Exercise 1 (10 points).
Prove that there exists no decomposition of the form
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
\ast & 0 \\
\ast & \ast
\end{pmatrix} \begin{pmatrix}
\ast & \ast \\
\ast & 0
\end{pmatrix},
\]
where the stars are complex constants.

On the other hand, prove that \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
can be approximated arbitrarily closely by such decompositions.

Exercise 2 (10 points).
Let \(U_n\) denote the group of \(n \times n\) upper triangular matrices with 1s on the main diagonal. For \(1 \leq i < j \leq n\), \(\alpha \in \mathbb{C}\), let \(x_{ij}(\alpha) \in U_n\) denote the identity matrix with an entry \(\alpha\) in row \(i\) and column \(j\). Prove that \(U_n\) is generated as a group by the set \(\{x_{ij}(\alpha) \mid 1 \leq i < j \leq n, \alpha \in \mathbb{C}\}\).

Exercise 3 (10 points).
Let \(V\) be a polynomial \(\text{GL}_n\)-representation. For \(i < j\) let \(x_{ij}(\alpha)\) be defined as in Exercise 2. The raising operator \(E_{ij} : V \to V\) is a map defined via
\[
E_{ij}(v) = \lim_{\varepsilon \to 0} \varepsilon ((x_{ij}(\varepsilon)v) - v).
\]
Prove that \(E_{ij}\) is well-defined, i.e., that the limit exists. Furthermore, prove that \(E_{ij} : V \to V\) is a linear map.

Exercise 4 (10 points).
Let \(V\) be a polynomial \(\text{GL}_n\)-representation and let \(v \in V\) be of weight \(\lambda\) for \(\lambda \in \mathbb{N}^n\). Prove that \(E_{ij}(v) = \sum_{\mu \vdash \lambda} v_{\mu}\), where each \(v_{\mu}\) is either zero or a weight vector of weight \(\mu\).