Discrepancy in Different Numbers of Colors

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Abstract

In this article we investigate the interrelation between the discrepancies of a given hypergraph in different numbers of colors. Being an extreme example we determine the multi-color discrepancies of the $k$-balanced hypergraph $\mathcal{H}_{nk}$ on partition classes of (equal) size $n$. Let $c, k, n \in \mathbb{N}$. Set $k_0 := k \mod c$ and $b_{nk,c} := \left( n - \left\lfloor \frac{n}{t} \right\rfloor \right) \frac{k}{c}$. For the discrepancy in $c$ colors we show

$$b_{nk,c} \leq \text{disc}(\mathcal{H}_{nk}, c) < b_{nk,c} + 1,$$

if $k_0 \neq 0$, and $\text{disc}(\mathcal{H}_{nk}, c) = 0$, if $c$ divides $k$. This shows that in general there is little correlation between the discrepancies of $\mathcal{H}_{nk}$ in different numbers of colors. If $c$ divides $k$ though, $\text{disc}(\mathcal{H}, c) \leq \frac{k}{c} \text{disc}(\mathcal{H}, k)$ holds for any hypergraph $\mathcal{H}$.

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1 Introduction and Results

The basic problem of combinatorial discrepancy theory is to partition the vertices of a hypergraph in such a way that all hyperedges are split into roughly equal parts by this partition. Discrepancy measures the deviation of an optimal solution to this ideal. Usually the maximum occurring imbalance is used as a measure.

Up to very recently, only partitions into two classes have been studied. For a beautiful survey on this see [BS95]. The very recent development can be found in [Mat99]. As the partitions are normally represented by colorings, we call this part of the theory “two-color discrepancy theory”. In 1999, A. Srivastav and the author introduced the notion of multi-color discrepancies in [DS99]. They defined the c-color discrepancy of a hypergraph $\mathcal{H}$ by

$$\text{disc}(\mathcal{H}, c) := \min_{\chi : \chi \to [c]} \max_{E \in \mathcal{E}} \left| \chi^{-1}(i) \cap E \right| - \frac{|E|}{c}$$

and show that many classical results hold also in $c$ colors. Some results are even independent of the number of colors (e.g. the Beck–Fiala theorem bounding the discrepancy by twice the maximum degree).

In this article we will investigate the opposite phenomenon: We will show that there are hypergraphs that have very different discrepancies in different numbers of colors. We determine the discrepancy of the hypergraph $\mathcal{H}_{nk}$ on $nk$ vertices which can be partitioned into $k$ equal-sized classes such that a set $E$ of vertices is an edge if and only if it contains the same number of vertices in each class. Obviously, $\text{disc}(\mathcal{H}_{nk}, k) = 0$. Let $c, k, n \in \mathbb{N}$. Set $k_0 := k \mod c$. If $k_0 \neq 0$, we show

$$\text{(*)} \quad \text{disc}(\mathcal{H}_{nk}, c) = \left( n - \left\lfloor \frac{n}{\left\lfloor \frac{k_0}{c} \right\rfloor} \right\rfloor \right) \frac{k_0}{c} + \varepsilon,$$

for some $\varepsilon \in [0, 1]$. Comparing this to the maximum possible $c$-color discrepancy $\left\lceil \frac{n}{k} \right\rceil (1 - \frac{1}{c})$ of a hypergraph on $nk$ vertices, we see that there is little correlation between $\text{disc}(\mathcal{H}_{nk}, c)$ and $\text{disc}(\mathcal{H}_{nk}, k)$. We will also see that the optimal colorings satisfying (**) are not at all unique but can look quite different. There is though some correlation if $c$ divides $k$: In this situation $\text{disc}(\mathcal{H}, c) \leq \frac{k}{c} \text{disc}(\mathcal{H}, k)$ holds for any hypergraph $\mathcal{H}$. 

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2 Definition of Multi-Color Discrepancies

For a nonnegative integer $n$ we set $[n] := \{i \in \mathbb{N} | i \leq n\}$. Let $\mathcal{H} = (V,E)$ be a hypergraph, i.e. $V$ is a finite set and $E \subseteq 2^V$. A $c$-coloring of $\mathcal{H}$ is a mapping $\chi : X \to [c]$. The basic idea (measuring the deviation from the ideal) motivates the definitions of the discrepancy of an edge $E \in E$ in color $i \in [c]$ with respect to $\chi$ by

\[
\text{disc}_{\chi,i}(E) := \left| \chi^{-1}(i) \cap E \right| - \frac{|E|}{c},
\]

the discrepancy of $\mathcal{H}$ with respect to $\chi$ by

\[
\text{disc} \!(\mathcal{H},\chi) := \max_{i \in [c], E \in E} \text{disc}_{\chi,i}(E)
\]

and the discrepancy of $\mathcal{H}$ in $c$ colors by

\[
\text{disc}(\mathcal{H}, c) := \min_{\chi : X \to [c]} \text{disc} \!(\mathcal{H},\chi).
\]

This is all we need for our purpose. For a deeper insight in multi-color discrepancies see [DS99]. To get some intuition and comparison let us determine the multi-color discrepancies of the complete hypergraph, which is of course the worst case.

**Lemma 2.1.** Let $n \in \mathbb{N}$. The complete hypergraph $\mathcal{H} = ([n], 2^{[n]})$ on $n$ vertices has $c$-color discrepancy

\[
\text{disc}(\mathcal{H}) = \left(1 - \frac{1}{c}\right) \left\lfloor \frac{n}{c} \right\rfloor.
\]

**Proof.** Let $\chi : [n] \to [c]$ be any coloring. Then there is a color $j \in [c]$ such that $E := \chi^{-1}(j)$ has at least $\left\lceil \frac{n}{c} \right\rceil$ points. As $E$ is an edge of $\mathcal{H}$, we have $\text{disc}(\mathcal{H}) \geq \left| E \cap \chi^{-1}(j) \right| - \frac{1}{c} |E| = \left(1 - \frac{1}{c}\right) |E| \geq \left(1 - \frac{1}{c}\right) \left\lfloor \frac{n}{c} \right\rceil$. For the upper bound consider a coloring such that all color classes deviate in size by at most 1, i.e., for all $j \in [c]$ we have $\left\lfloor \frac{n}{c} \right\rceil \leq \chi^{-1}(j) \leq \left\lceil \frac{n}{c} \right\rceil$. From the definition of multi-color discrepancy it is clear that we just have to consider the two extreme cases of an edge $E^+$ having exactly $\left\lceil \frac{n}{c} \right\rceil$ points in one color $j^+$ and an edge $E^-$ having $n - \left\lceil \frac{n}{c} \right\rceil$ points and avoiding one color class $\chi^{-1}(j^-)$. We compute $\text{disc}_{\chi,j^+}(E^+) = \left(1 - \frac{1}{c}\right) \left\lceil \frac{n}{c} \right\rceil$ and $\text{disc}_{\chi,j^-}(E^-) = \frac{1}{c} \left(n - \left\lceil \frac{n}{c} \right\rceil\right)$. Hence $\text{disc}(\mathcal{H}) \leq \max\{\text{disc}_{\chi,j^+}(E^+), \text{disc}_{\chi,j^-}(E^-)\} = \left(1 - \frac{1}{c}\right) \left\lfloor \frac{n}{c} \right\rceil$. This shows the claim. $\square$
3 Proofs

If $c_2$ divides $c_1$, then the discrepancy in $c_2$ colors can be bounded in terms of the one in $c_1$ colors:

**Lemma 3.1.** Let $\mathcal{H}$ be any hypergraph. If $c_2$ divides $c_1$, then $\text{disc}(\mathcal{H}, c_2) \leq \frac{c_1}{c_2} \text{disc}(\mathcal{H}, c_1)$.

**Proof.** Let $\chi : X \to [c_1]$ be an optimal $c_1$–coloring of $\mathcal{H}$. Set $q = \frac{c_1}{c_2}$. Define $\chi_2 : X \to [c_2]$ by $\chi_2(x) = \lfloor \frac{\chi(x)}{q} \rfloor + 1$. For an edge $E \in \mathcal{E}$ and a color $j \in [c_2]$ we compute

$$
\text{disc}_{\chi_2,j}(E) = \left| |E \cap \chi_2^{-1}(j)| - \frac{1}{c_2} |E| \right|
= \left| \left| E \cap \bigcup_{k=1}^{q} \chi^{-1}((j-1)q + k) \right| - \frac{q}{c_1} |E| \right|
= \sum_{k=1}^{q} \left| \left| E \cap \chi^{-1}((j-1)q + k) \right| - \frac{1}{c_1} |E| \right|
\leq \sum_{k=1}^{q} \text{disc}_{\chi,(j-1)q+k}(E).
$$

This marks the extreme case where the correlation between different numbers of colors is high. For arbitrary numbers of colors a quite different phenomenon can be observed. We investigate the opposite behavior through the following class of hypergraphs. Fix $n, k \in \mathbb{N}$. For $l \in [k]$ set $L_l := [n] \times \{l\}$ and call these sets *lines*. Set $\mathcal{E}_{n,k} := \{ E \subseteq [n] \times [k] \}$. Call this set the (modulo isomorphism) unique hypergraph on $nk$ vertices of $k$–color discrepancy zero with maximal number of edges. We determine the discrepancy of this hypergraph in any number of colors. This will show that there is no correlation apart from the case exhibited in Lemma 3.1.
Theorem 3.2. Let $k_0 := k \mod c$. Set $b_{nk} = \left(n - \left\lfloor \frac{n}{t} \right\rfloor \right) \frac{k}{c}$. Then

$$b_{nk} \leq \text{disc}(\mathcal{H}_{nk}, c) < b_{nk} + 1,$$

if $k_0 \neq 0$, and $\text{disc}(\mathcal{H}_{nk}, c) = 0$, if $c$ divides $k$.

It will be convenient to consider the case $c > kn$ ("more colors than points") separately. In this case, any coloring $\chi$ avoids some colors, and for such a color $j$ we have $\text{disc}_{\chi,j}([n] \times [k]) = \frac{nk}{c}$. For an occurring color $j$ there is an edge $E$ of size $k$ containing at least one point colored $j$. Hence $\text{disc}_{\chi,j}(E) \geq 1 - \frac{k}{c}$. On the other hand, a coloring such that every color occurs at most once shows that we actually have $\text{disc}(\mathcal{H}_{nk}, c) = \max\{\frac{nk}{c}, 1 - \frac{k}{c}\}$. As $b_{nk} \leq \max\{\frac{nk}{c}, 1 - \frac{k}{c}\} < 1$, the case $c > kn$ is settled.

The case that $c$ divides $k$ is solved by Lemma 3.1, so let us assume that $k_0 \neq 0$. To prove the theorem we start with an easy observation:

**Lemma 3.3.** In the notation of Theorem 3.2 we have

$$\text{disc}(\mathcal{H}_{nk}, c) \leq \text{disc}(\mathcal{H}_{nk}, c).$$

**Proof.** Let $\chi_0$ be an optimal $c$-coloring for $\mathcal{H}_{nk}$. Set $X_0 = [n] \times \{c \left\lfloor \frac{k}{c} \right\rfloor + 1, \ldots, c \left\lfloor \frac{k}{c} \right\rfloor + k_0 = k\}$ and $\sigma : X_0 \to [n] \times [k_0]$; $(i, l) \mapsto (i, l - c \left\lfloor \frac{k}{c} \right\rfloor)$. Then $\sigma$ is a hypergraph isomorphism from $\mathcal{H}_{nk}|X_0$ to $\mathcal{H}_{nk_0}$. Define a coloring $\chi : [n] \times [k] \to [c]$ by

$$\chi((i, l)) := \begin{cases} 
\chi_0(\sigma(i, l)) & \text{if } (i, l) \in X_0 \\
\left\lfloor \frac{l-1}{\frac{k}{c}} \right\rfloor + 1 & \text{else}.
\end{cases}$$
Then for any edge \( E \in \mathcal{E} \) and any color \( j \in [c] \) we have

\[
\text{disc}_{\chi,j}(E) = \left| \frac{1}{c} |E \cap \chi^{-1}(j)| - \frac{1}{c} |E| \right| \\
= \left| \frac{1}{c} |E \cap X_0 \cap \chi^{-1}(j)| - \frac{1}{c} |E \cap X_0| + \left| \frac{1}{c} |(E \setminus X_0) \cap \chi^{-1}(j)| - \frac{1}{c} |(E \setminus X_0)| \right| \right| \\
= \left| \frac{1}{c} \sigma(E \cap X_0) \cap \chi_0^{-1}(j)| - \frac{1}{c} \sigma(E \cap X_0)| + \left| E \cap (\lfloor n \rfloor \times \{ j-1 \} , \lfloor \frac{n}{c} \rfloor + 1, \ldots, j \lfloor \frac{n}{c} \rfloor \} \right) \right| \\
= \left| \frac{1}{c} \sigma(E \cap X_0) \cap \chi_0^{-1}(j)| - \frac{1}{c} \sigma(E \cap X_0)| + 0 \right| \\
= \text{disc}_{\chi_0,j}(\sigma(E \cap X_0)).
\]

Since \( \sigma(E \cap X_0) \) is an edge of \( \mathcal{H}_{nk_0} \) and \( \chi_0 \) an optimal \( c \)-coloring for \( \mathcal{H}_{nk_0} \), we conclude \( \text{disc}(\mathcal{H}_{nk}, c) \leq \text{disc}(\mathcal{H}_{nk_0}, c) \). \( \square \)

It would save us some problems if we could show that an optimal coloring for \( \mathcal{H}_{nk} \) in the case \( c < k \) has to be of the kind constructed in Lemma 3.3. Unfortunately, this is not true:

Set \( k = 5 \) and \( c = 3 \). Let \( n \) be a non-negative integer divisible by 6. Let \( \chi \) be such that the color classes intersect the lines as described in the table below (this defines \( \chi \) up to permutations inside the lines which have no influence on the discrepancy).

| Line \( l \) | \( \frac{1}{c} |L_l \cap \chi^{-1}(1)| \) | \( \frac{1}{c} |L_l \cap \chi^{-1}(2)| \) | \( \frac{1}{c} |L_l \cap \chi^{-1}(3)| \) |
|-----------|-----------------|-----------------|-----------------|
| 1         | 1               | 0               | 0               |
| 2         | 0               | 1               | 0               |
| 3         | 1/6             | 0               | 5/6             |
| 4         | 0               | 1/6             | 5/6             |
| 5         | 1/2             | 1/2             | 0               |

We determine the discrepancy of this coloring: An edge \( E \) such that \( \text{disc}_{\chi,1}(E) \) is maximal has to have \( \frac{5}{6} n \) points and either contains all or none of the 1-colored points of line 3 and 5. Both cases yield a discrepancy of \( \frac{1}{6} n \) in color 1. The situation for color 2 is the same. For color 3 the extreme edges look like this: \( E \) has either no points in color 3 and size \( \frac{5}{6} n \) and thus \( \text{disc}_{\chi,3}(E) = \frac{5}{18} n \), or \( E \) contains all points in this color, has size \( \frac{5}{6} n \) and thus \( \text{disc}_{\chi,3}(E) = \frac{5}{18} n \).
From Theorem 3.2 we see that $\chi$ is an optimal coloring, but $\chi$ is not of the kind of coloring used to prove Lemma 3.3. This shows that the optimal coloring is not unique modulo permutations of lines and permutations inside lines.

Proof of Theorem 3.2. We investigate the case $c > k$ first, which implies $k = k_0$.

Case $c > k$, upper bound: For the upper bound we construct a coloring $\chi$. Start with all points being uncolored. For each color $j$, color $\left\lfloor \frac{n}{\left\lceil \frac{c}{k} \right\rceil} \right\rfloor$ points in this color. Do so in a way that points of the same color are in the same line. This is possible as each line can hold up to $\left\lceil \frac{c}{k} \right\rceil$ such color classes. The remaining points color in any way such that all color classes differ in size by at most one.

We calculate the discrepancy of $\mathcal{H}$ with respect to $\chi$. Let $j \in [c]$. Let $E$ be an edge such that $d_j(E) := |E \cap \chi^{-1}(j)| - \frac{1}{c}|E|$ is maximal. Assume that there is a point in color $j$ that is not contained in $E$. Let $F$ be a set of points disjoint from $E$ containing one point of every line, at least one of these colored $j$. Then

$$d_j(E \cup F) = d_j(E) + d_j(F) \geq d_j(E) + 1 - \frac{k}{c} > d_j(E),$$

a contradiction. Hence $\chi^{-1}(j) \subseteq E$ and $|E| \geq k \left\lfloor \frac{n}{\left\lceil \frac{c}{k} \right\rceil} \right\rfloor$ by construction. This yields

$$\text{disc}_{\chi,j}(E) = |E \cap \chi^{-1}(j)| - \frac{1}{c}|E| \leq \left\lfloor \frac{n}{c} \right\rfloor - \frac{k}{c} \left\lfloor \frac{n}{\left\lceil \frac{c}{k} \right\rceil} \right\rfloor. \quad (1)$$

Now let $E$ be an edge such that $d_j(E) := |E \cap \chi^{-1}(j)| - \frac{1}{c}|E|$ is minimal. By a similar argument as above we see that $E$ contains no point colored $j$. As $c \leq kn$, we have $|E| \leq \left( nk - k \left\lfloor \frac{n}{\left\lceil \frac{c}{k} \right\rceil} \right\rfloor \right)$ and hence

$$\text{disc}_{\chi,j}(E) = \left| |E \cap \chi^{-1}(j)| - \frac{1}{c}|E| \right| = \frac{1}{c}|E| \leq \frac{1}{c} \left( nk - k \left\lfloor \frac{n}{\left\lceil \frac{c}{k} \right\rceil} \right\rfloor \right). \quad (2)$$
From (1) and (2) we conclude

\[ \text{disc}(H_{nk}, c) \leq \text{disc}_\chi(H_{nk}) \leq \left\lceil \frac{nk}{c} \right\rceil - \frac{k}{c} \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil < b_{nkc} + 1. \]

**Case** \( c > k \) **lower bound:** For the lower bound let \( \chi \) be any coloring. Let \( m : [c] \to [k] \) such that \( |L_h \cap \chi^{-1}(j)| \leq |L_m(j) \cap \chi^{-1}(j)| \) for all \( h \in [k] \) and \( j \in [c] \). From the pigeon-hole principle we get a line number \( l \in [k] \) such that \( |m^{-1}(l)| \geq \left\lceil \frac{c}{k} \right\rceil \). Let \( j \in m^{-1}(l) \) such that \( |L_l \cap \chi^{-1}(j)| \) is minimal. Then for all \( h \in [k] \) we have \( |L_h \cap \chi^{-1}(j)| \leq |L_l \cap \chi^{-1}(j)| \leq \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \). Thus there is an edge of size \( \left( n - \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \right) k \) having no point in this color. We conclude

\[ \text{disc}_\chi(H_{nk}) \geq \left( n - \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \right) \frac{k}{c} = b_{nkc}. \]

**Case** \( c < k \): We now turn to the case that \( c < k \) and \( c \) does not divide \( k \). Lemma 3.3 proves the upper bound. For the lower bound let \( \chi \) be any \( c \)-coloring of \( H_{nk} \). Let \( i : [c] \times [k] \to [k] \) be such that \( |L_{i(j,p)} \cap \chi^{-1}(j)| \geq |L_{i(j,p+1)} \cap \chi^{-1}(j)| \) for all \( j \in [c], p \in [k - 1] \). Hence \( i(j,p) \) denotes the index of a line with \( p \)-th largest number of points in color \( j \).

Assume first that there exists a \( j \in [c] \) such that \( \sum_{p=1}^{[\frac{k}{c}]} |L_{i(j,p)} \cap \chi^{-1}(j)| \leq \left\lceil \frac{k}{c} \right\rceil n + \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \). If \( |L_{i(j,[\frac{k}{c}])} \cap \chi^{-1}(j)| < \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \), then there is an edge \( E \) such that \( |E| = k \left( n - \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \right) \) and \( |E \cap \chi^{-1}(j)| \leq \left\lceil \frac{k}{c} \right\rceil \left( n - \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \right) \), hence \( \text{disc}_\chi,j(E) \geq b_{nkc} \). If \( |L_{i(j,[\frac{k}{c}])} \cap \chi^{-1}(j)| \geq \left\lceil \frac{n}{\lfloor \frac{c}{k} \rfloor} \right\rceil \), then there is an edge \( E \) such that \( |E \cap L_l| = n - |L_{i(j,[\frac{k}{c}])} \cap \chi^{-1}(j)| \) for all \( l \in [k] \) and
\[ |E \cap \chi^{-1}(j)| \leq \left\lfloor \frac{k}{c} \right\rfloor n + \left\lfloor \frac{n}{\left\lfloor \frac{c}{k_0} \right\rfloor} \right\rfloor - \left\lfloor \frac{k}{c} \right\rfloor |L_{i,j\in[j]} \cap \chi^{-1}(j)|. \] In this case also

\[
\text{disc}_{\chi,j}(E) \geq \frac{k}{c} \left( n - \left| L_{i,j\in[j]} \cap \chi^{-1}(j) \right| \right) - \left\lfloor \frac{k}{c} \right\rfloor n \\
- \left\lfloor \frac{n}{\left\lfloor \frac{c}{k_0} \right\rfloor} \right\rfloor + \left\lfloor \frac{k}{c} \right\rfloor \left| L_{i,j\in[j]} \cap \chi^{-1}(j) \right| \\
= \frac{k_0}{c} n + (1 - \frac{k_0}{c}) \left| L_{i,j\in[j]} \cap \chi^{-1}(j) \right| - \left\lfloor \frac{n}{\left\lfloor \frac{c}{k_0} \right\rfloor} \right\rfloor \\
\geq \left( n - \left\lfloor \frac{n}{\left\lfloor \frac{c}{k_0} \right\rfloor} \right\rfloor \right) \frac{k_0}{c} = b_{n,k_0}
\]

holds. So let us assume from now on

\[
\sum_{p=1}^{\left\lfloor \frac{k}{c} \right\rfloor} |L_{i,j\in[j], p} \cap \chi^{-1}(j)| > \left\lfloor \frac{k}{c} \right\rfloor n + \left\lfloor \frac{n}{\left\lfloor \frac{c}{k_0} \right\rfloor} \right\rfloor \text{ for all } j \in [c]. \tag{3}
\]

Assume that there exists a color \( j \in [c] \) such that \( |L_{i,j\in[j]} \cap \chi^{-1}(j)| \leq n - \left\lfloor \frac{n}{\left\lfloor \frac{c}{k_0} \right\rfloor} \right\rfloor - 1 \). From (3) we conclude \( |L_{i,j\in[j]} \cap \chi^{-1}(j)| \geq 2 \left\lfloor \frac{n}{\left\lfloor \frac{c}{k_0} \right\rfloor} \right\rfloor + 2 =: m \) and \( k_0 \leq \frac{1}{3} c \). Thus there is an edge \( E \) such that \( |E \cap L_t| = m \) for all \( t \in [k] \) and \( |E \cap \chi^{-1}(j)| \geq \left\lfloor \frac{k}{c} \right\rfloor m \). Our assumptions yield \( \frac{1}{\left\lfloor \frac{c}{k_0} \right\rfloor} \geq \frac{3k_0}{4c} \). If \( \left\lfloor \frac{c}{k_0} \right\rfloor \) does
not divide \( n \), we have

\[
\text{disc}_{\chi,j}(E) \geq \left\lfloor \frac{k}{c} \right\rfloor m - \frac{km}{c}
\]

\[
= 2 \left\lfloor \frac{n}{\frac{c}{k_0}} \right\rfloor \left( 1 - \frac{k_0}{c} \right)
\]

\[
= (2 - \frac{k_0}{c}) \left\lfloor \frac{n}{\frac{c}{k_0}} \right\rfloor - \frac{k_0}{c} \left\lfloor \frac{n}{\frac{c}{k_0}} \right\rfloor
\]

\[
\geq (2 - \frac{k_0}{c}) \left( \frac{n}{\frac{c}{k_0}} - 1 \right) - \frac{k_0}{c} \left\lfloor \frac{n}{\frac{c}{k_0}} \right\rfloor
\]

\[
\geq (2 - \frac{1}{3}) \frac{3nk_0}{4c} + \frac{k_0}{2c} - \frac{3k_0^2}{4c^2} - \left( \frac{n}{\frac{c}{k_0}} \right) \frac{k_0}{c}
\]

\[
\geq \left( n - \left\lfloor \frac{n}{\frac{c}{k_0}} \right\rfloor \right) \frac{k_0}{c} = b_{nk_0c}.
\]

The calculation for the case that \( \left\lfloor \frac{c}{k_0} \right\rfloor \) is a divisor of \( n \) is about the same, a little easier actually, as we do not have to care about the fractional parts.

It remains to look at the case that for all \( j \in [c] \) we have \( \left| L_{i(j, \left\lceil \frac{c}{k_0} \right\rceil)} \cap \chi^{-1}(j) \right| > n - \left\lfloor \frac{n}{\frac{c}{k_0}} \right\rfloor - 1 \). In this case \( i([c] \times \left\lceil \frac{c}{k_0} \right\rceil) \) is injective and \( i([c] \times \left\lceil \frac{c}{k_0} \right\rceil) \cap i([c] \times \left\{ \left\lceil \frac{c}{k_0} \right\rceil \}) = \emptyset \). Thus \( i \) maps \([c] \times \left\lceil \frac{c}{k_0} \right\rceil\) into a set of size \( k_0 \). From the pigeonhole principle we conclude that there is a line \( L_l \) such that \( i(j, \left\lceil \frac{c}{k_0} \right\rceil) = l \) for \( \frac{c}{k_0} \) different colors \( j \). Again from the pigeon-hole principle we see that for at least one of these colors we have \( \left| L_{i(j, \left\lceil \frac{c}{k_0} \right\rceil)} \cap \chi^{-1}(j) \right| \leq \frac{n}{\frac{c}{k_0}} \) contradicting (3). This ends the proof of Theorem 3.2.

\[\square\]

**Remark:** The coloring \( \chi \) constructed in the proof above can be improved in some cases. According to the values of \( n, k \) and \( c \) it may be possible to
color what we called ‘the remaining points’ in such a way that for all colors classes with \( \left\lfloor \frac{n}{e} \right\rfloor \) points there is a line with \( \left\lfloor \frac{n}{e} \right\rfloor \) points in this color. This can improve the discrepancy of \( \chi \) up to another \( \frac{k}{e} \). We are not that much interested in this very last bit to give all the details.

4 Conclusion

Above we gave a class of hypergraphs which have very different discrepancies in different numbers of colors. If \( c \mid k \), \( \operatorname{disc}(\mathcal{H}_{nk}, c) = 0 \), if \( c \) does not divide \( k \), then \( \operatorname{disc}(\mathcal{H}_{nk}, c) \geq \left(n - \left\lfloor \frac{n}{\left\lfloor \frac{k}{c} \right\rfloor} \right\rfloor \right) \ln c \). In particular for \( k < c \ll nk \), \( \operatorname{disc}(\mathcal{H}_{nk}, c) \) is more than half the maximum discrepancy a hypergraph on \( nk \) vertices can have. This means that the correlation between the discrepancies in different numbers of colors is small.

Contrary to this, there are examples of hypergraphs that display a rather regular behavior. For hypergraphs having \( n \) vertices and \( n \) edges, an upper bound of \( O\left(\sqrt{\frac{n}{c}} \log c\right) \) was proven in [DS01]. On the other hand, an example due to Spencer constructed from Hadamard matrices was shown to have \( c \)-color discrepancy \( \Omega\left(\sqrt{\frac{n}{c}}\right) \) in [DS99]. The hypergraph \( \mathcal{H}_n \) of arithmetic progressions in \([n]\) fulfills \( C_1 c^{-0.5} \sqrt[4]{n} \leq \operatorname{disc}(\mathcal{H}_n, c) \leq C_2 c^{-0.16} \sqrt[4]{n} \) for constants \( C_1, C_2 > 0 \) independent of \( c \) and \( n \) (cf. [DS99, DS01]).

It seems to be an interesting phenomenon that some hypergraphs have a nice behavior like this and others do not. Apart from the general methods given in the papers cited, we do not know very much about this. Actually, also for the two examples above a full understanding is missing, as the gaps between the lower and upper bound show. We believe that these are questions worth some further research. A good understanding of these problems might also give some more insight in traditional two-color discrepancies.
References


