Assignment 3  
due on Wednesday, May 2, 2018

Name: 

Exercise 1 (10 points).
Consider the action of $\mathbb{C}^{N \times N}$ on $\mathbb{C}[X_1, \ldots, X_N]_d$ defined in the lecture. Compute the following polynomial in the standard monomial basis:

$$
\begin{pmatrix}
2 & 3 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
(X_1^2 X_2 + X_3)
$$

Exercise 2 (15 points).
Prove that the Waring rank of a polynomial is always finite by proving that

$$
X_1 X_2 \cdots X_d = \frac{1}{d! 2^{d-1}} \sum_{\alpha \in \{-1, 1\}^{d-1}} \left( \prod_{i=1}^{d-1} \alpha_i \right) (\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_{d-1} X_{d-1} + X_d)^d.
$$

Exercise 3 (15 points).
Let $\text{GL}_n$ denote the group of invertible complex $n \times n$ matrices. Let $G = \text{GL}_n \times \text{GL}_n$ and let $V = \mathbb{C}^{n \times n}$. Define an action of $G$ on $V$ by

$$(g_1, g_2)v := g_1 \cdot v \cdot g_2^t,$$

where “$\cdot$” is the product of matrices. Let $v \in V$ have matrix rank $\text{rk}(v) = k$. Prove that

$$
Gv = \{ w \in V | \text{rk}(w) = k \}.
$$