Glivenko and Kuroda for Simple Type Theory

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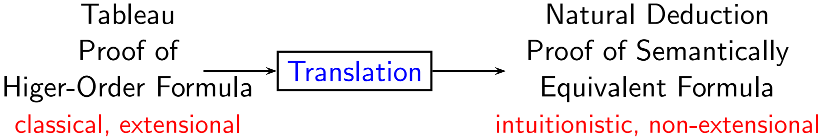
Joint work with: Chad E. Brown (Saarland University)

Student Session at POPL 2013
25.01.2013

What Did We Do?



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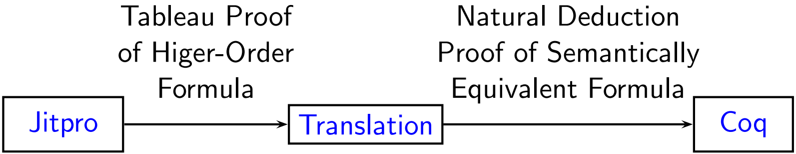

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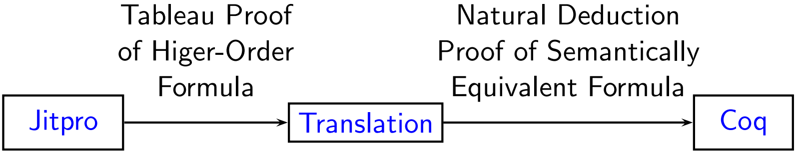

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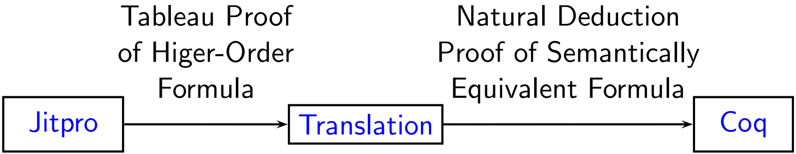

- Jitpro is a new interactive tableau prover
- Coq well known tool, could be used as a proof checker
- Aim: use Coq to check proofs by Jitpro

What will we Talk about Today?

Classical Proof of Higer-Order Formula → Translation → Intuitionistic Proof of Semantically Equivalent Formula

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- Intuitionistic logic does not contain the axiom $p \vee \neg p$ or equivalently $\neg\neg p \rightarrow p$.

- In classical logic more can be proven but less can be expressed.
- Intuitionistic proof of an existence statement gives a witness for the statement.

Example of Existence in the Classical Sense

- Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{I}$ be the set of irrational numbers.
- Consider the statement $\exists x, y.(x \in \mathbb{I}) \wedge (y \in \mathbb{I}) \wedge (x^y \in \mathbb{Q})$.

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  - This formula has an intutionistic proof.
- **Aim:** Given a formula and a classical proof of the formula, find a (classically) equivalent formula and an intuitionistic proof if it.

Short Logic Recap

- Propositional logic
  - No quantifiers
  - $p \vee \neg p$
- First-order logic
  - Quantification only over predicates
  - $\forall p.p \vee \neg p$
- Higher-order logic (Simple Type Theory)
  - Quantification also over functions
  - $\forall f.\forall g.fx = gx$

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- First-Order Logic
  - Kolmogorov negative translation
  - Gödel-Gentzen negative translation
  - Kuroda negative translation

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- Higher-Order Logic
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Result

We prove that:

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Result

We prove that:

- Glivenko's translation
  - Extends to higher-order logic without $\forall$.
  - Does not extend when $\forall$ is added.
- Kuroda's negative translation
  - Extends to higher-order logic
- Kolmogorov and Gödel-Gentzen negative translations do not extend to higher-order logic.

Extensionality Axioms

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- Propositional extensionality
  - If two propositions are equivalent then they are equal.
  - $\forall p, q.(p \equiv q) \rightarrow (p = q)$

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- Propositional extensionality
  - If two propositions are equivalent then they are equal.
  - $\forall p, q.(p \equiv q) \to (p = q)$
- Functional extensionality
  - Two functions are equal if given the same input they always produce the same output.
  - $\forall f, g.(\forall x. f\ x = g\ x) \to (f = g)$

Recall

We prove that:

- Glivenko's translation
  - Extends to higher-order logic without $\forall$.
  - Does not extend when $\forall$ is added.
- Kuroda's negative translation
  - Extends to higher-order logic

More Detailed Result

We prove that:

- Glivenko's translation
  - Extends to non-extensional higher-order logic without $\forall$.
  - Does not extend when $\forall$ is added.
- Kuroda's negative translation
  - Extends to non-extensional higher-order logic

More Detailed Result

We prove that:

- Glivenko's translation
  - Extends to non-extensional higher-order logic without $\forall$.
  - Does not extend when $\forall$ is added.
- Kuroda's negative translation
  - Extends to non-extensional higher-order logic

Both extend when propositional extensionality is added.

More Detailed Result

We prove that:

- Glivenko's translation
  - Extends to non-extensional higher-order logic without $\forall$.
  - Does not extend when $\forall$ is added.
- Kuroda's negative translation
  - Extends to non-extensional higher-order logic

Both extend when propositional extensionality is added.
Neither extend when functional extensionality is added.

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Thanks for Listening

Questions?

Full Paper:

http://www.mpi-inf.mpg.de/~crizkall/Publications/BrownRizkallah2011.pdf

References

1. Berghofer, S. and Nipkow, T. 2000. Proof Terms for Simply Typed Higher Order Logic. In *Proceedings of the 13th International Conference on Theorem Proving in Higher Order Logic* (August 14 - 18, 2000). M. Aagaard and J. Harrison, Eds. Lecture Notes In Computer Science, vol. 1869. Springer-Verlag, London, 38-52.
2. Hindley, J. R. 1997 *Basic Simple Type Theory*. Cambridge University Press.
3. Girard, J., Taylor, P., and Lafont, Y. 1989 *Proofs and Types*. Cambridge University Press.
4. Troelstra, A. S. and Schwichtenberg, H. 1996 *Basic Proof Theory*. Cambridge University Press.