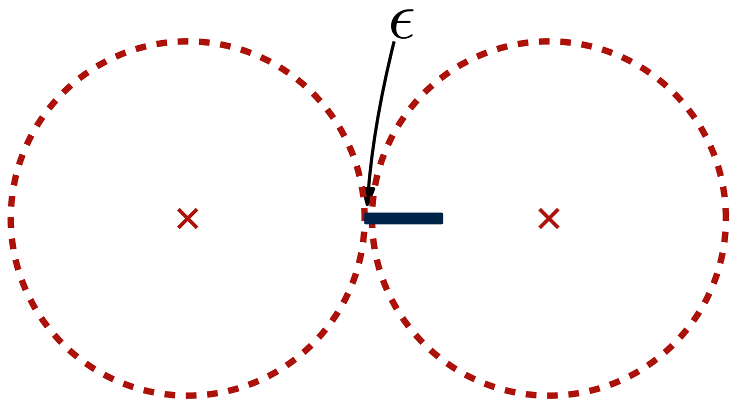
Conv3SUM $\leq \delta_H^T$ in L_2



$\tau_x \approx 2i\epsilon + x_i\epsilon^{1.5}$

$\tau_y \approx 2j\epsilon + x_j\epsilon^{1.5}$

Conv3SUM $\leq \delta_H^T$ in L_2



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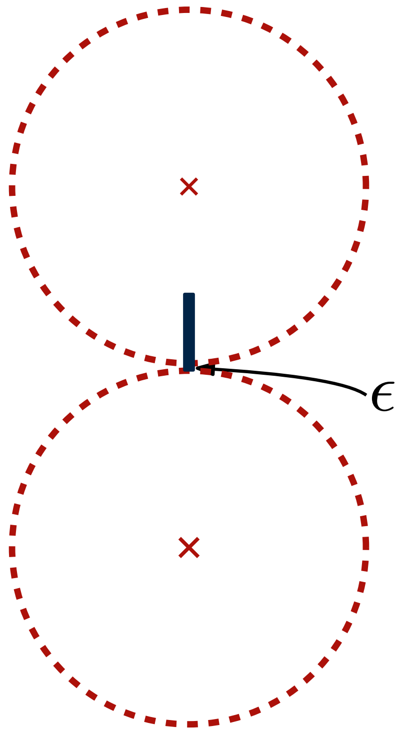

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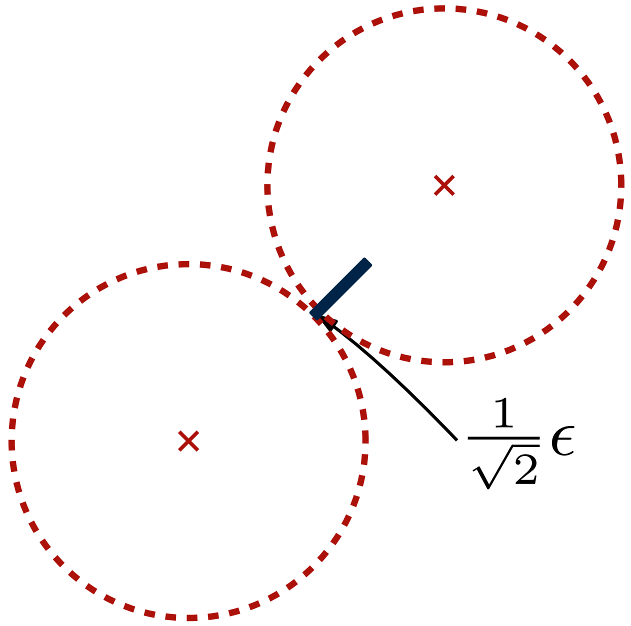

$\tau_x + \tau_y \approx 2k\epsilon + x_k\epsilon^{1.5}$

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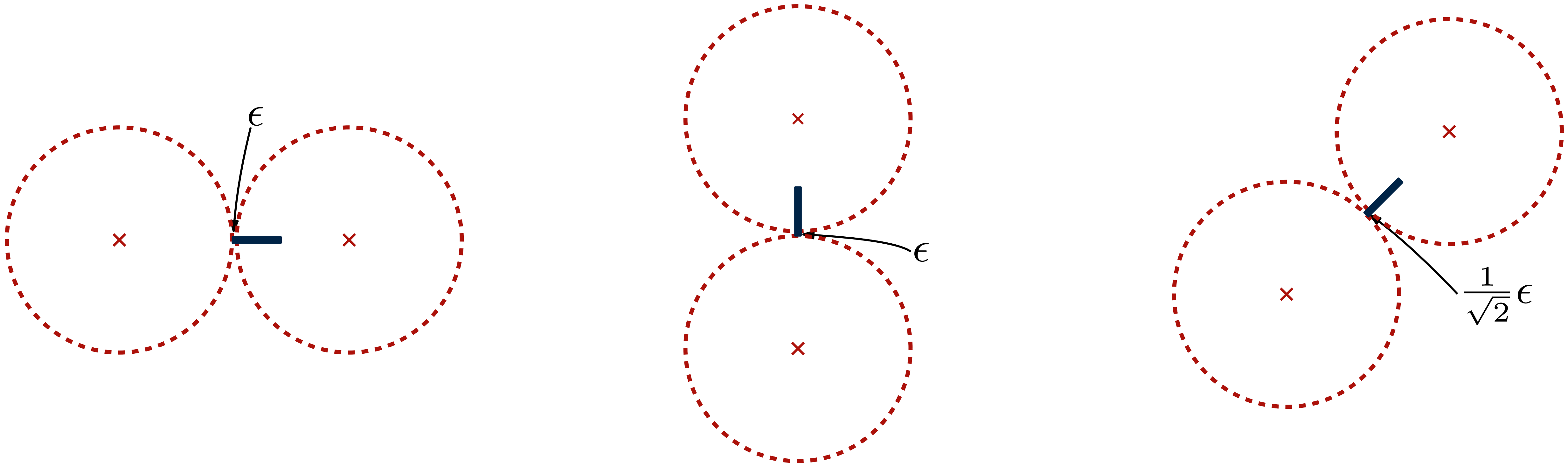
Conv3SUM $\leq \delta_H^T$ in L_2



$\tau_x \approx 2i\epsilon + x_i\epsilon^{1.5}$

$\tau_y \approx 2j\epsilon + x_j\epsilon^{1.5}$

$\tau_x + \tau_y \approx 2k\epsilon + x_k\epsilon^{1.5}$

$$\exists i,j,k \in [n] : 2i\epsilon + 2j\epsilon - 2k\epsilon + x_i\epsilon^{1.5} + x_j\epsilon^{1.5} - x_k\epsilon^{1.5} \approx 0$$

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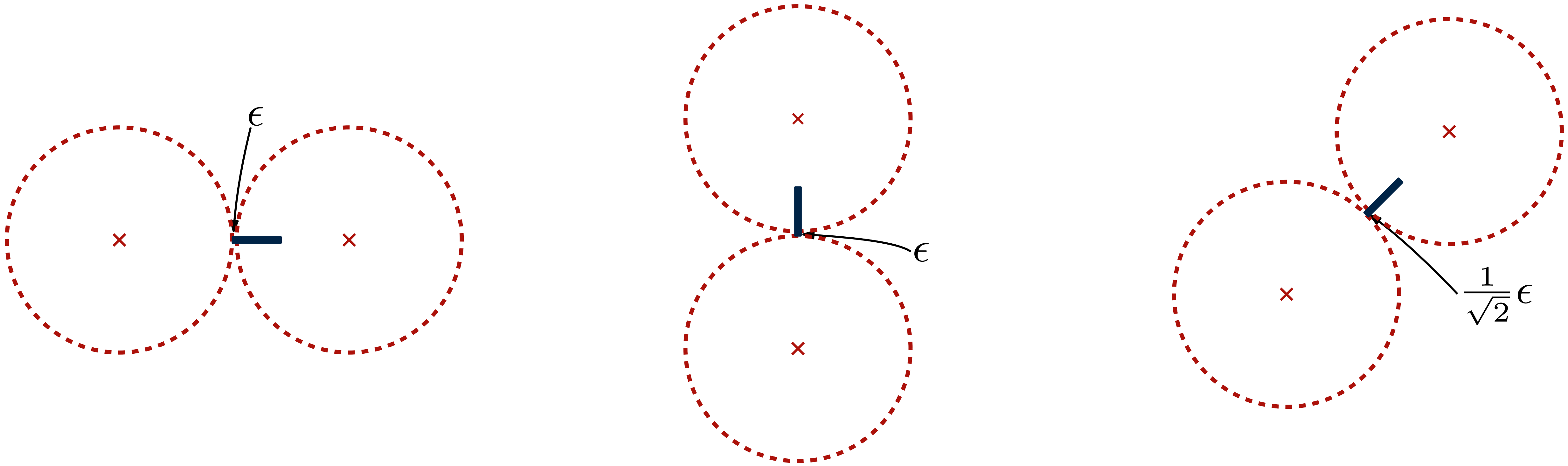
Conv3SUM $\leq \delta_H^T$ in L_2



$\tau_x \approx 2i\epsilon + x_i\epsilon^{1.5}$

$\tau_y \approx 2j\epsilon + x_j\epsilon^{1.5}$

$\tau_x + \tau_y \approx 2k\epsilon + x_k\epsilon^{1.5}$

$$\exists i, j, k \in [n] : 2i\epsilon + 2j\epsilon - 2k\epsilon + x_i\epsilon^{1.5} + x_j\epsilon^{1.5} - x_k\epsilon^{1.5} \approx 0$$

$$\iff \exists i, j, k \in [n] : i + j = k \text{ and } x_i + x_j = x_k$$

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Vielen Dank!

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