Fourier Analysis of Correlated Monte Carlo Importance Sampling

Gurprit Singh Kartic Subr David Coeurjolly Victor Ostromoukhov Wojciech Jarosz

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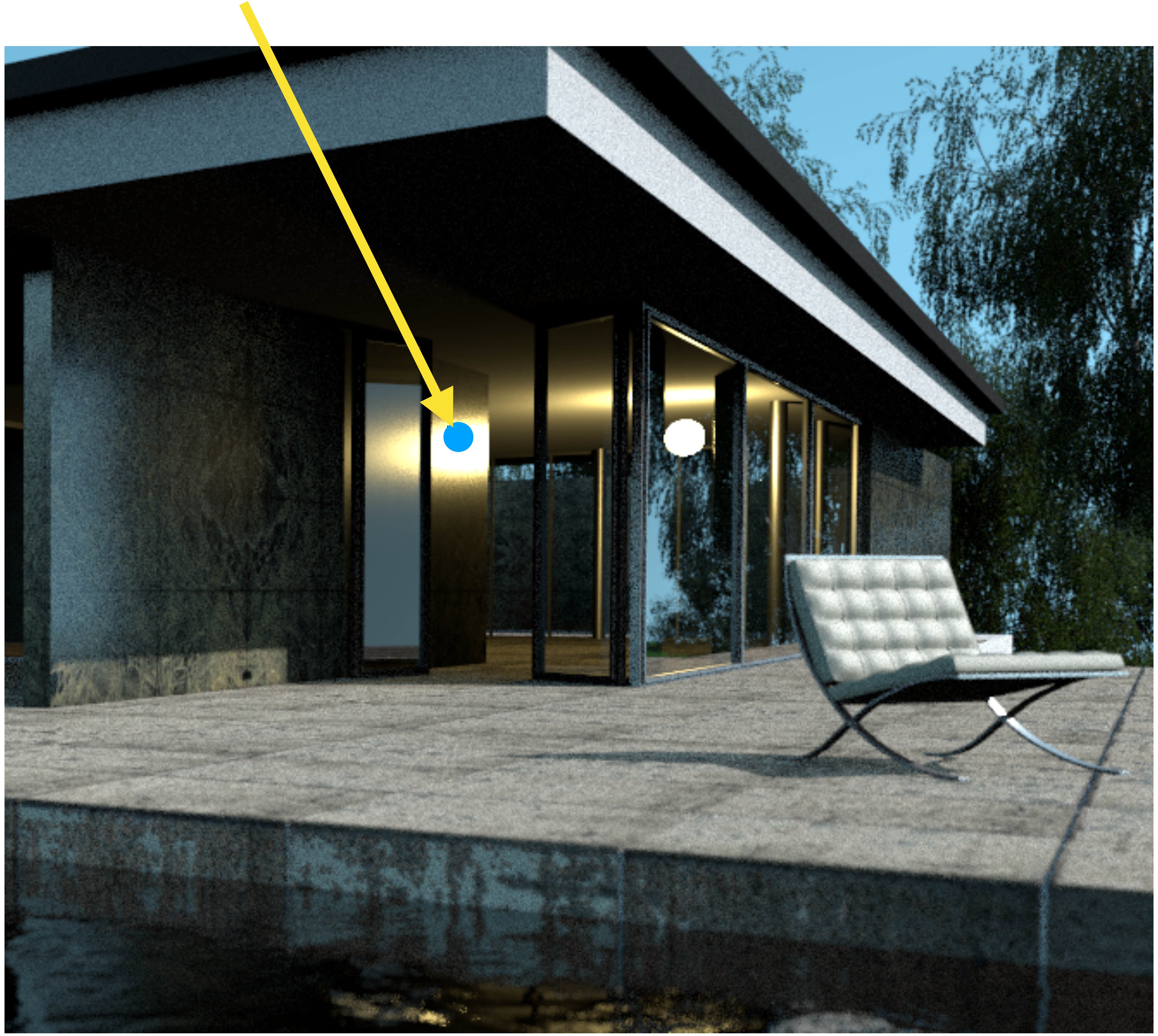
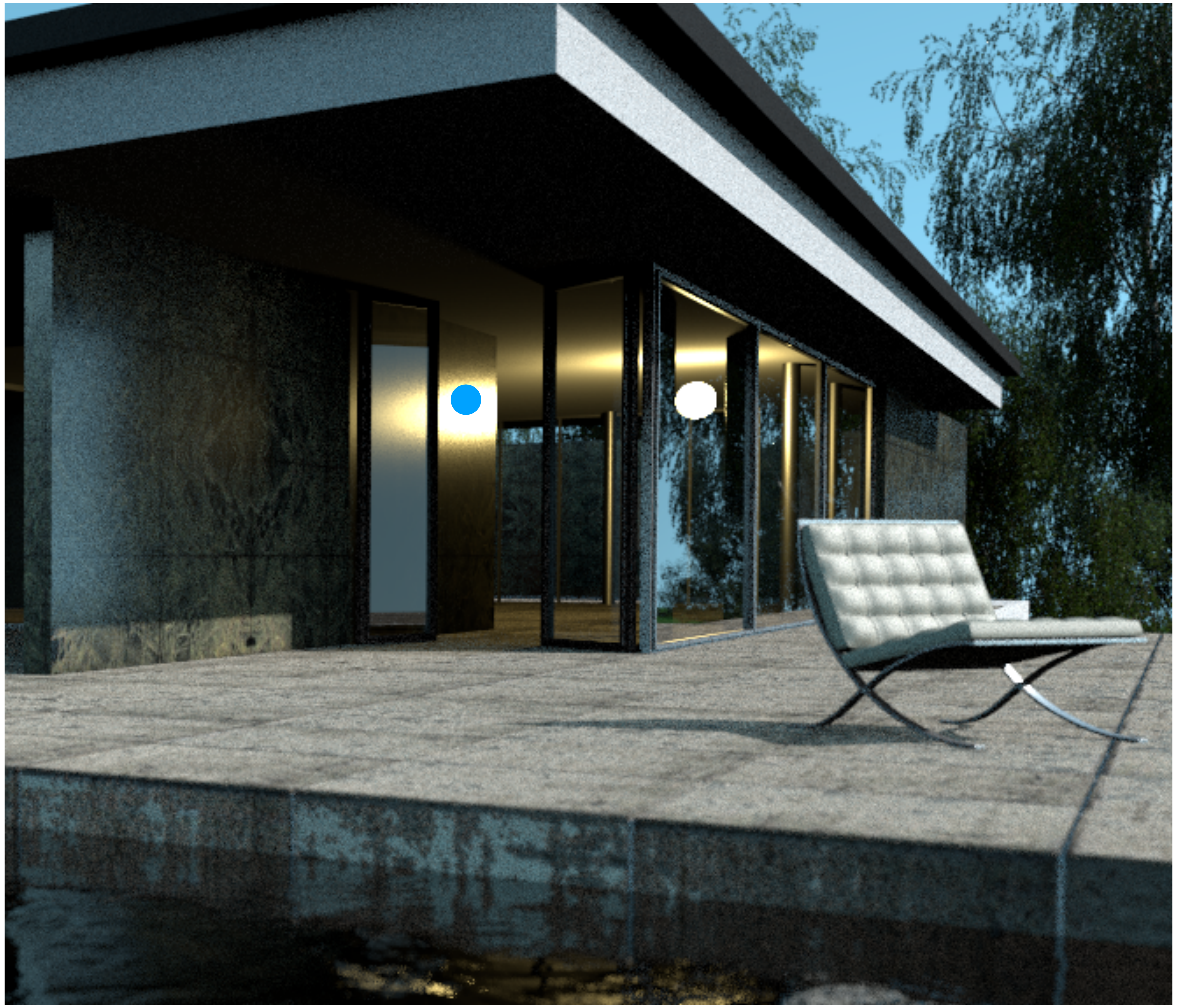
Computer Graphics Forum

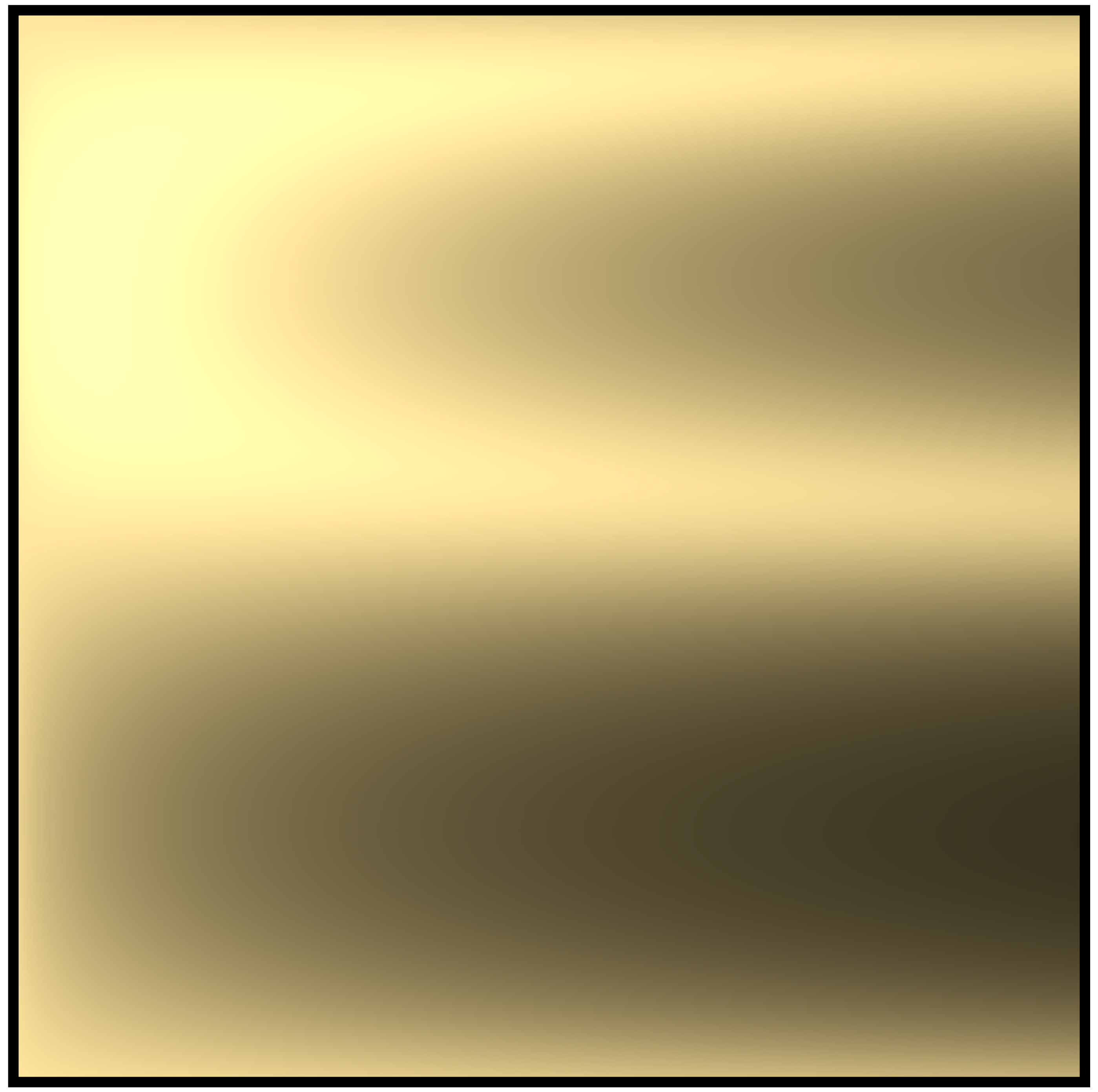
Monte Carlo Integration



Monte Carlo Integration

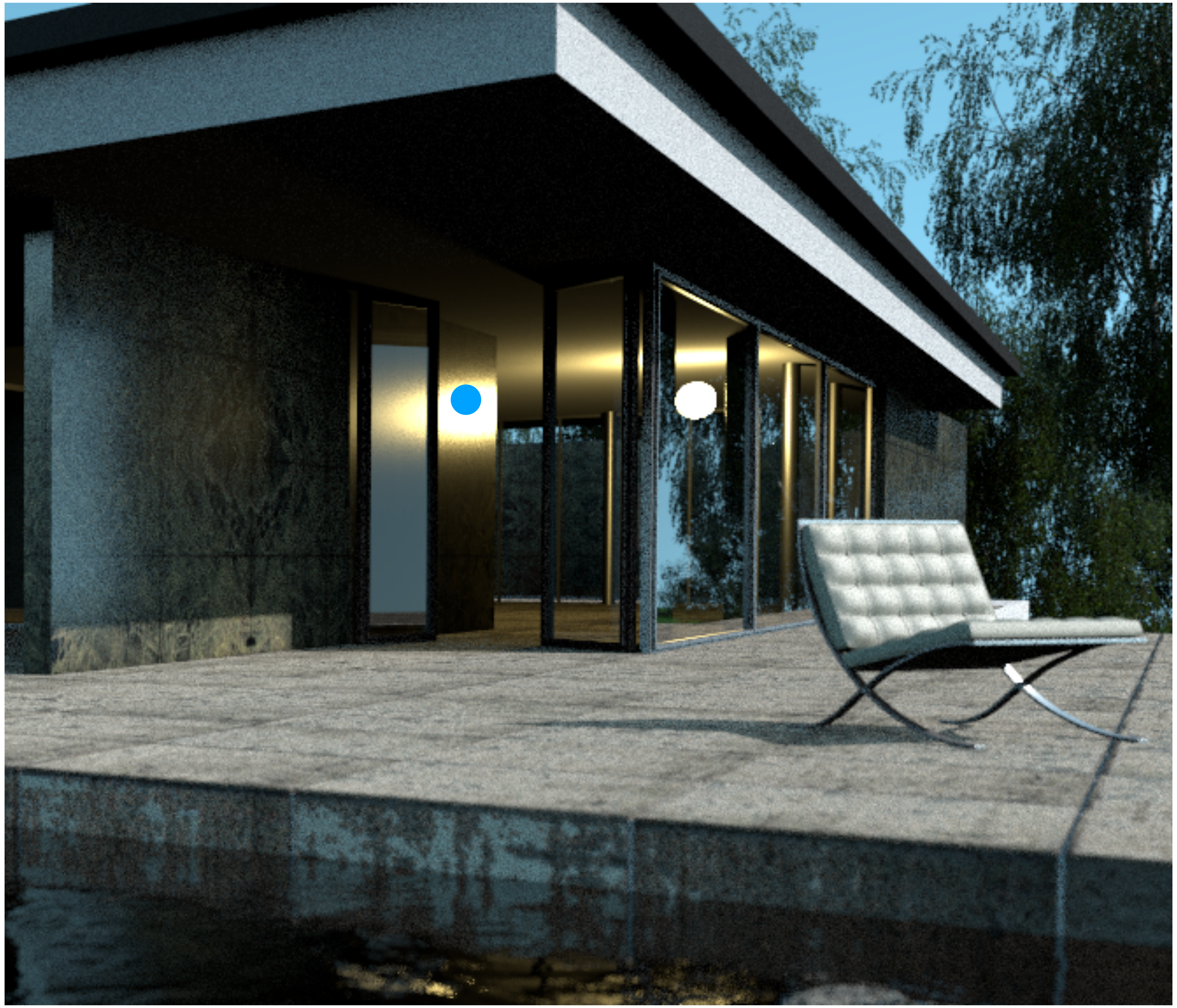


$f(\vec{x})$



$$I = \int_0^1 f(\vec{x})d\vec{x}$$

Monte Carlo Estimator



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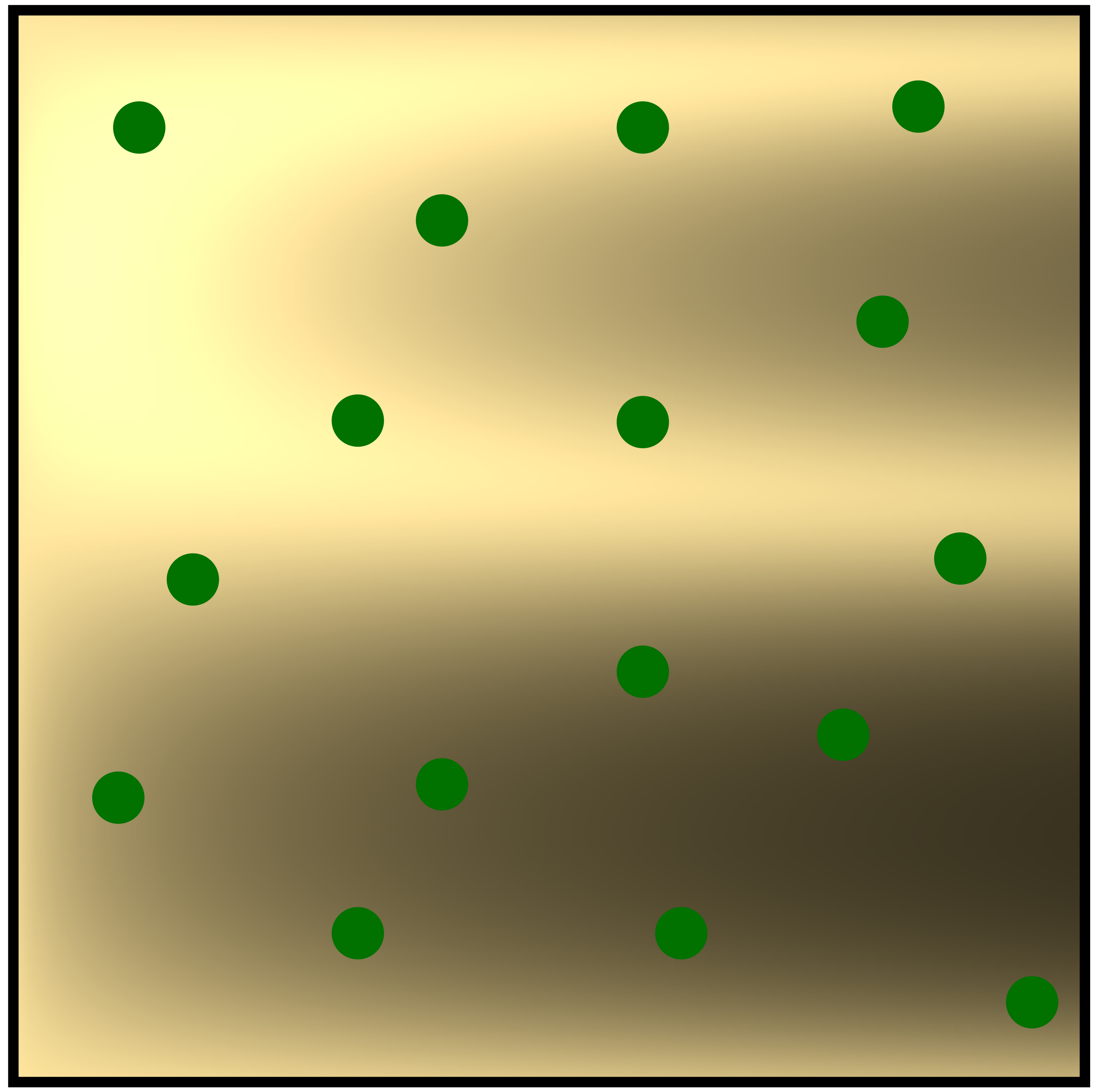
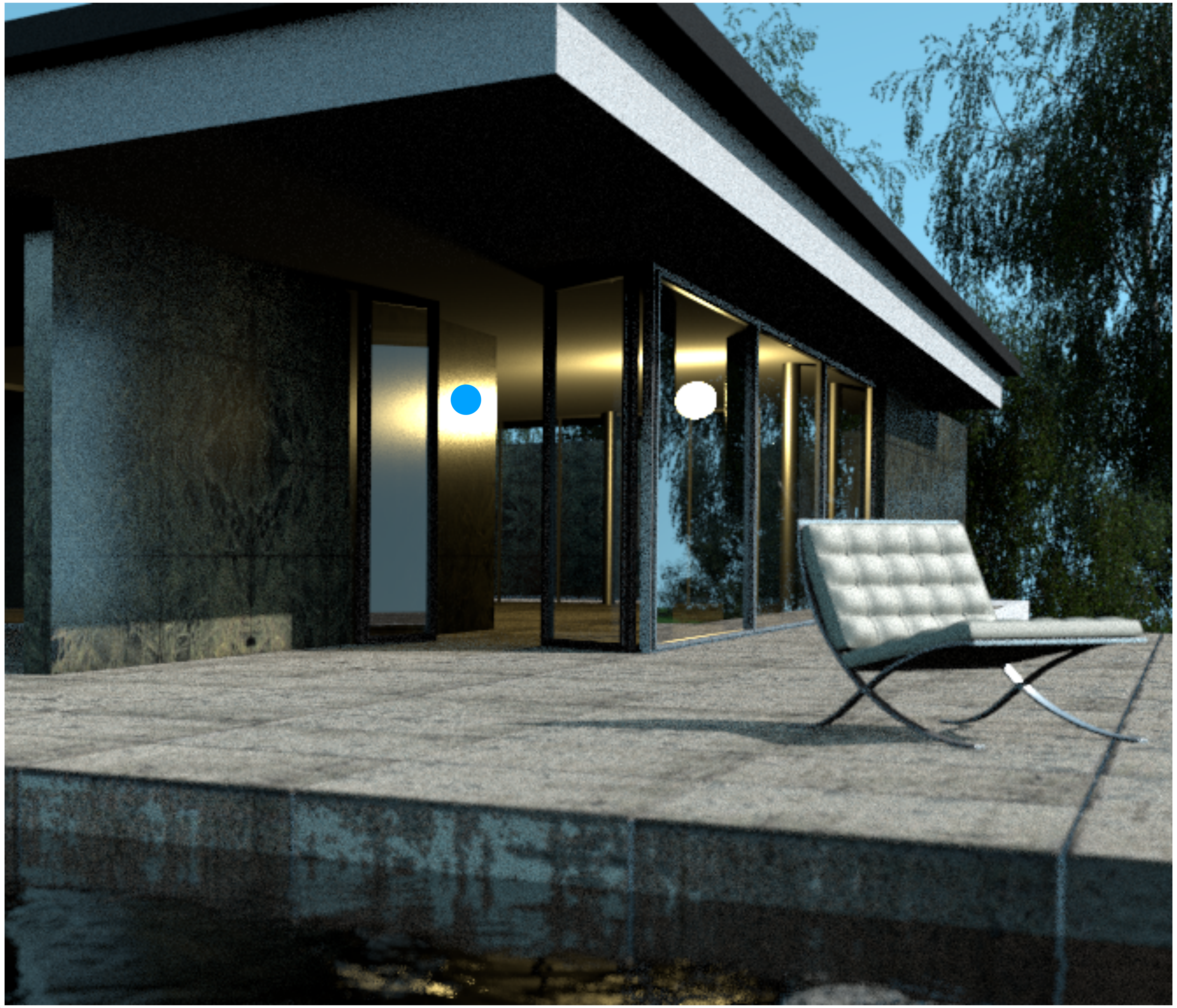


$$I_N = \frac{1}{N} \sum_{k=1}^{N} \frac{f(\vec{x}_k)}{p(\vec{x}_k)}$$

Monte Carlo Estimator



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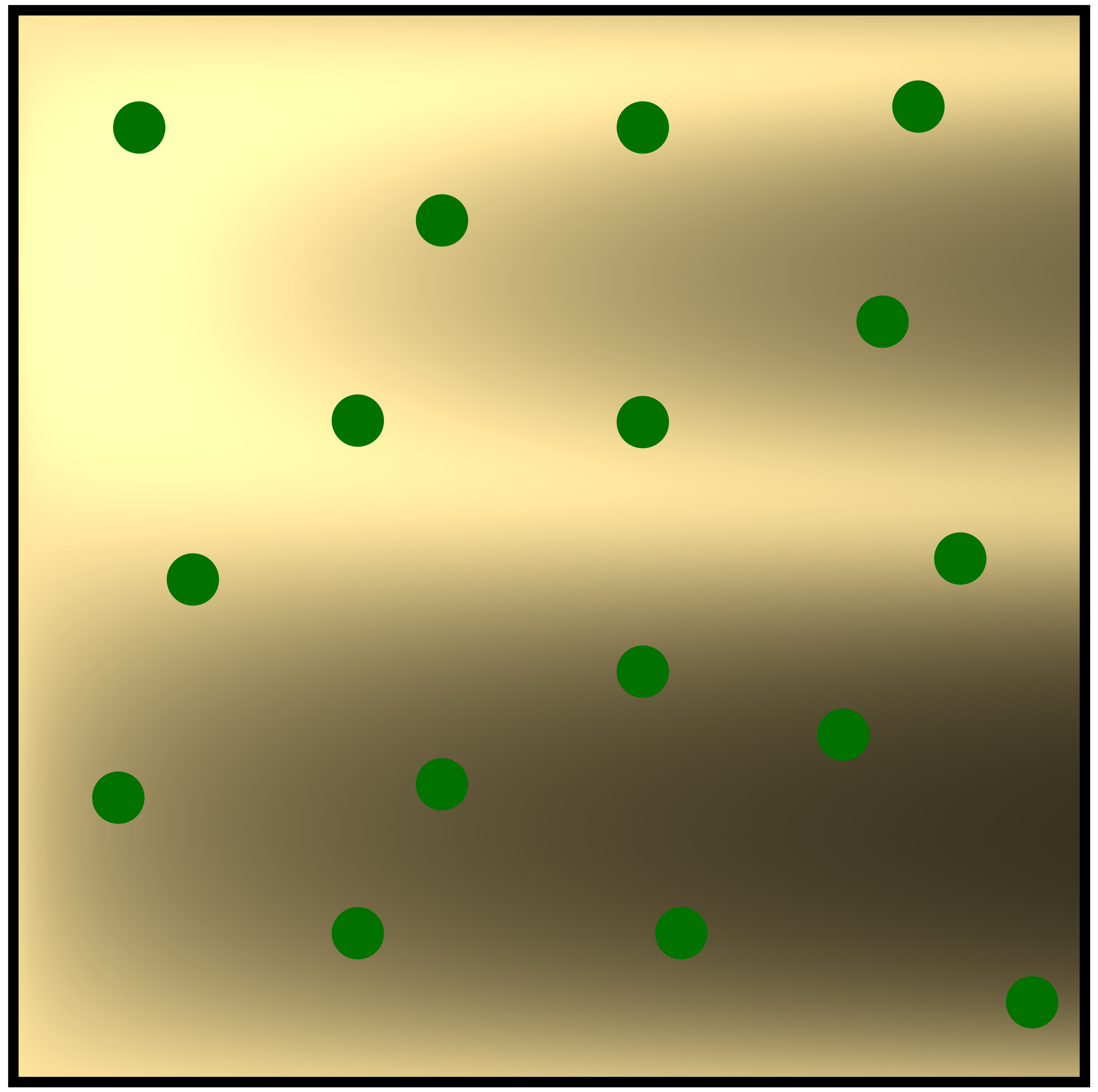

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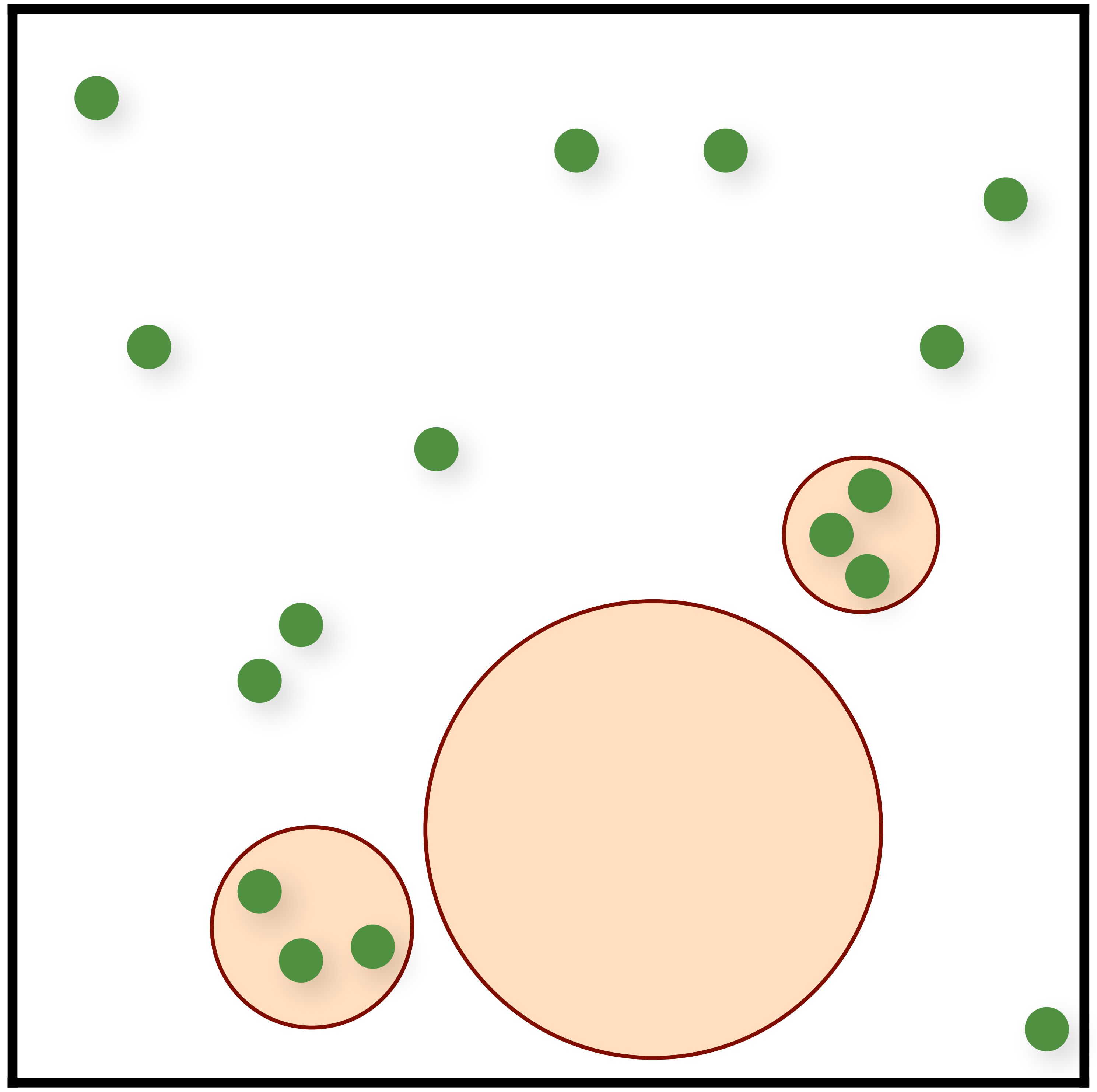
Random vs. Correlated Samples

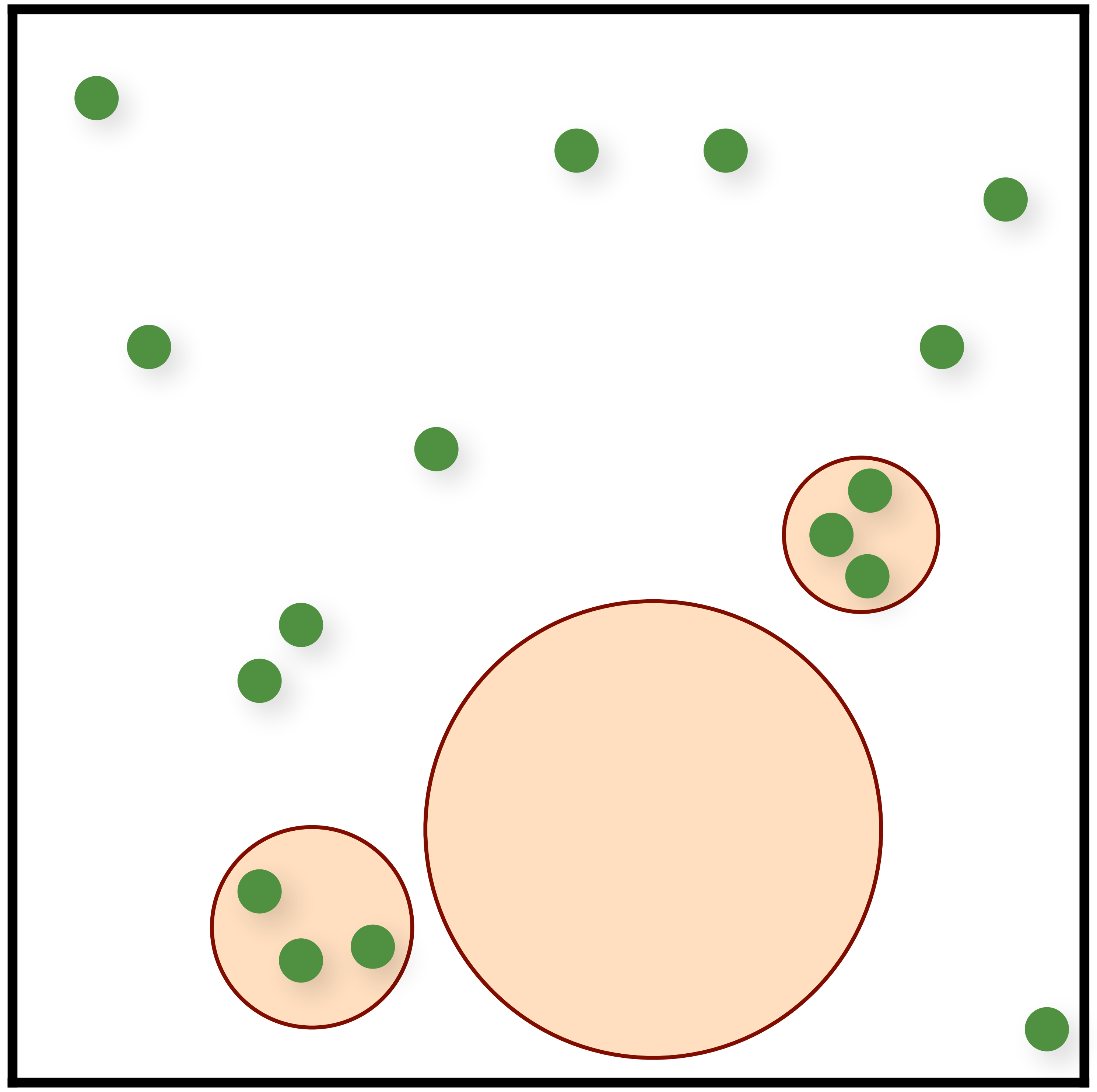


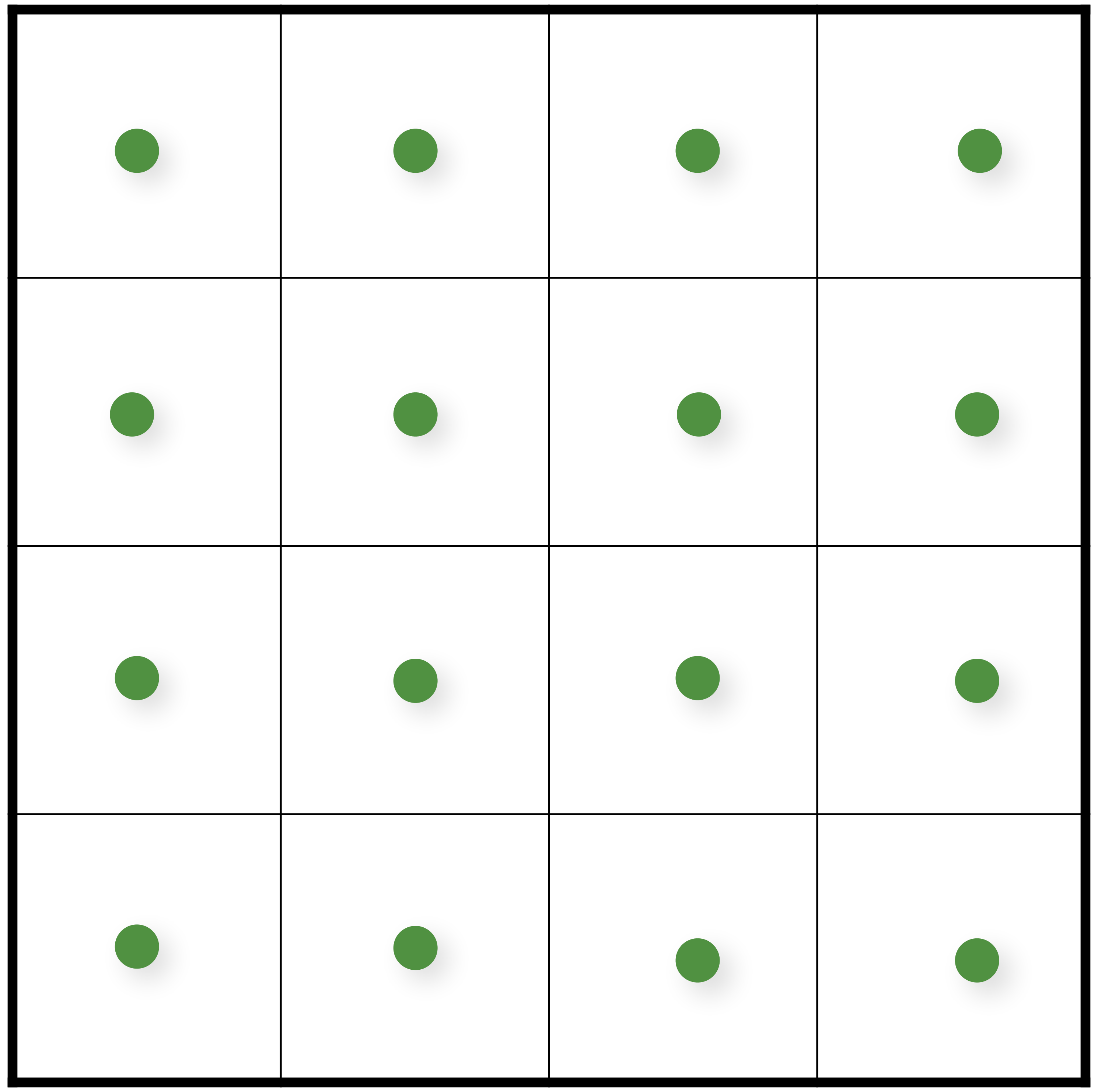
Random

$$S(\vec{x}) = \frac{1}{N} \sum_{k=1}^{N} \frac{\delta(\vec{x} - \vec{x}_k)}{p(\vec{x}_k)}$$

Slide after Wojciech Jarosz

Random vs. Correlated Samples

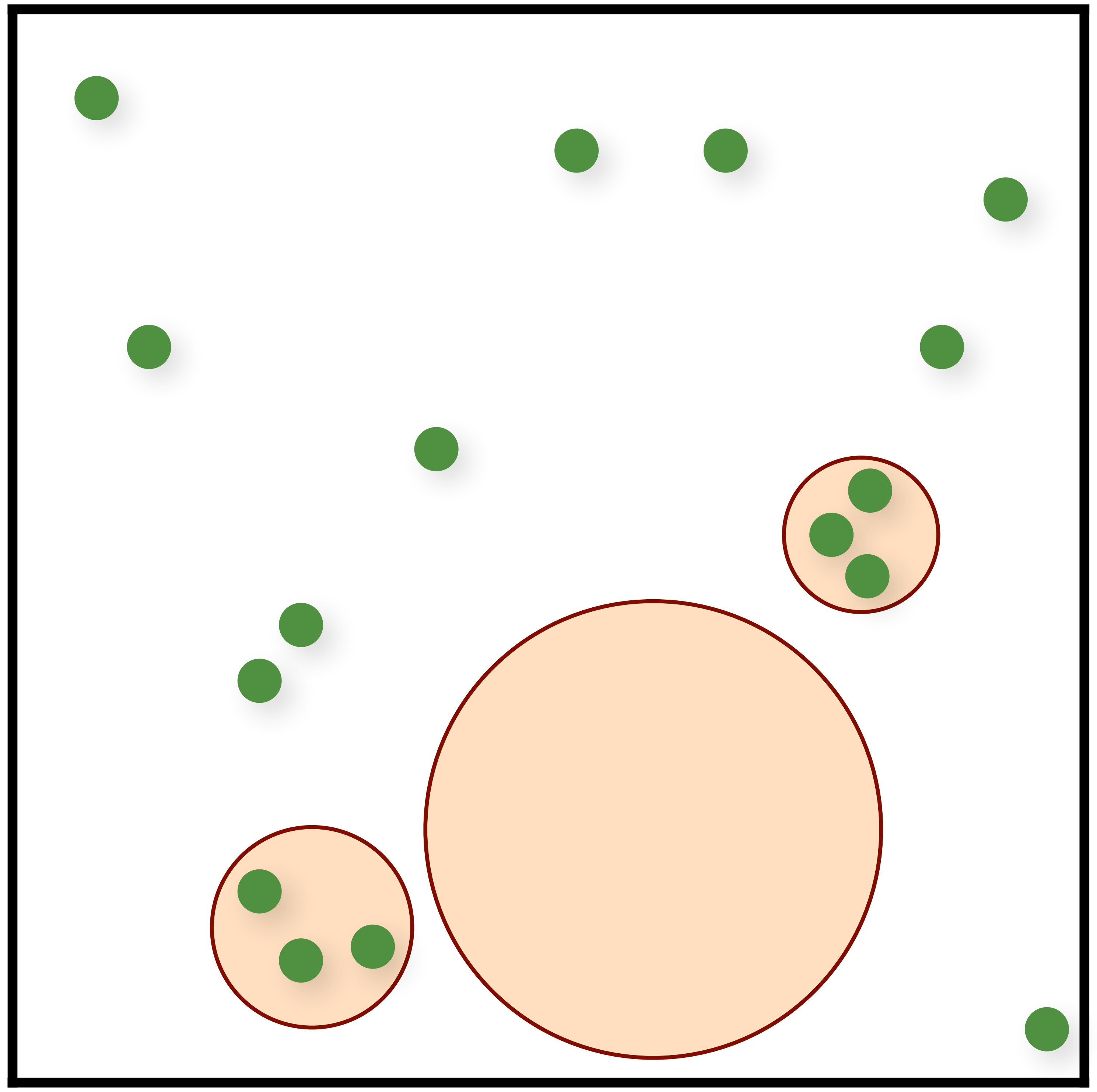

Random



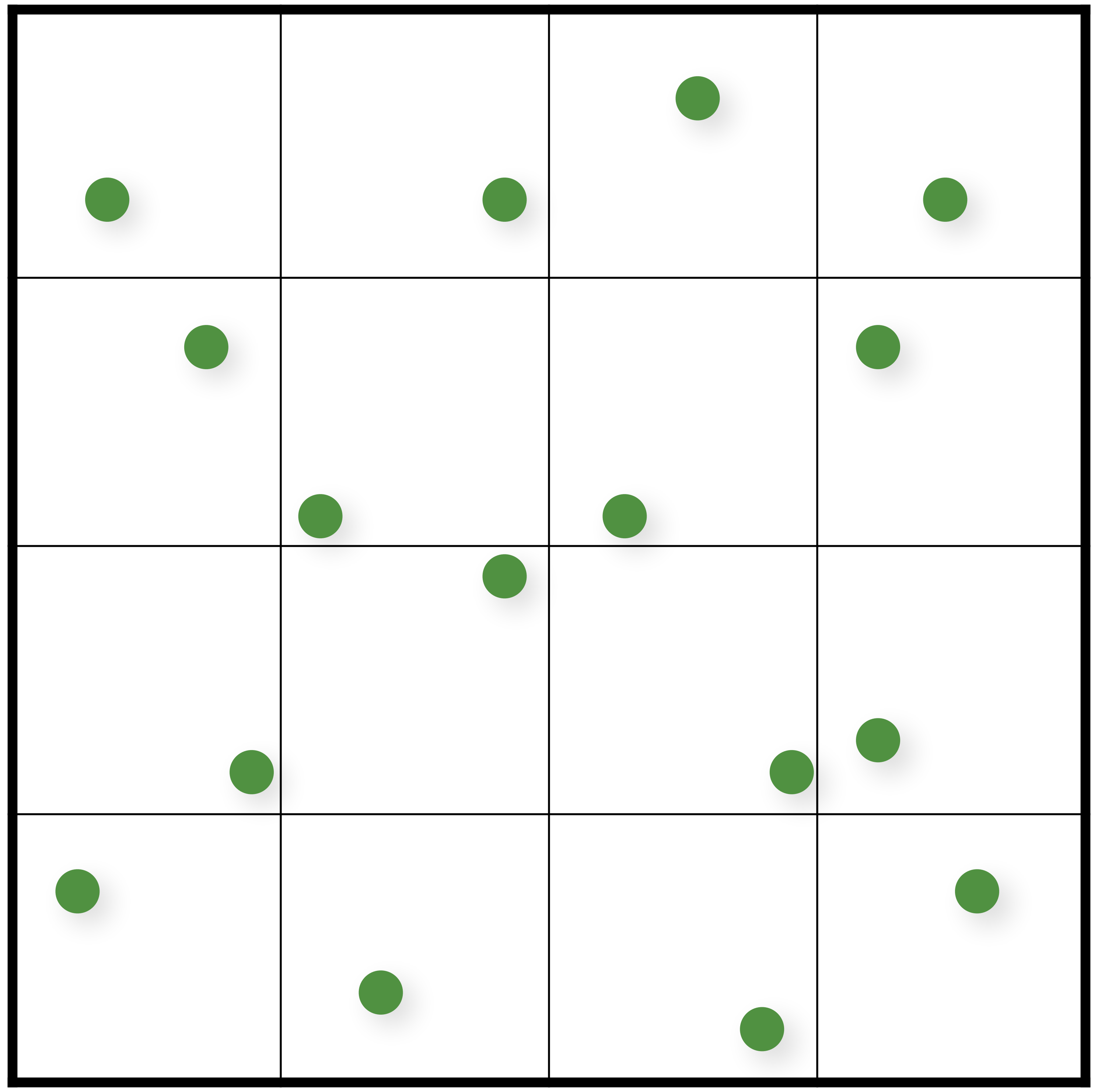
$$S(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N}\frac{\delta(\vec{x}-\vec{x}_k)}{p(\vec{x}_k)}$$

Slide after Wojciech Jarosz

Random vs. Correlated Samples



Random

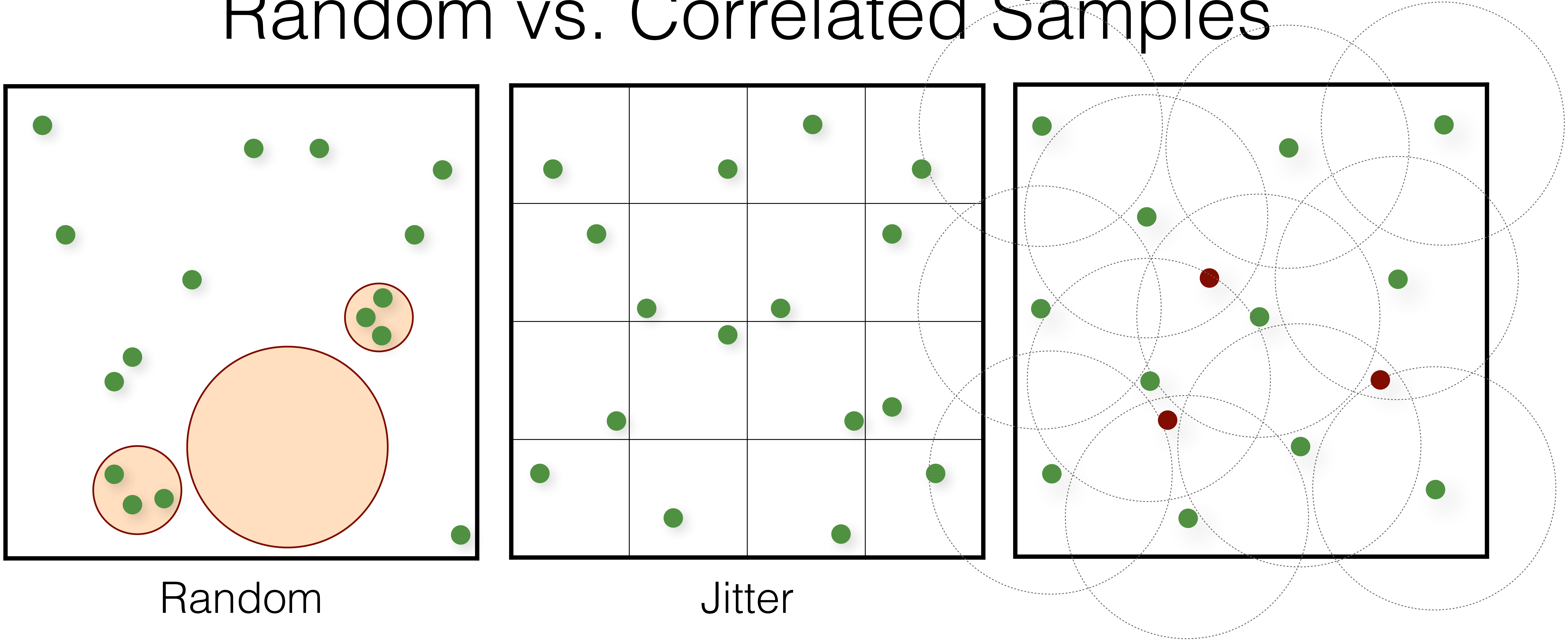


Jitter

$$S(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N}\frac{\delta(\vec{x}-\vec{x}_k)}{p(\vec{x}_k)}$$

Slide after Wojciech Jarosz

Random vs. Correlated Samples



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Slide after Wojciech Jarosz

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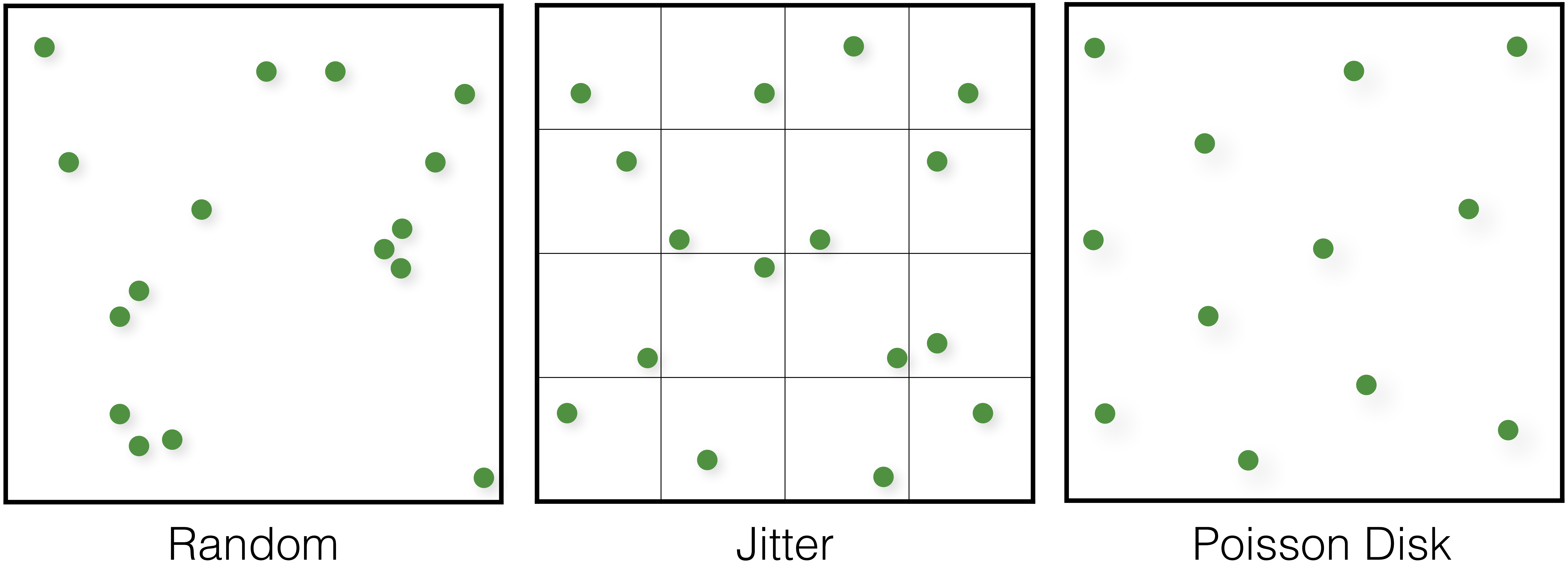

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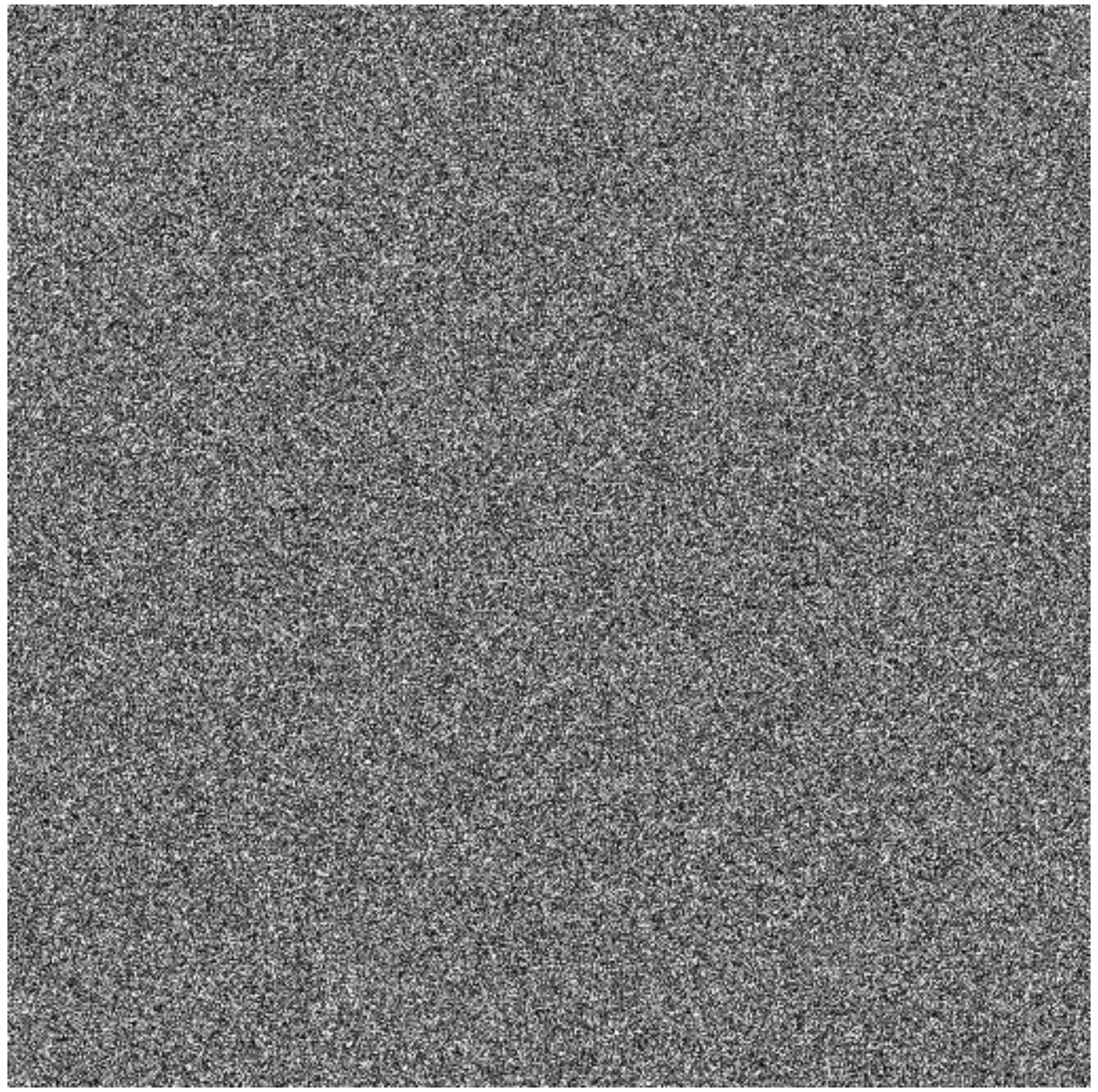
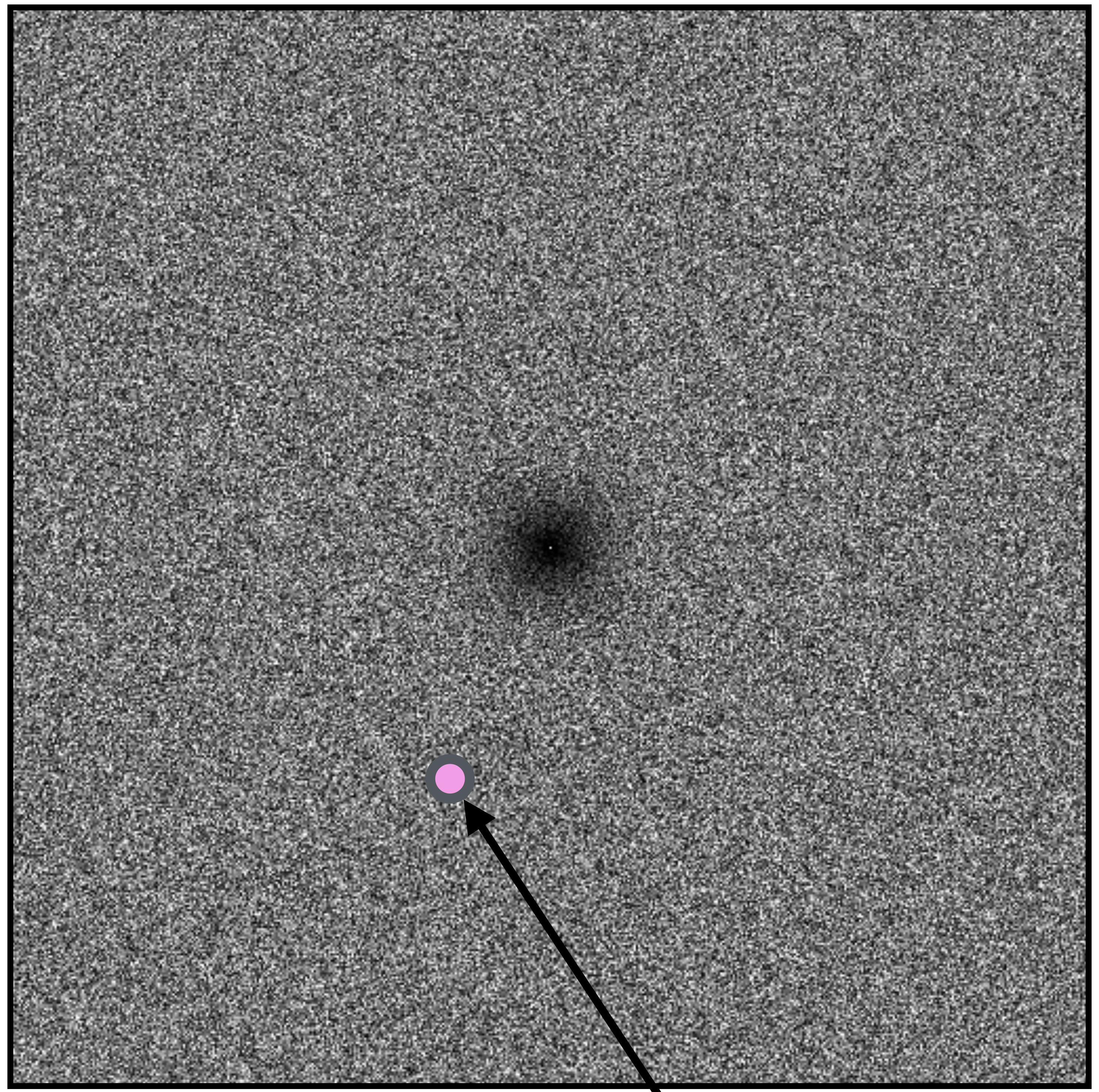
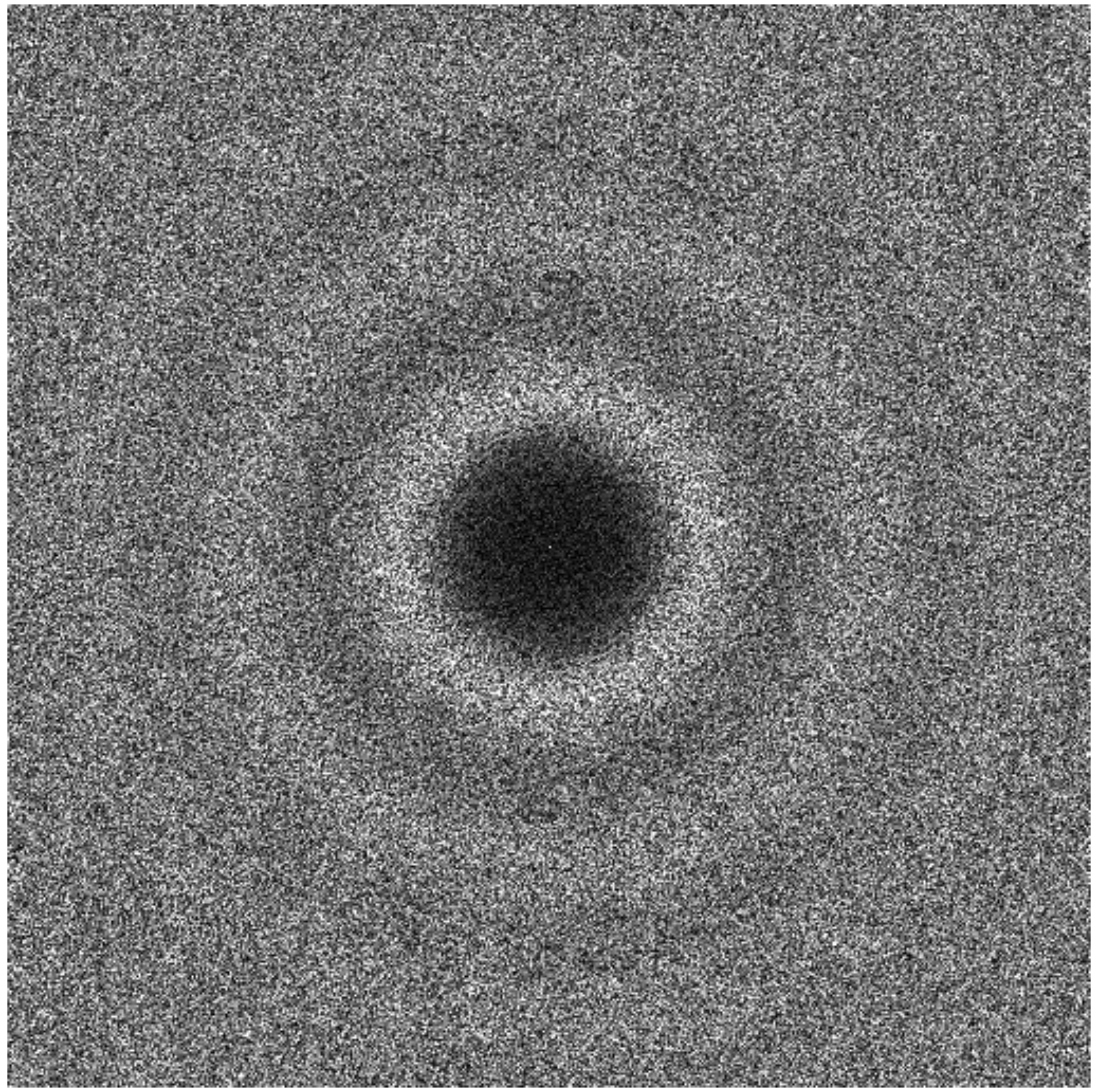
Slide after Wojciech Jarosz

Fourier Statistics: Power Spectrum

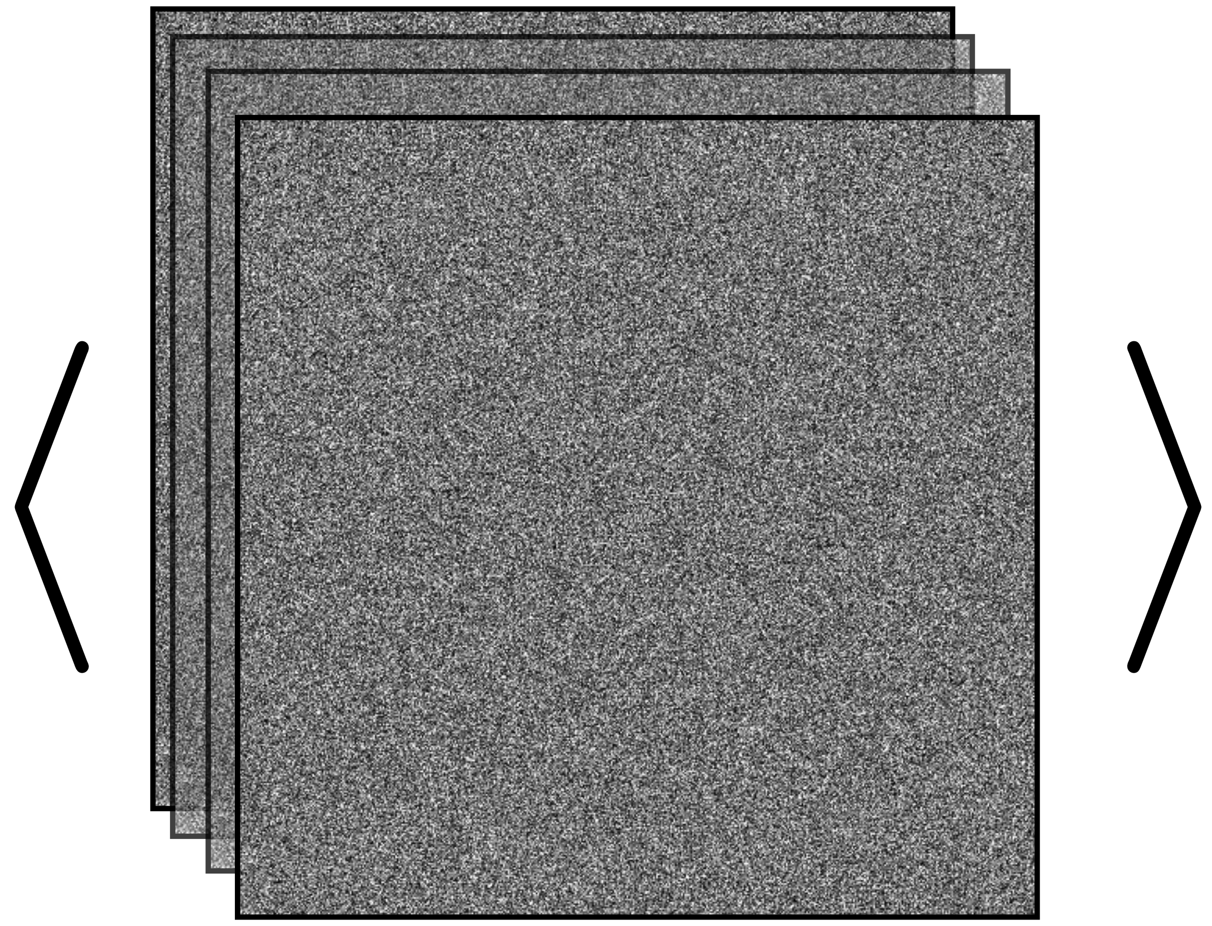

Random


Jitter

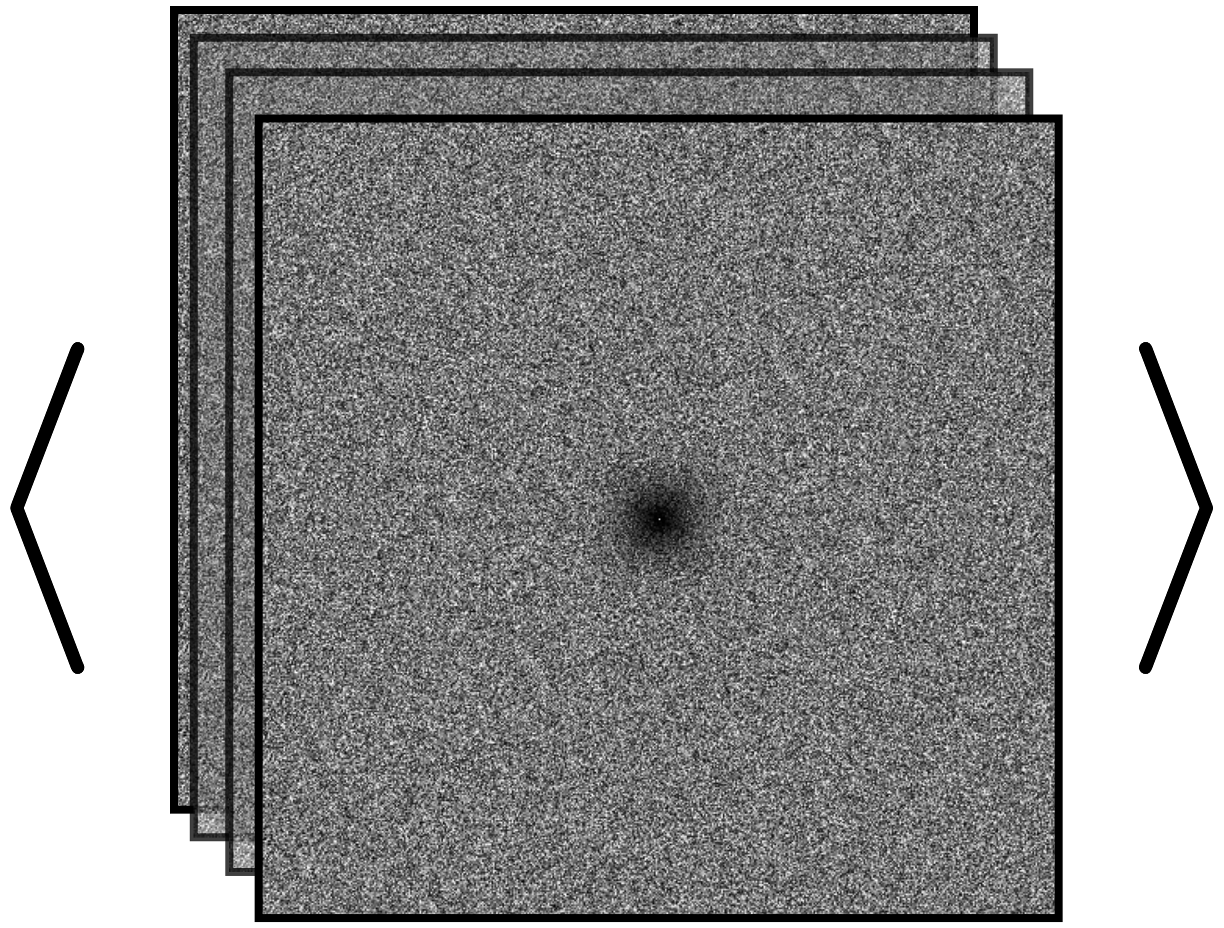

Poisson Disk

$$\mathcal{P}_S(\nu) = \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2$$

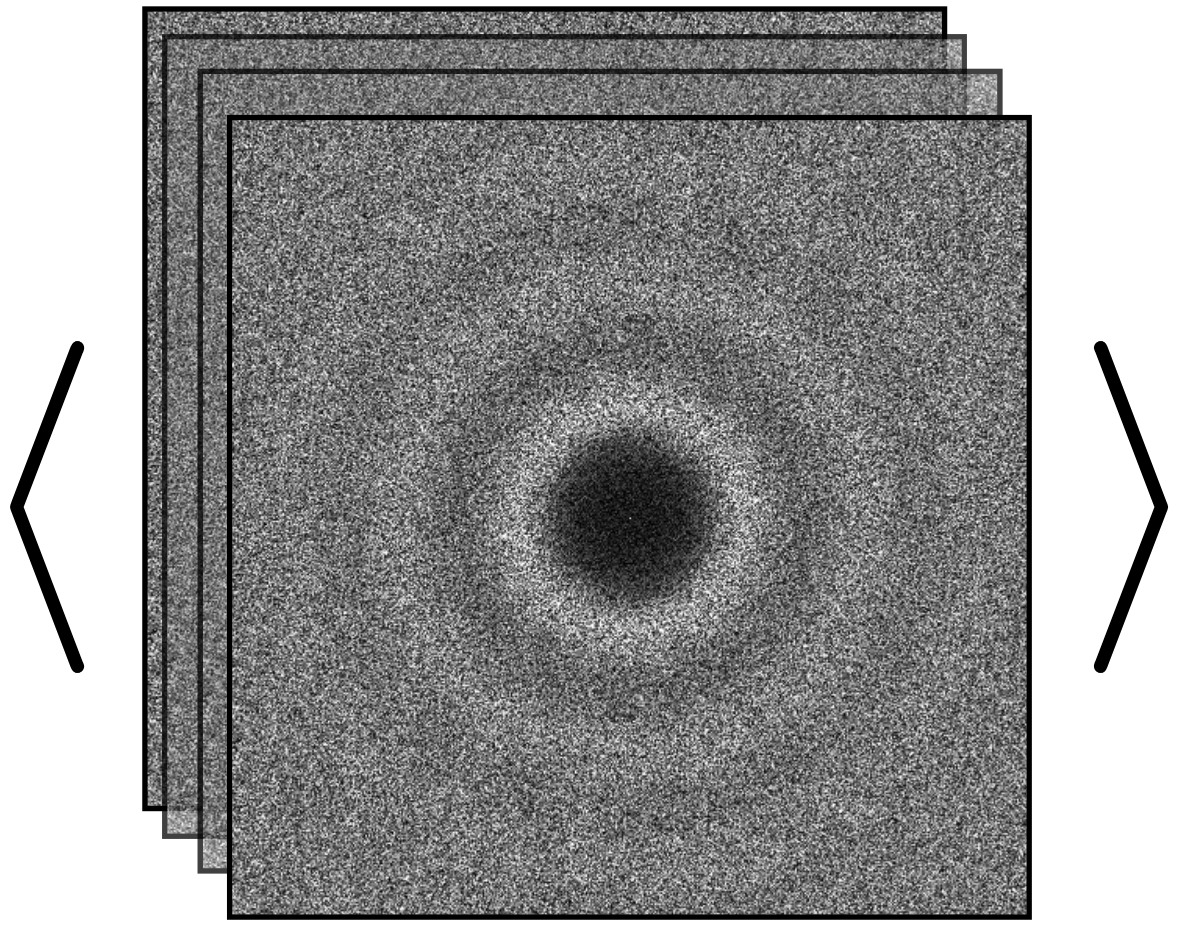
Point Samples' Expected Power Spectra



Random



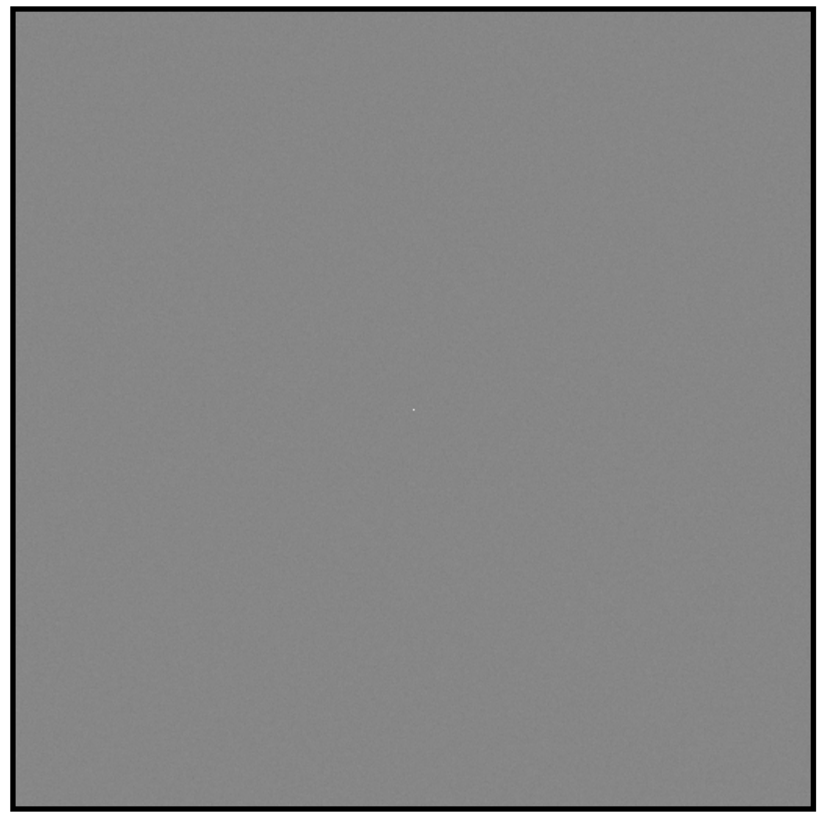
Jitter



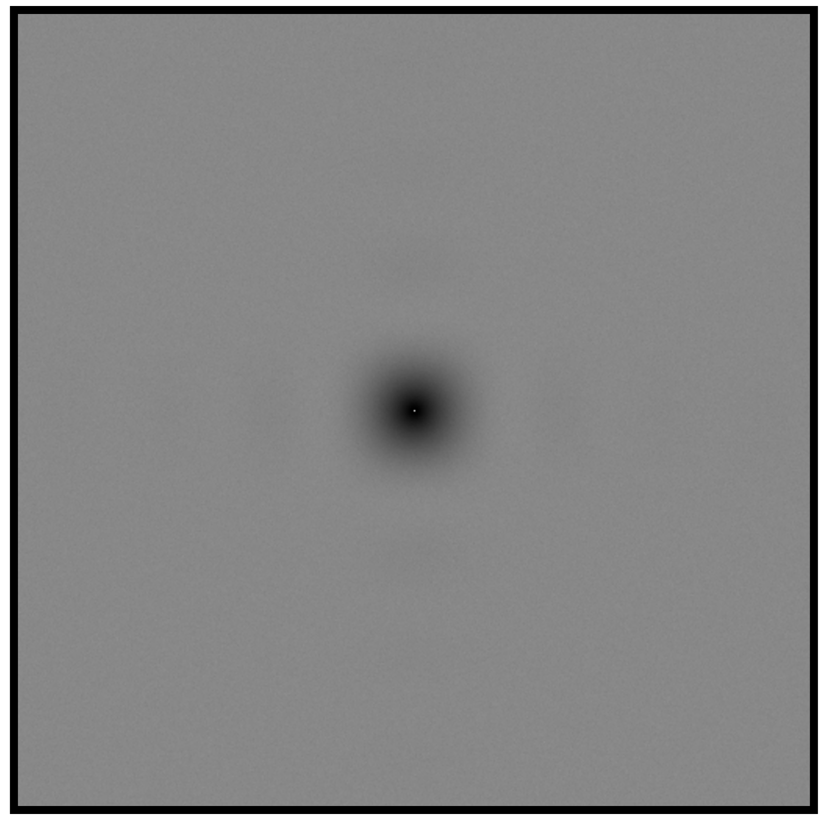
Poisson Disk

$$\left\langle \mathcal{P}_S(\nu) \right\rangle = \left\langle \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu \cdot \vec{x}_k} \right|^2 \right\rangle$$

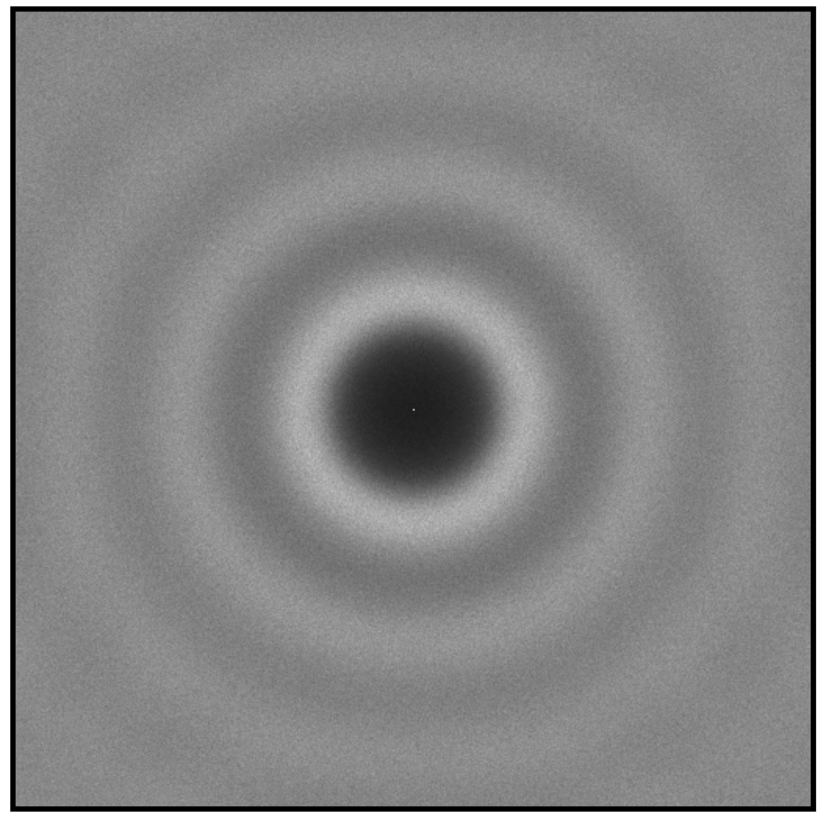
Point Samples' Expected Power Spectra



Random



Jitter



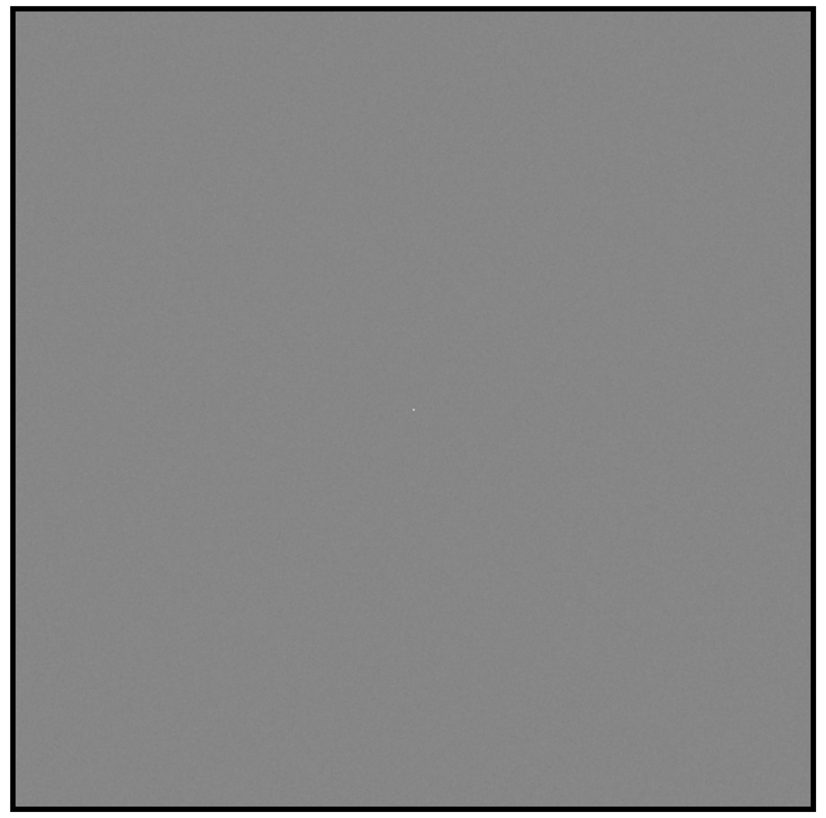
Poisson Disk

$$\langle \mathcal{P}_S(\nu) \rangle = \left\langle \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2 \right\rangle$$

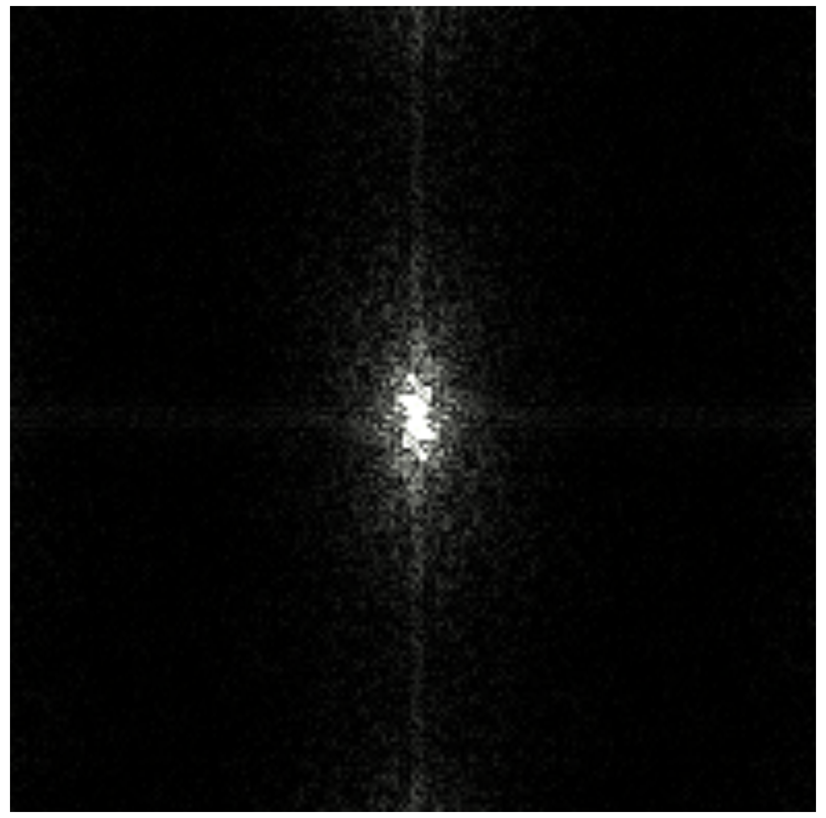
Monte Carlo Estimation Variance for Stationary Samples

$$\mathrm{Var}(I_N) = \int_\Omega \langle \mathcal{P}_{S_N}(\nu) \rangle \times \mathcal{P}_f(\nu) \, d\nu$$

$\langle \mathcal{P}_{S_N}(\nu) \rangle$



$\mathcal{P}_f(\nu)$

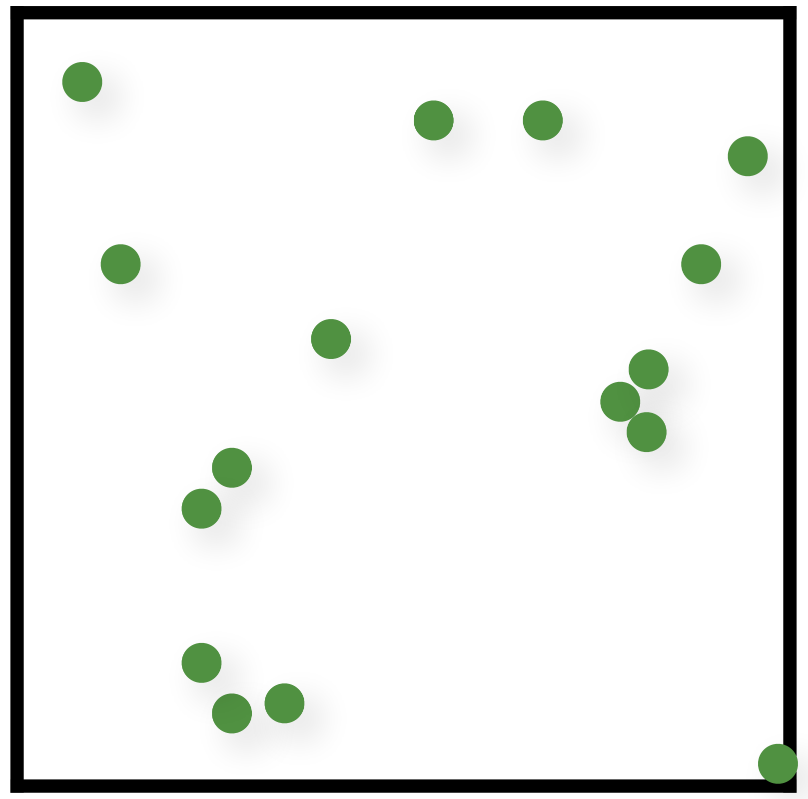


Fredo Durand [2011]

Subr & Kautz [2013]

$S_N(\vec{x})$

Random



$f(\vec{x})$

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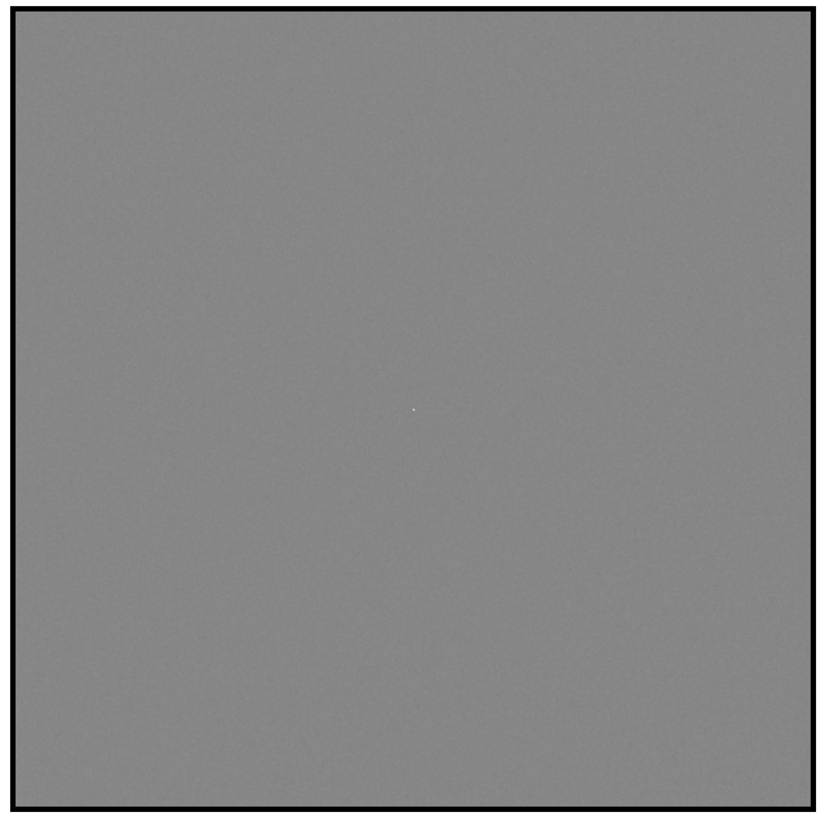
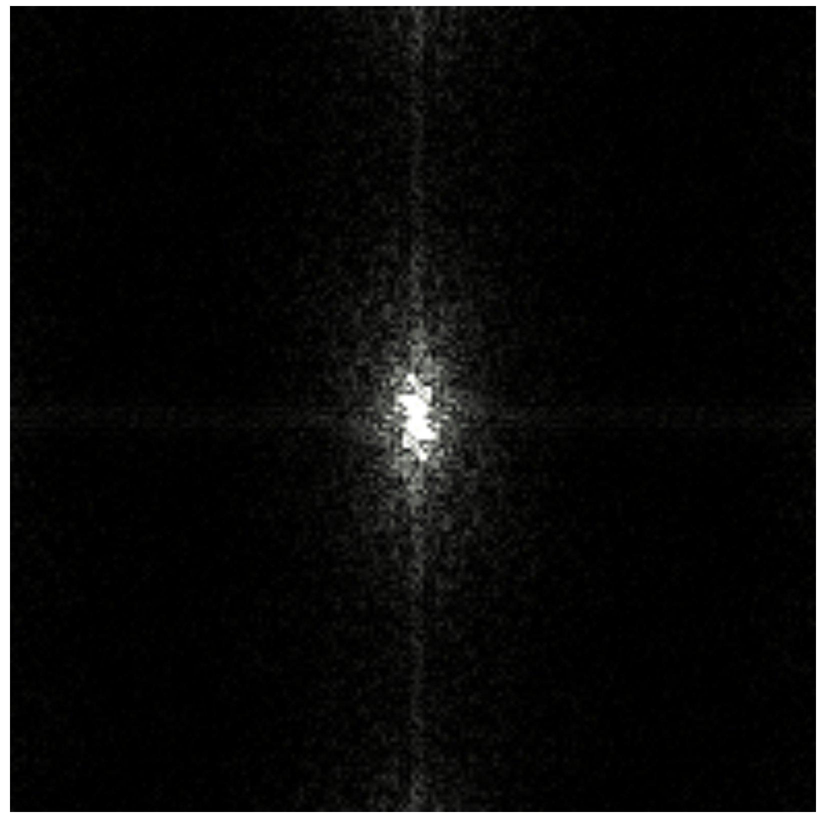


$\mathcal{P}_f(\nu)$

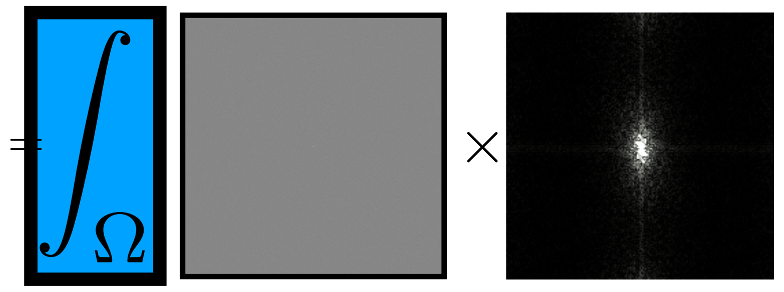


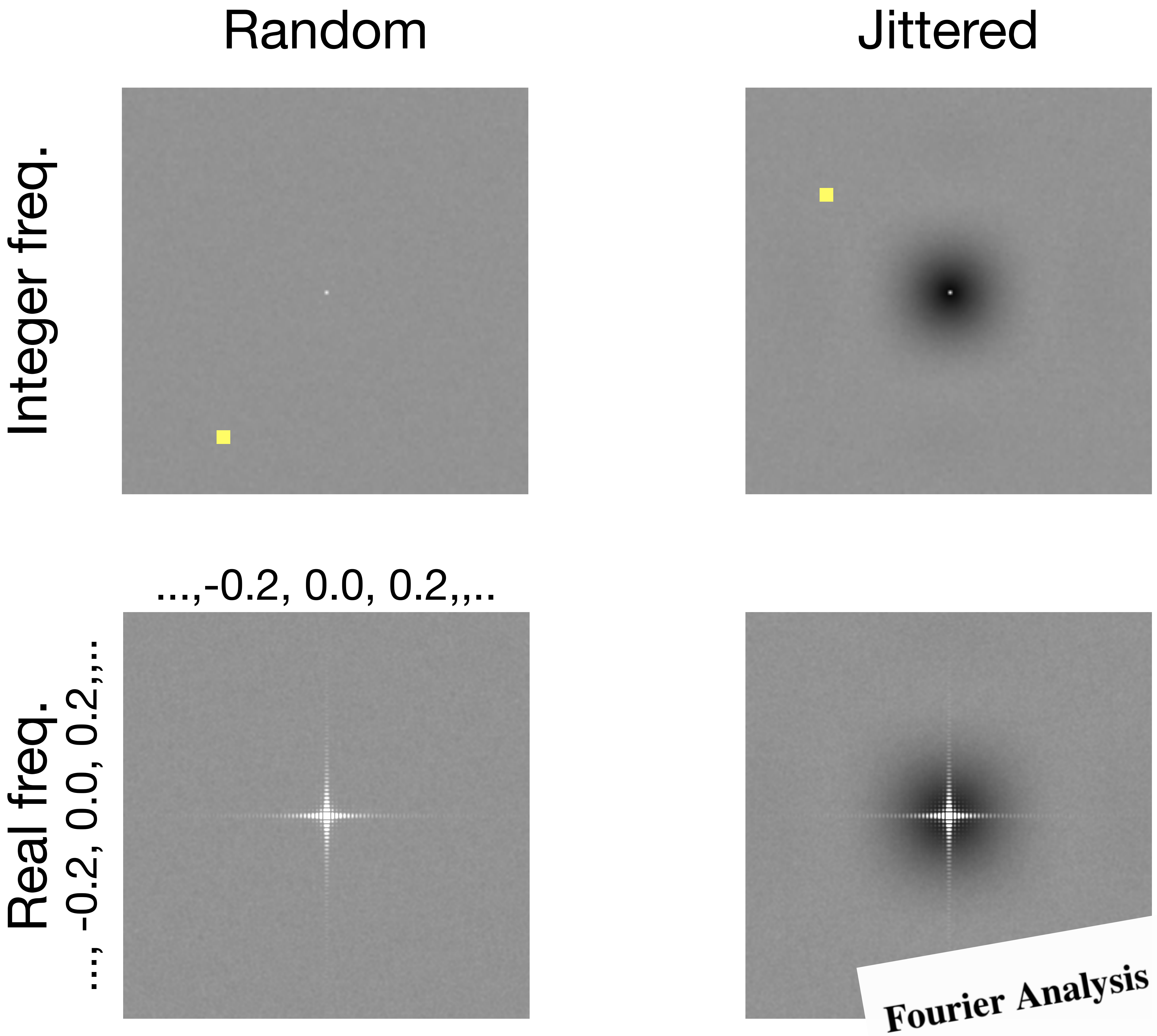
Fredo Durand [2011]

Subr & Kautz [2013]

Only valid for constant PDFs (uniformly distributed samples)

Real vs. Integer Frequencies





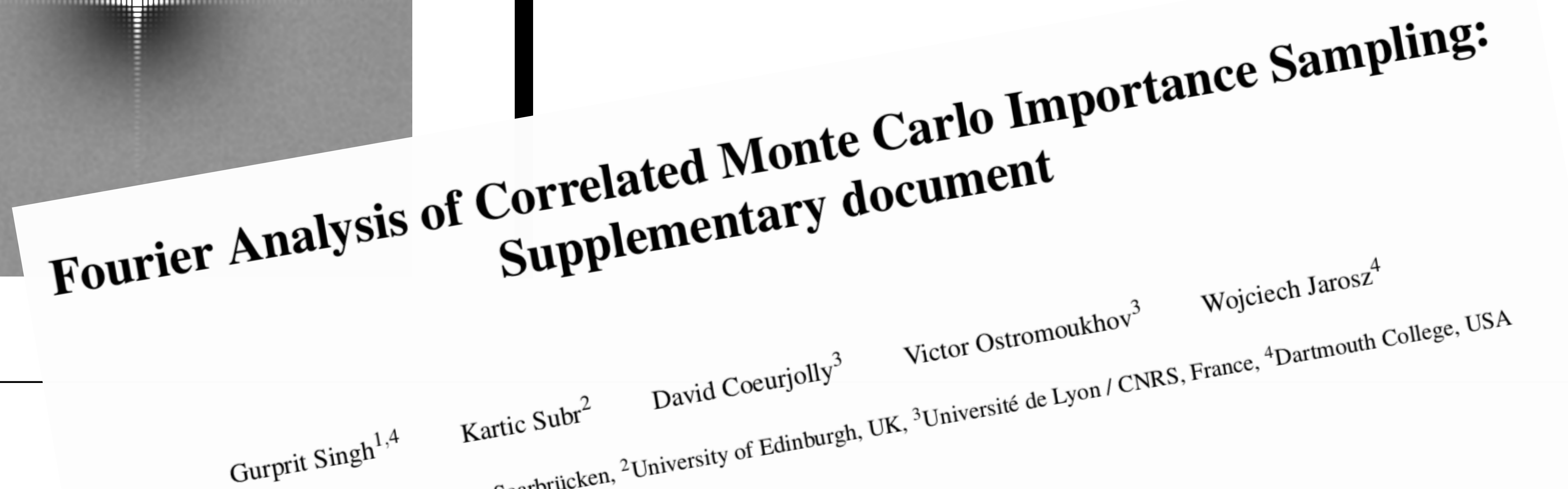
Expected power spectra

Random:

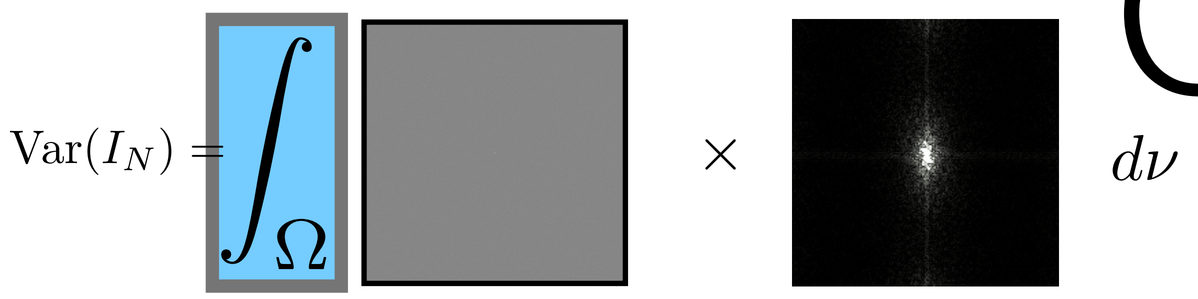
$$\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \begin{cases} 1 & m = 0 \\ \frac{1}{N} + \frac{N-1}{N}\mathrm{Sinc}(\pi m)^2 & m \neq 0 \end{cases}$$

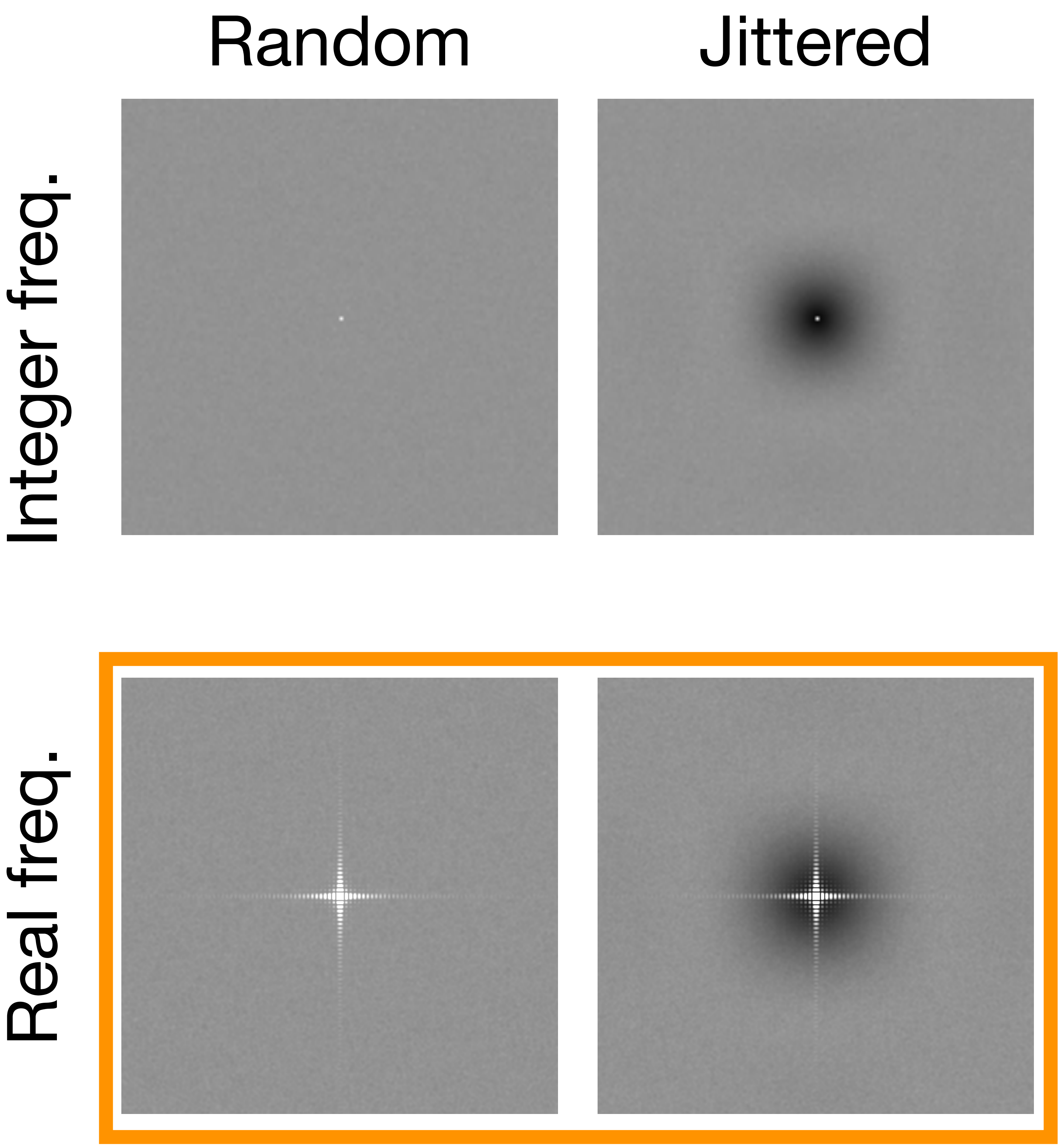
Jittered:

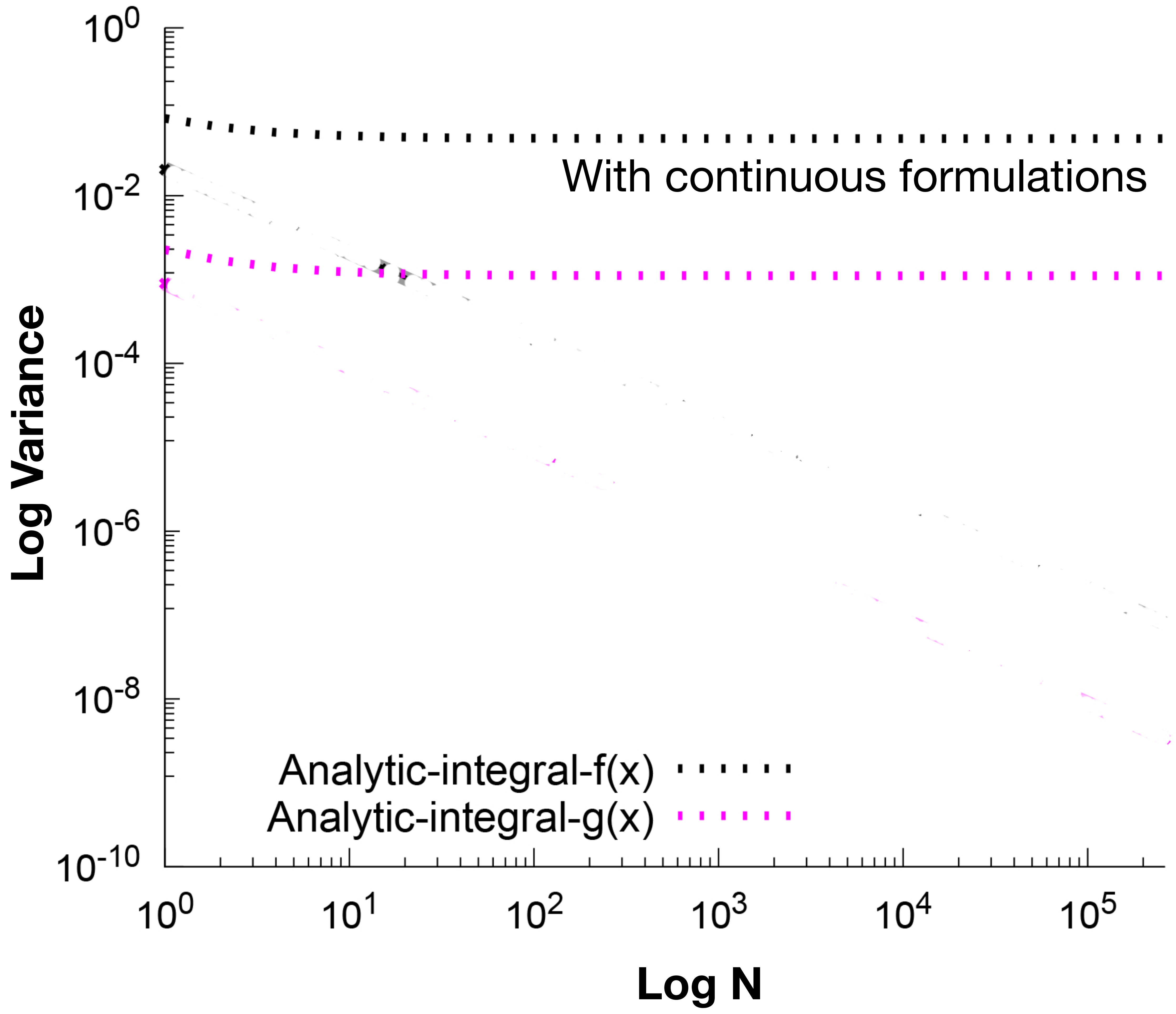
$$\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \frac{1}{N}\left(1 - \mathrm{Sinc}\left(\frac{\pi m}{N}\right)^2\right) + \mathrm{Sinc}(\pi m)^2$$

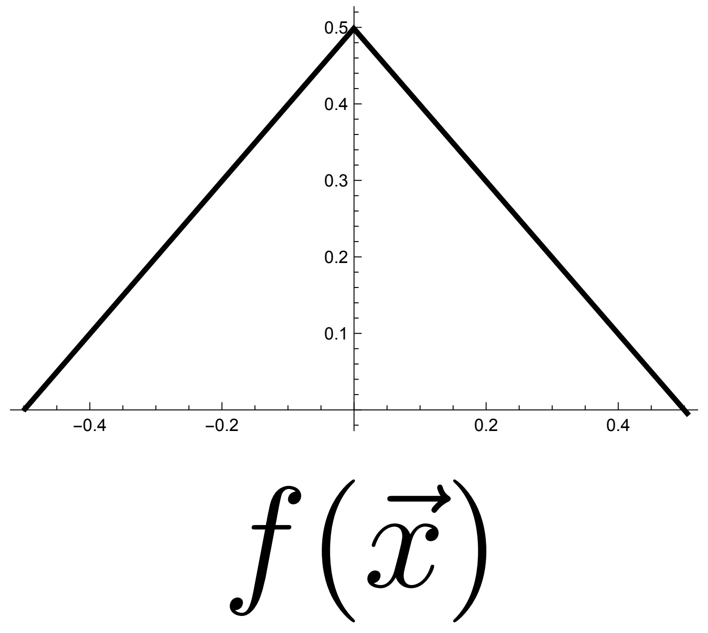

Fourier Analysis of Correlated Monte Carlo Importance Sampling:
Supplementary document
Gurprit Singh[1,4] Kartic Subr[2] David Coeurjolly[3] Victor Ostromoukhov[3] Wojciech Jarosz[4]
…arbrücken, [2]University of Edinburgh, UK, [3]Université de Lyon / CNRS, France, [4]Dartmouth College, USA

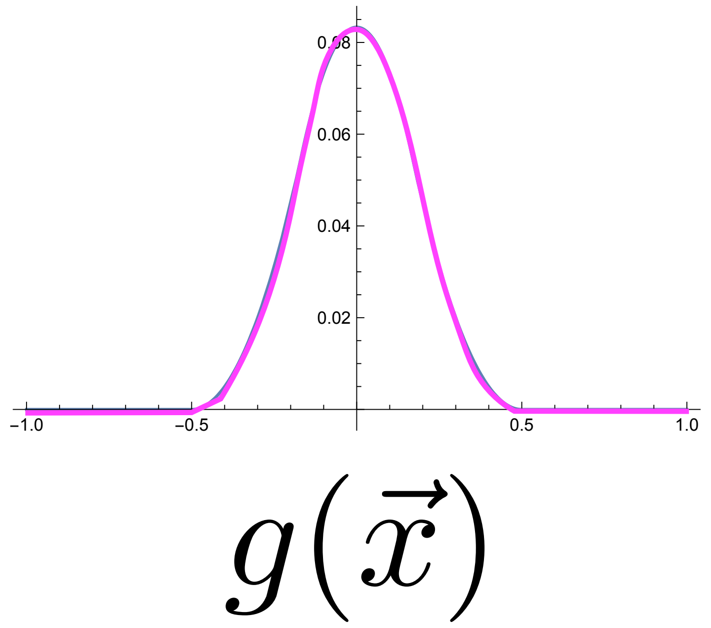
Convergence Rates Diverges



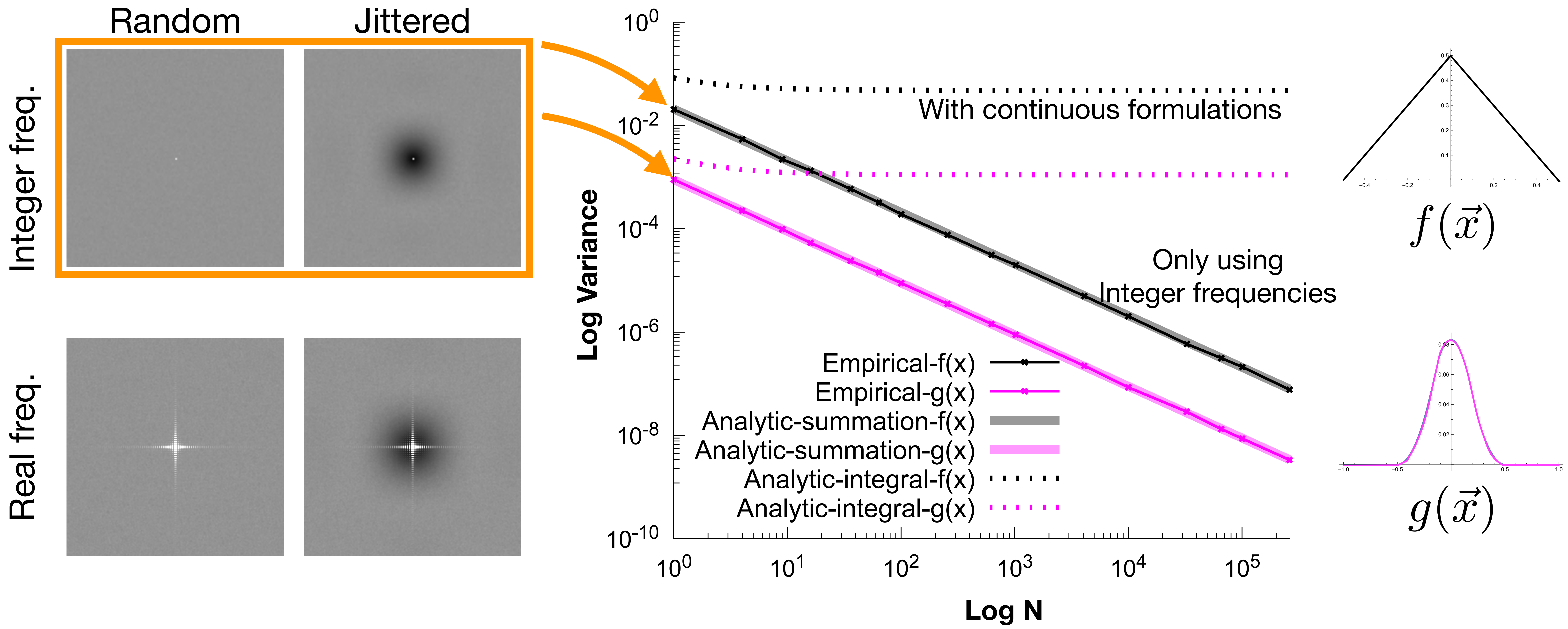








Convergence Rates Diverges at Real Frequencies



Monte Carlo Estimation Variance for Random Samples

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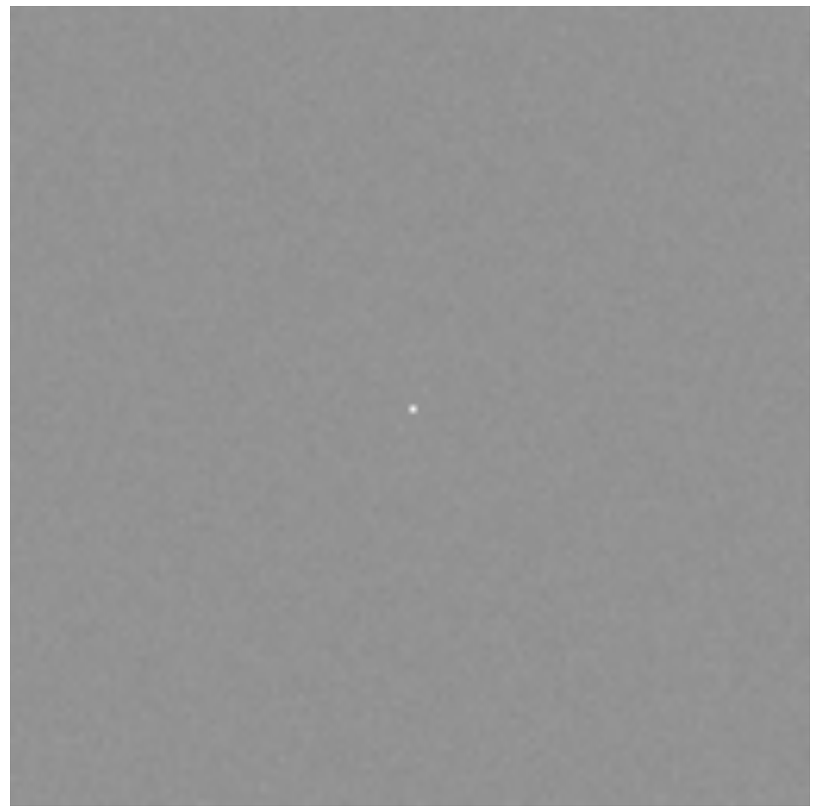
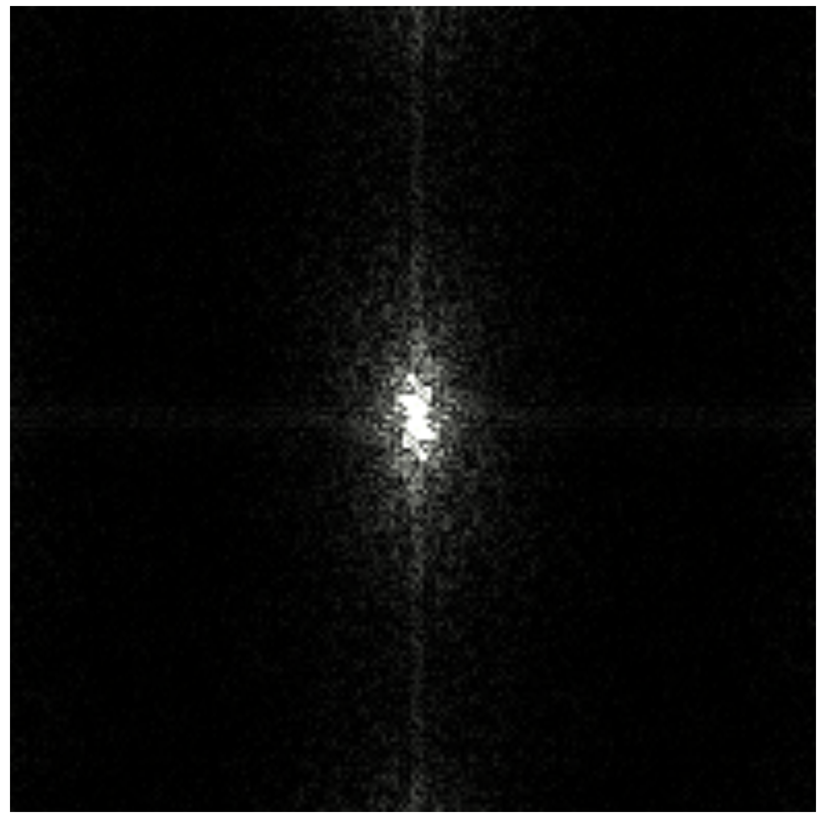
$\langle \mathcal{P}_{S_N}(\nu) \rangle$



$\mathcal{P}_f(\nu)$



Fredo Durand [2011]

Subr & Kautz [2013]

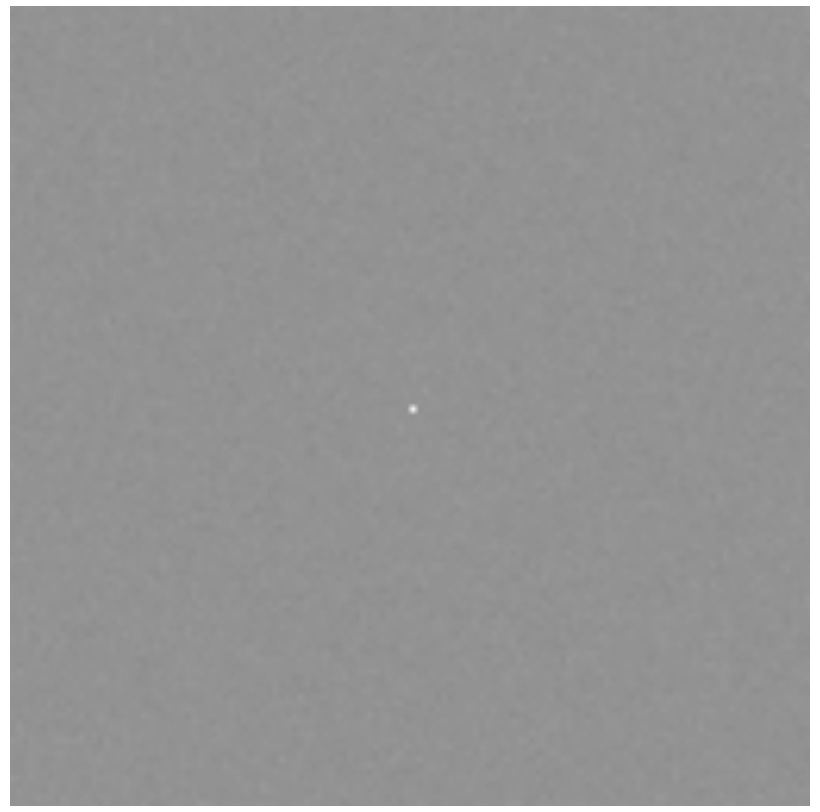
Pilleboue et al. [2015]

Only valid for constant PDFs (uniformly distributed samples)

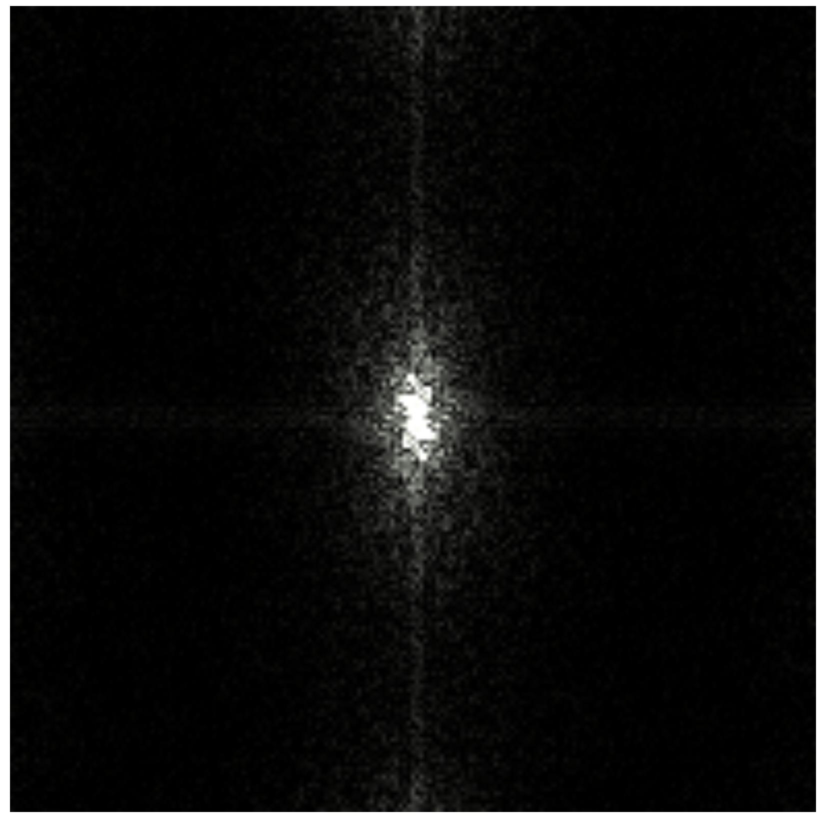
Finite sampling domain is not properly handled

Fourier series based Variance Formulation

$$\mathrm{Var}(I_N) = \sum_{m \in \mathbb{Z}/0}$$



$\times$



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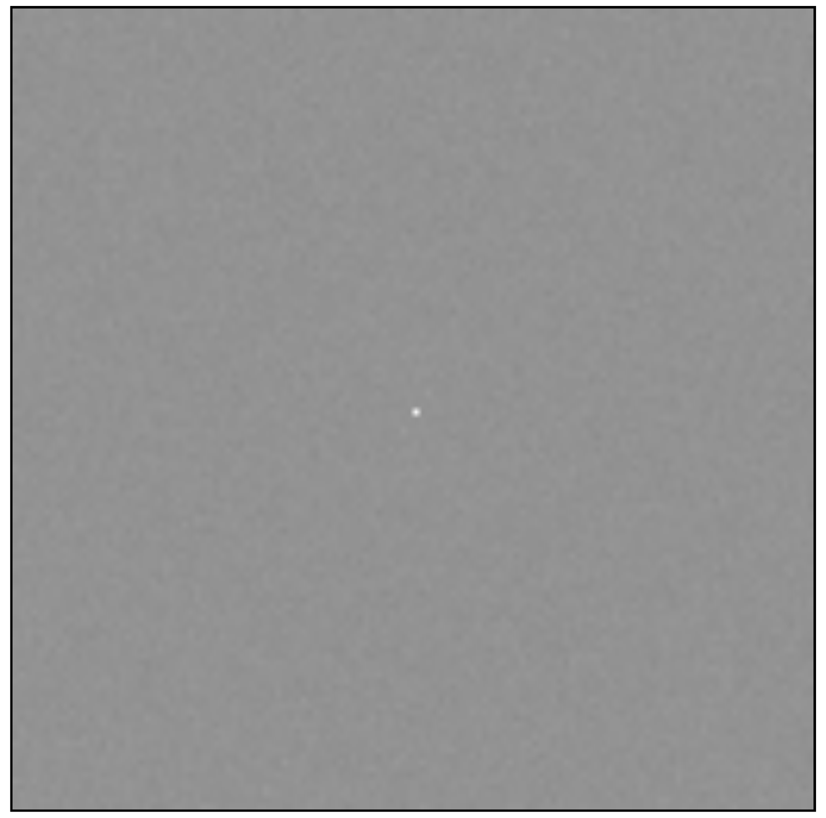

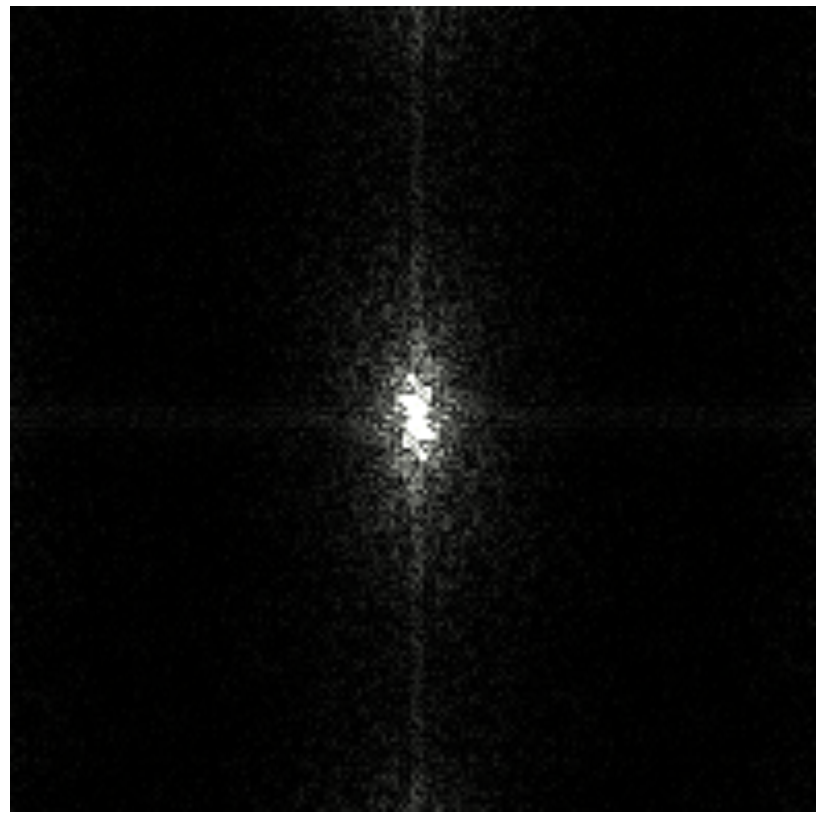

Stationarity can imposed using homogenization or Cranley-Patterson rotation for all samplers

Pilleboue et al. [2015]

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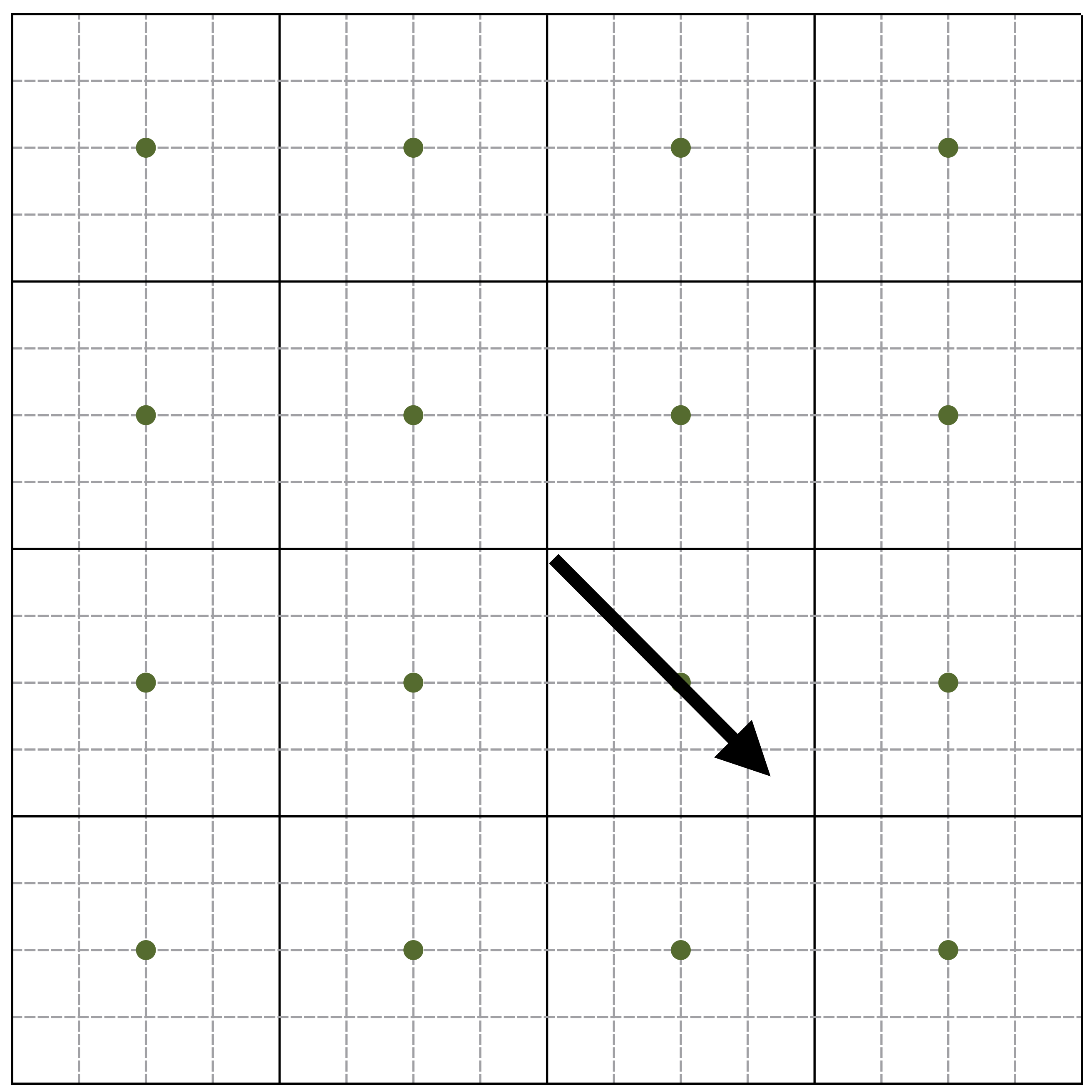
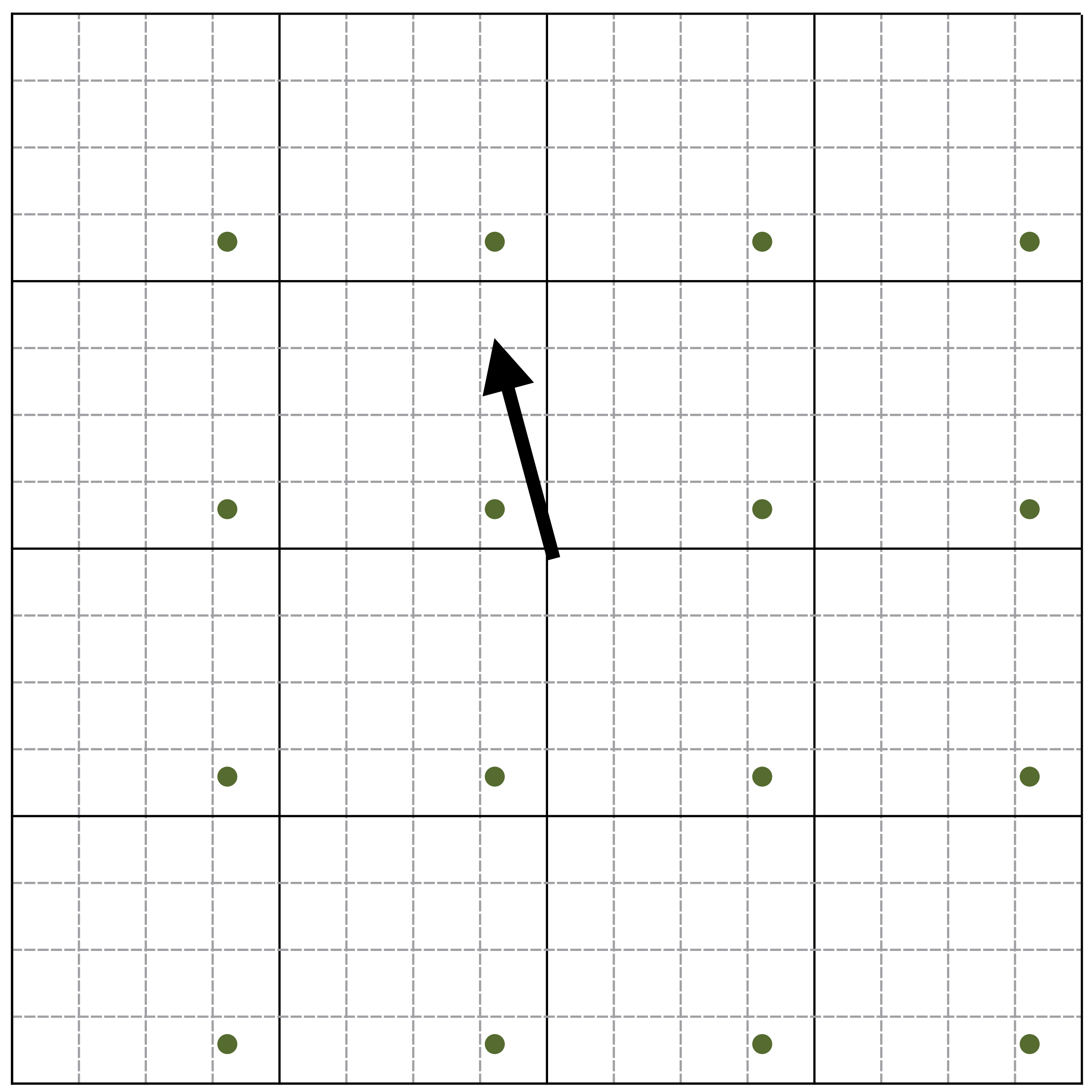
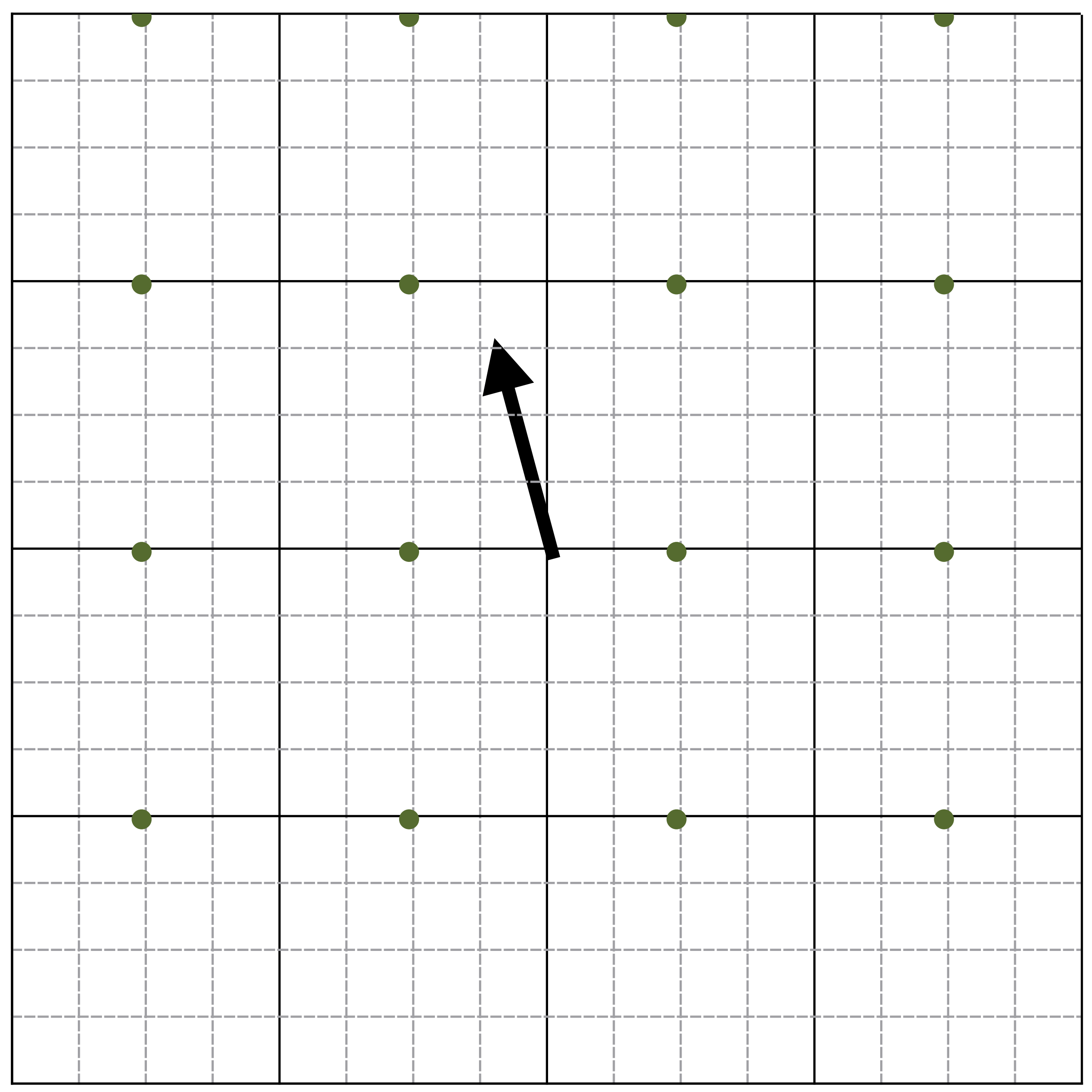
Homogenization or Cranley-Patterson rotation



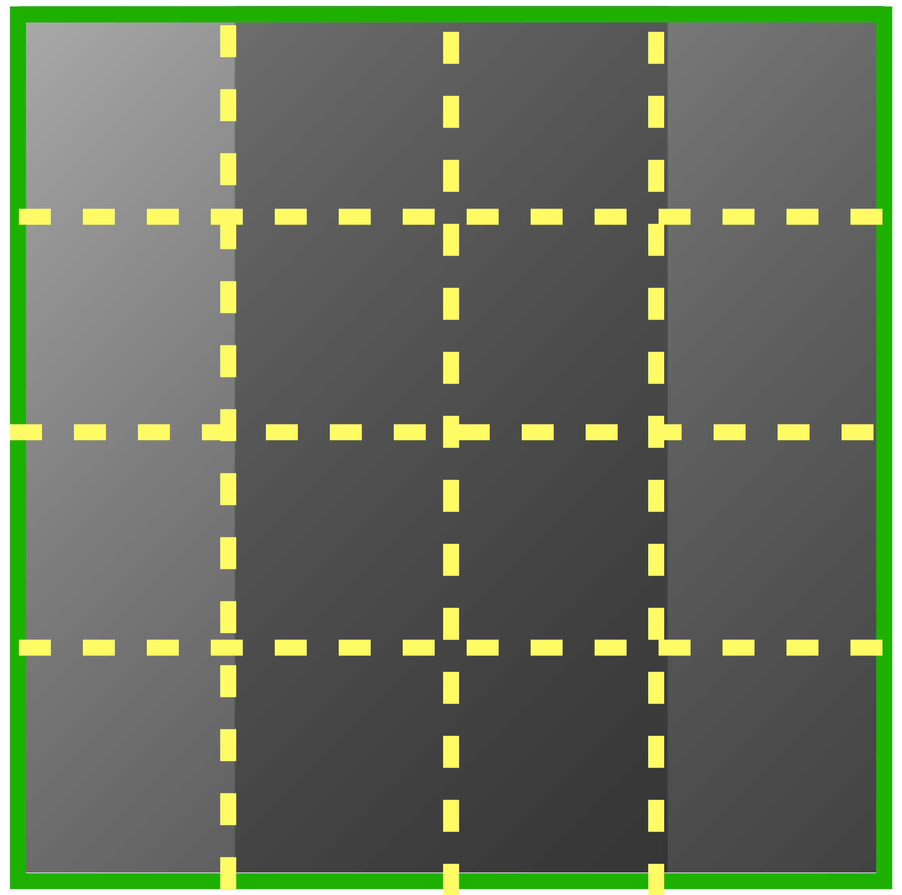
Homogenization or Cranley-Patterson rotation

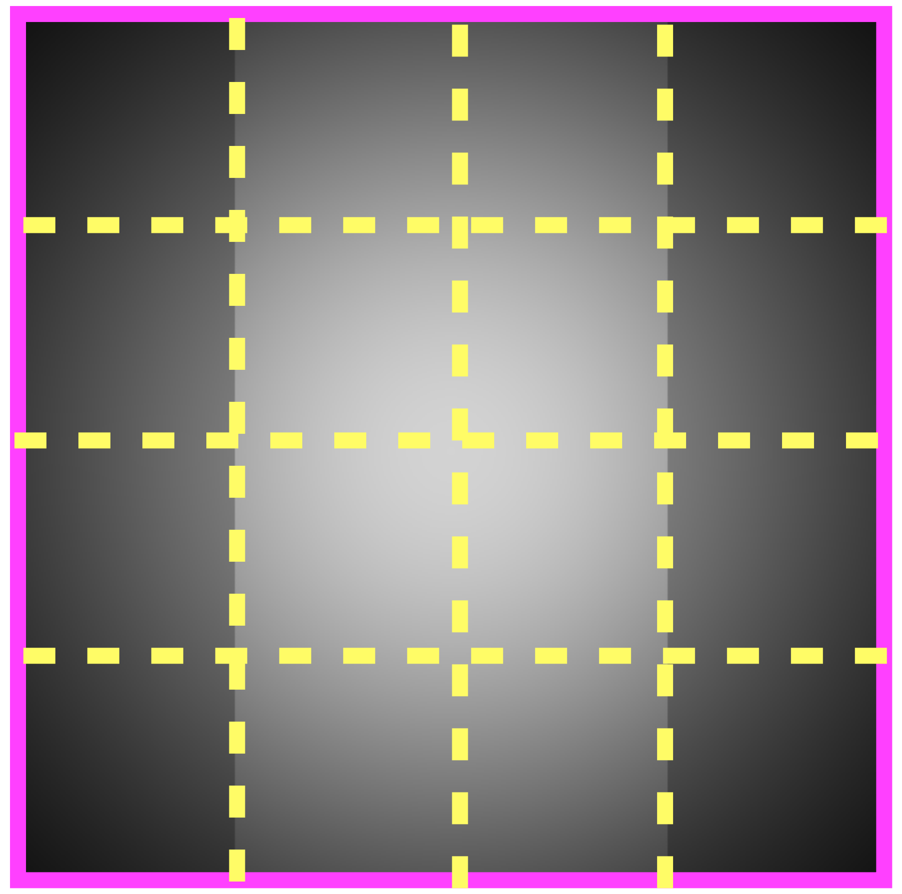


Homogenization or Cranley-Patterson rotation

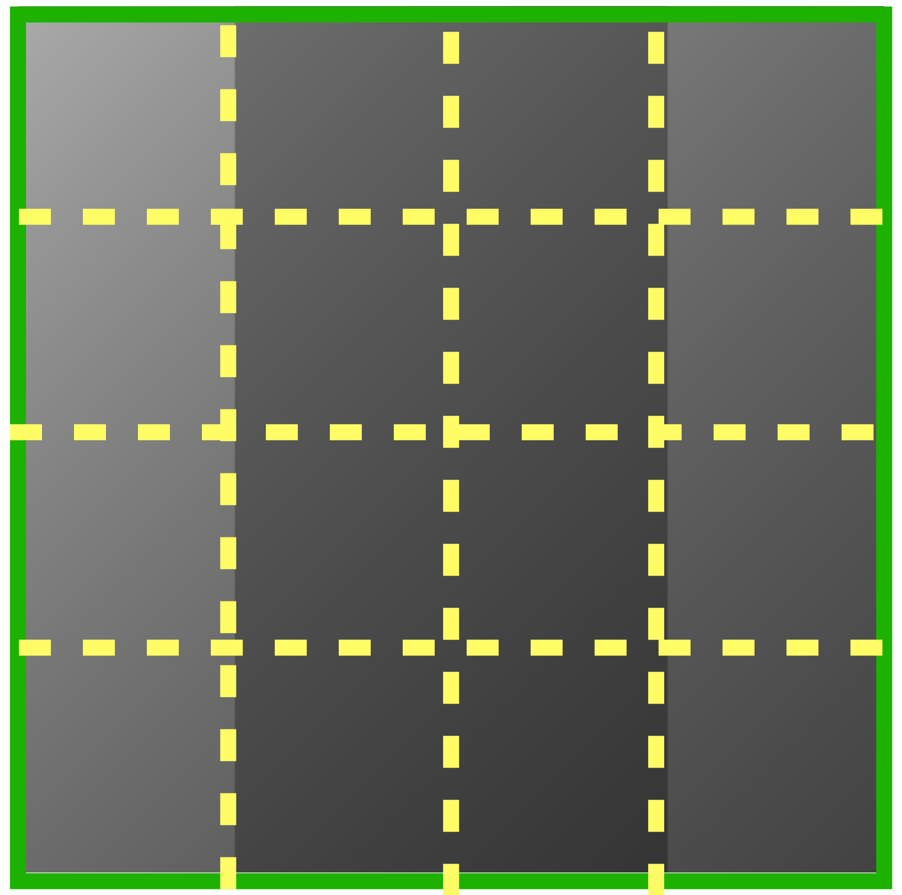


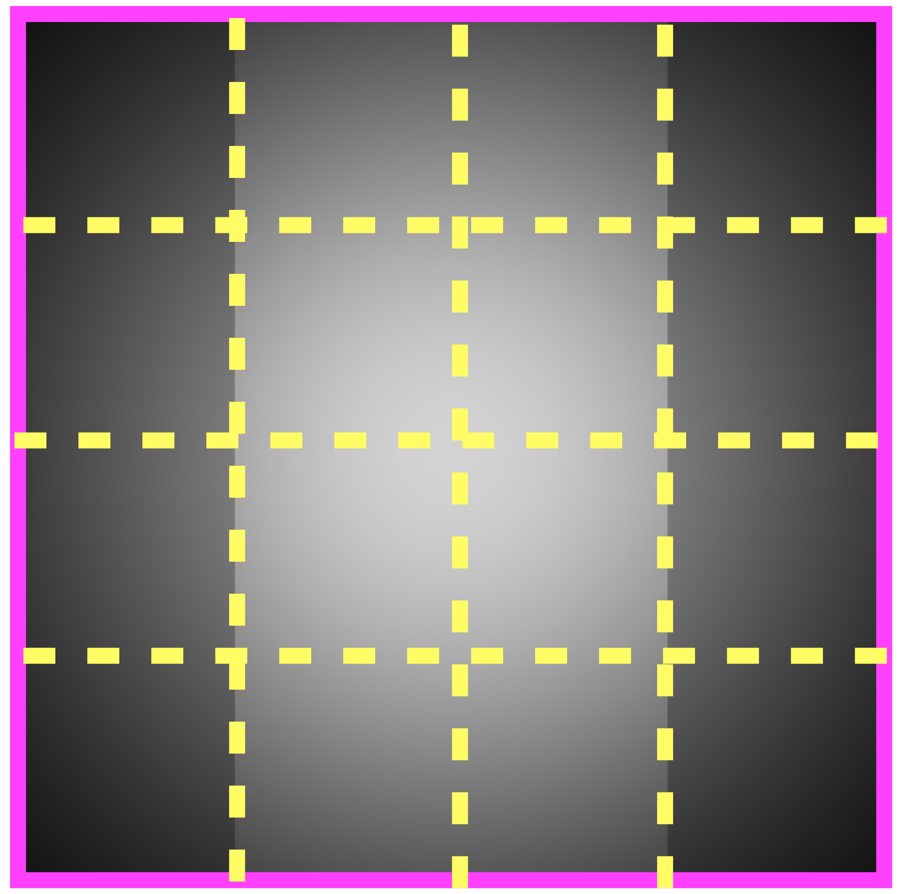
Homogenization affect Convergence

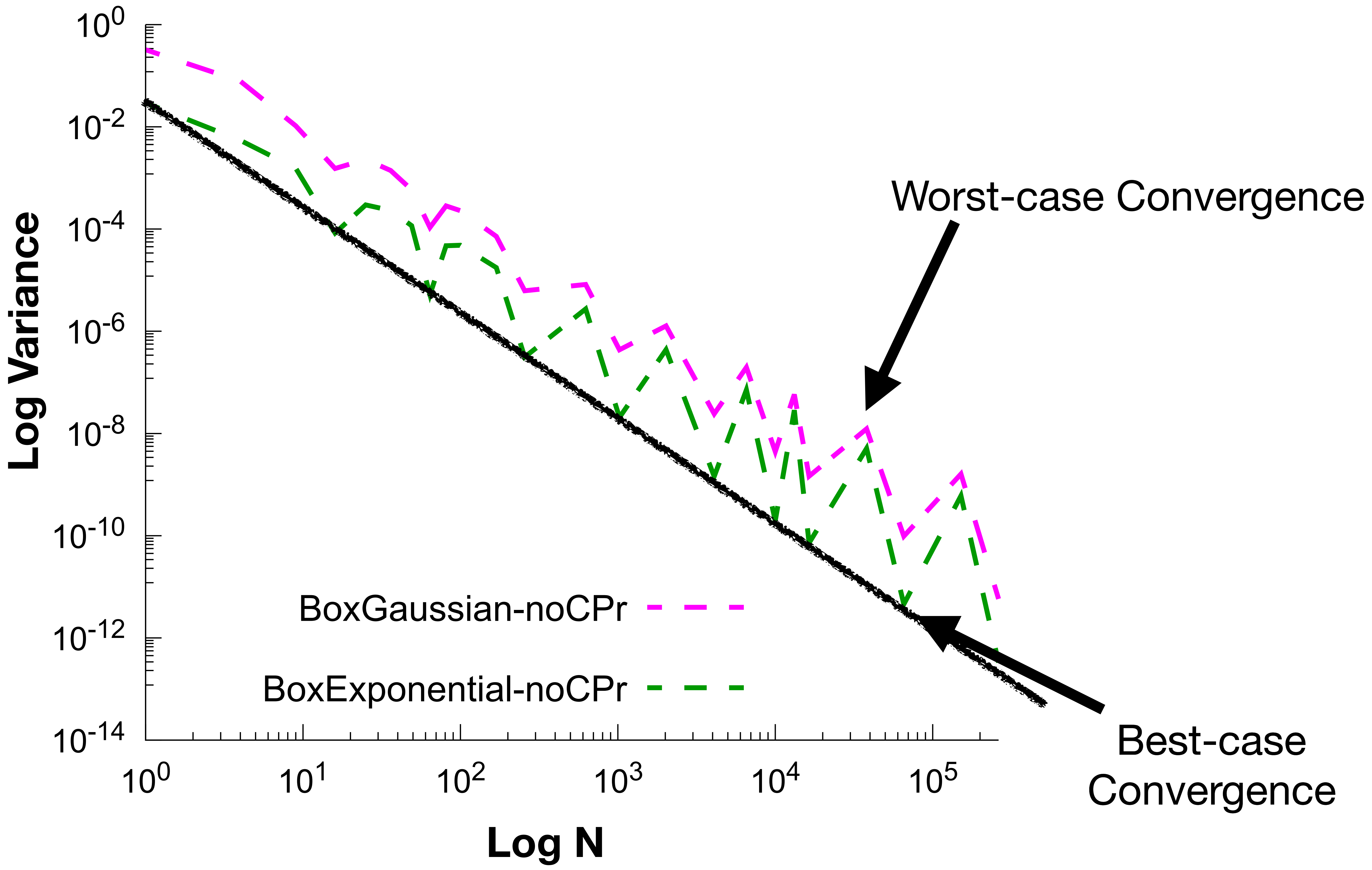




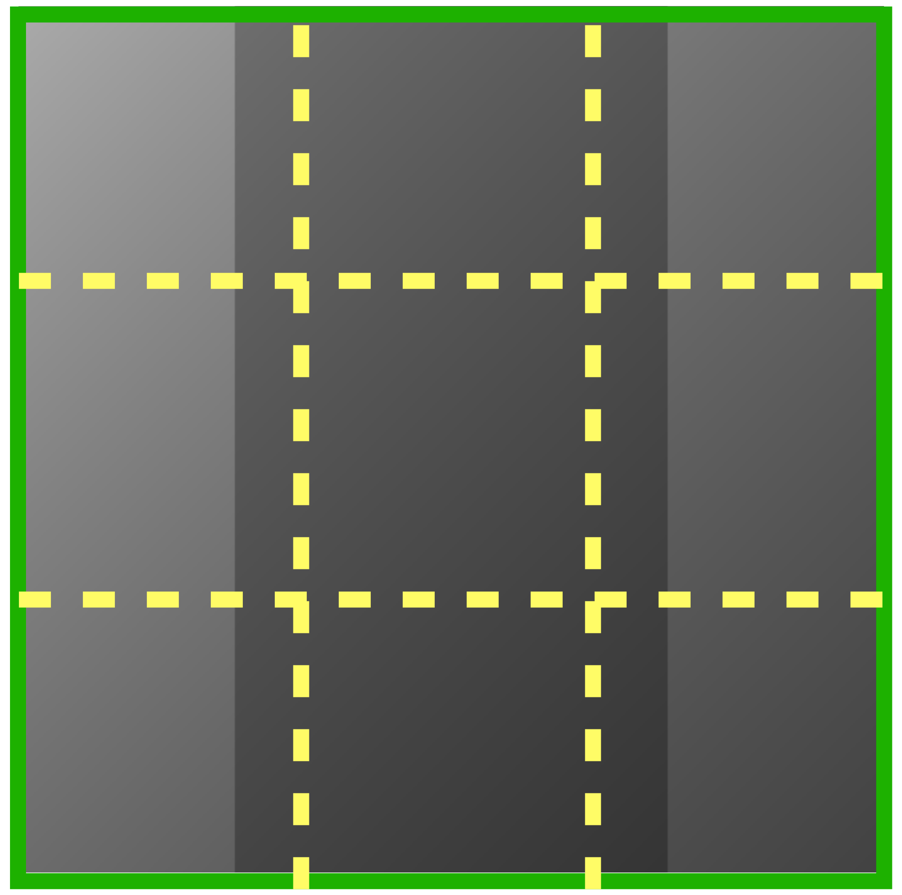
No Homogenization: Strata alignment helps

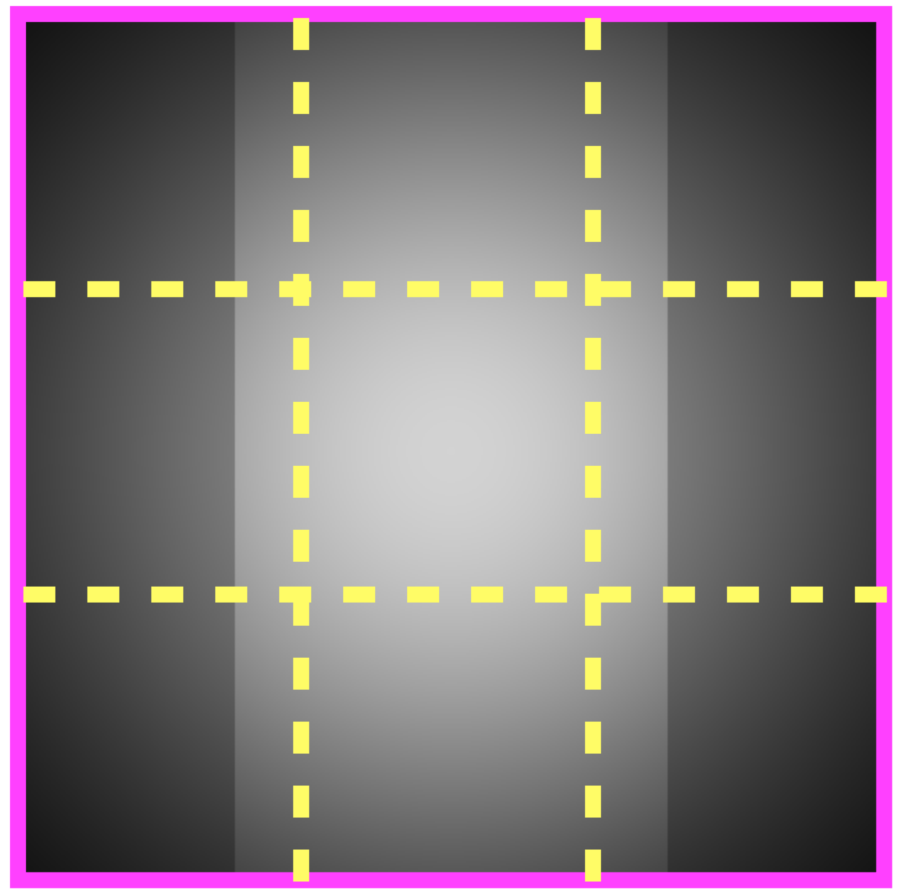


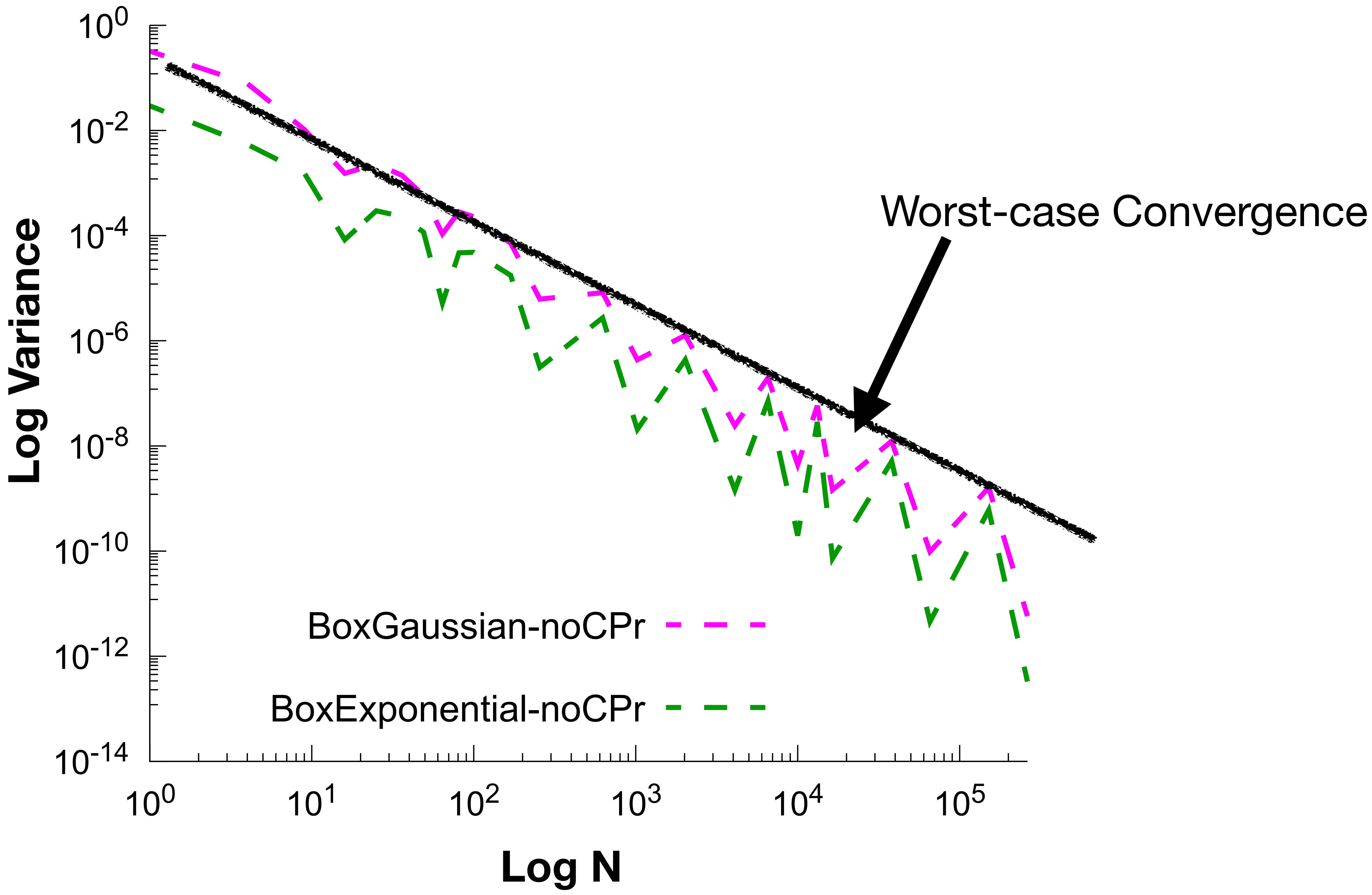




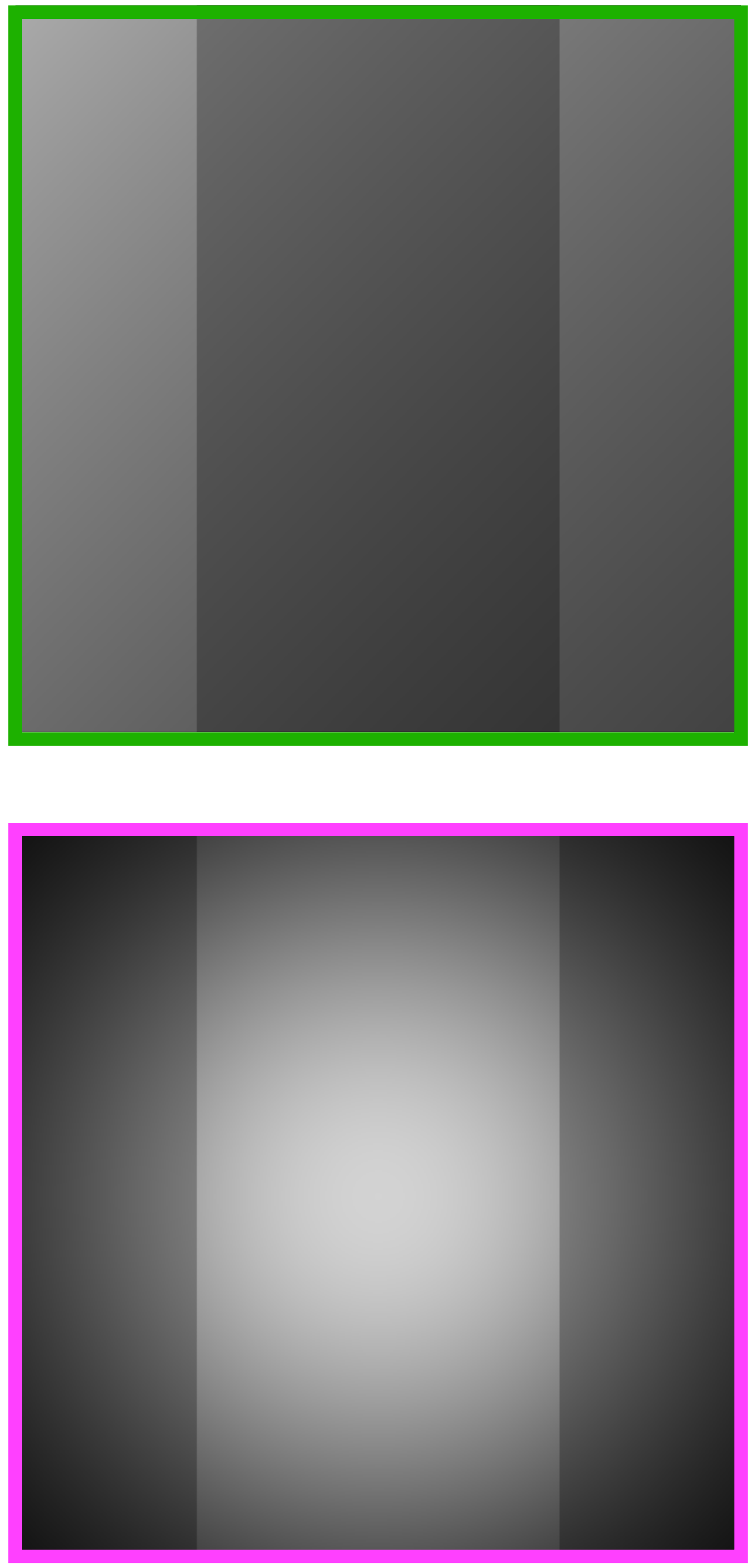
Strata-alignment affects Convergence

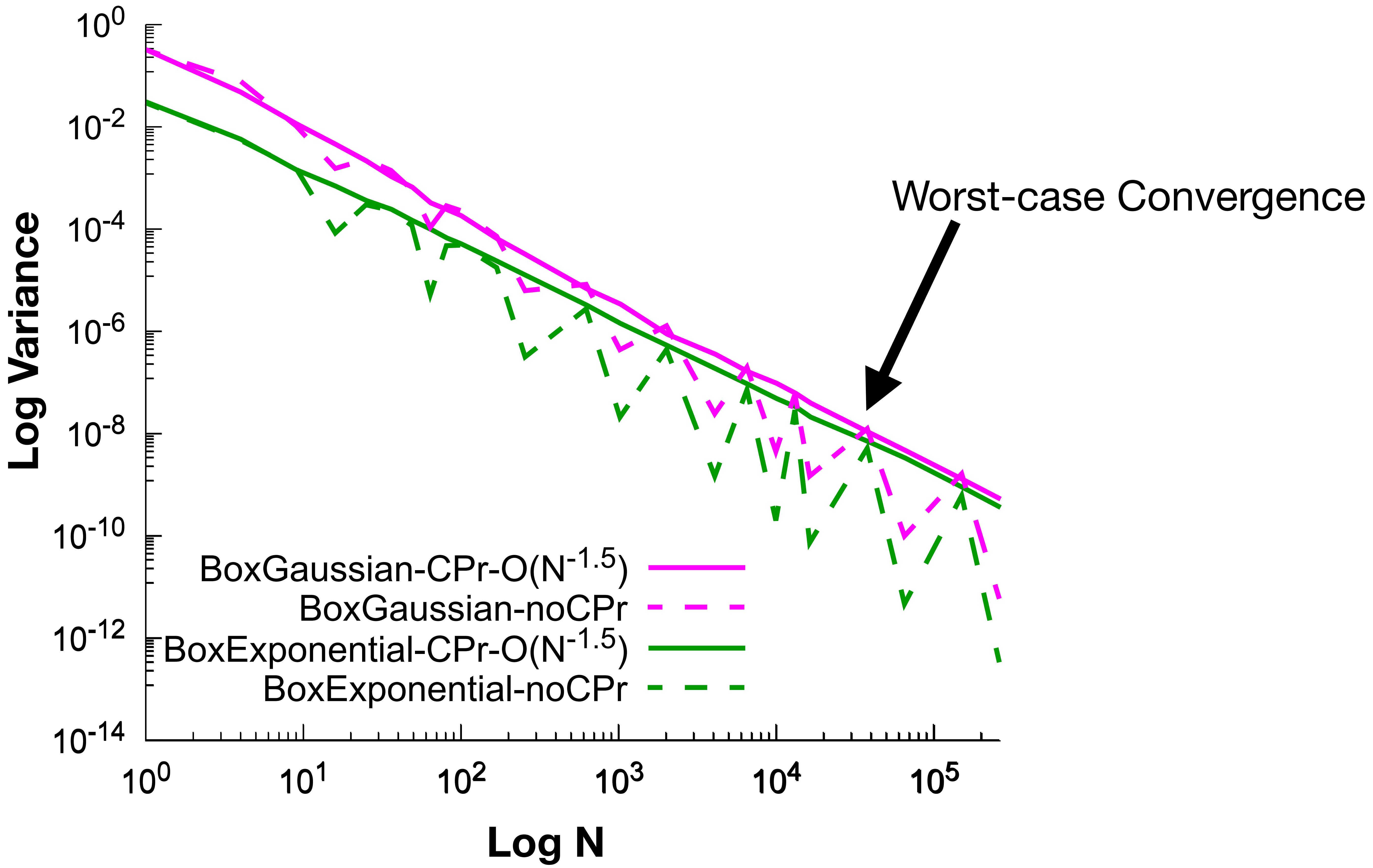






Homogenization Destroys Good Correlations





Fourier series based Variance Formulation

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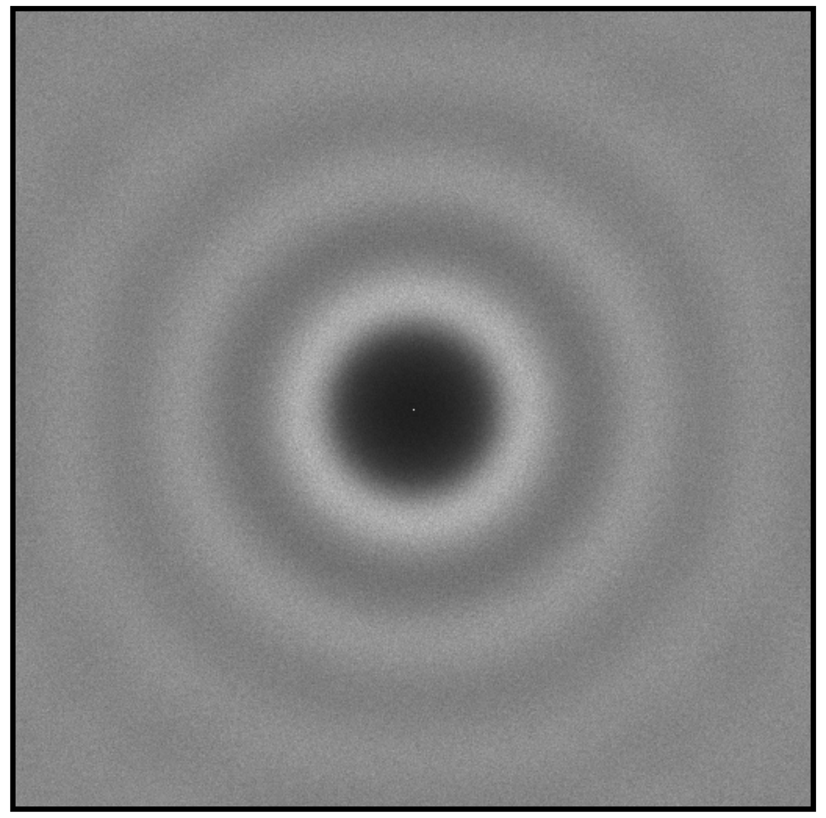

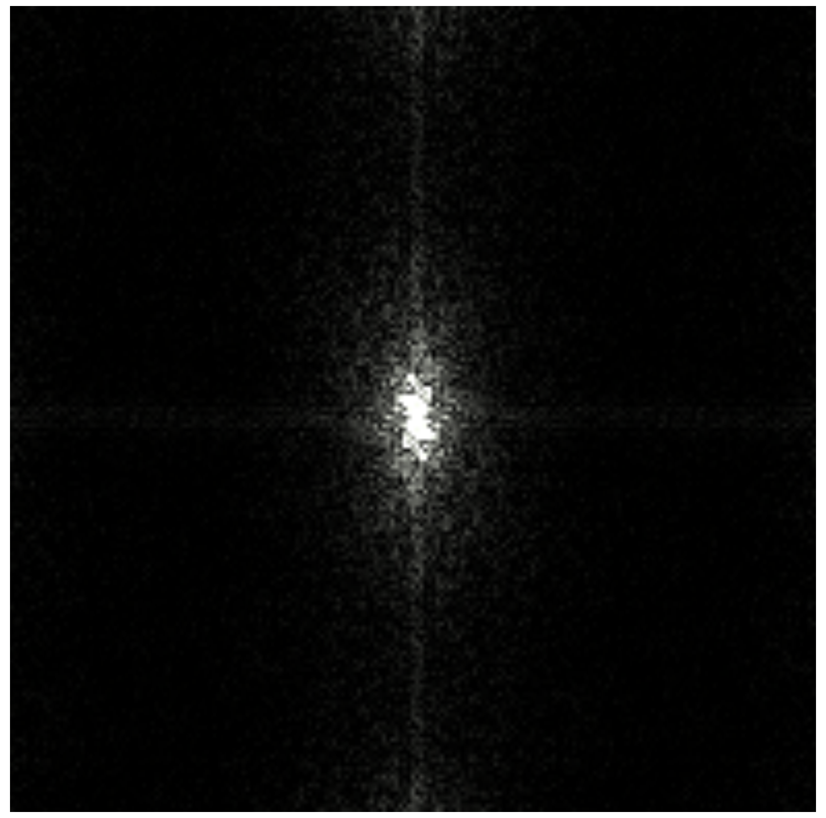

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Finite sampling domain is not properly handled

Homogenization could destroy good correlations

Generalized Variance Formulation

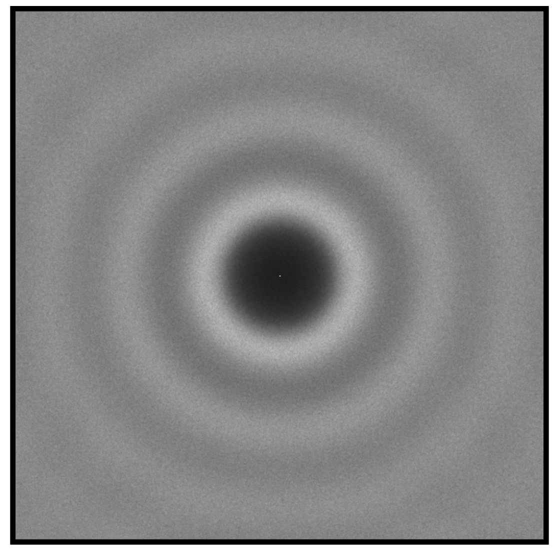
based on Fourier Series

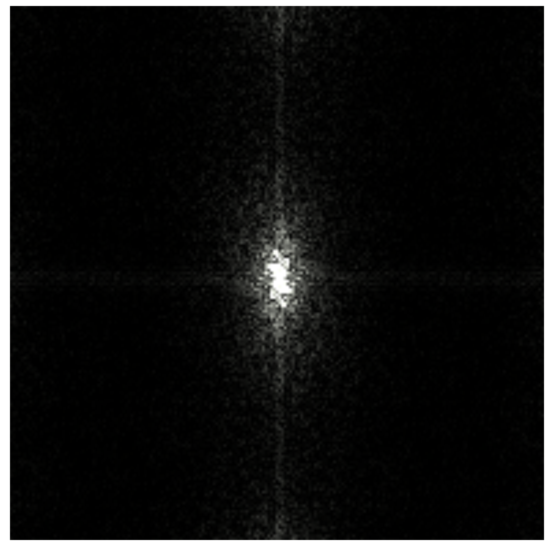
Generalized Variance Formulation

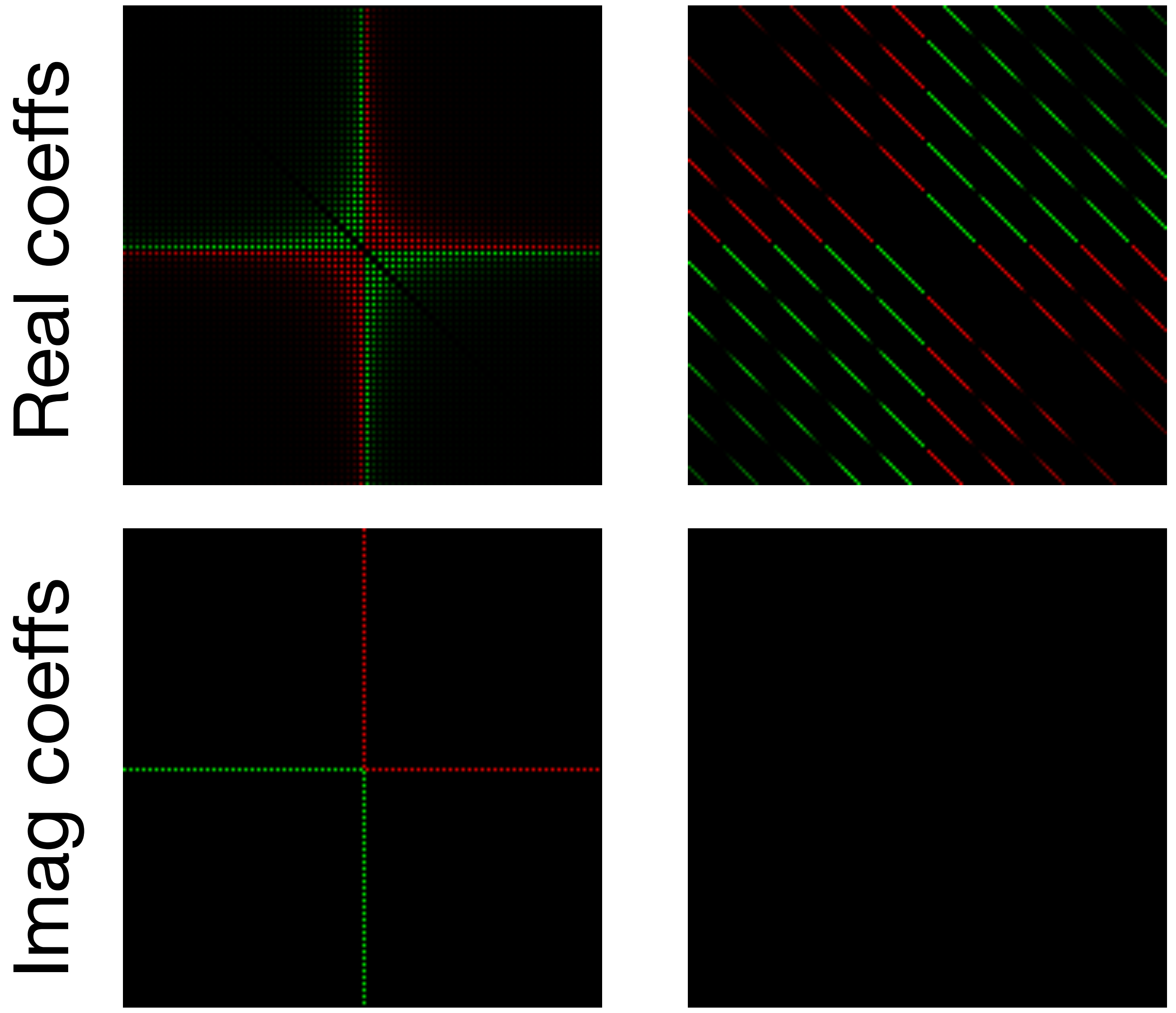
Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle \mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle \mathbf{S}_m^*\mathbf{S}_l\rangle$$

DC component

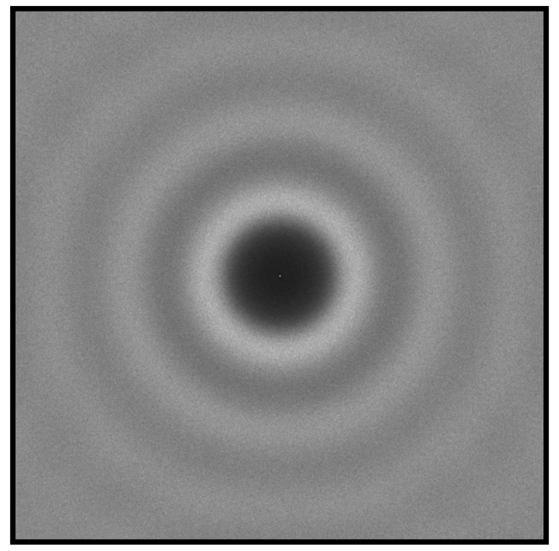


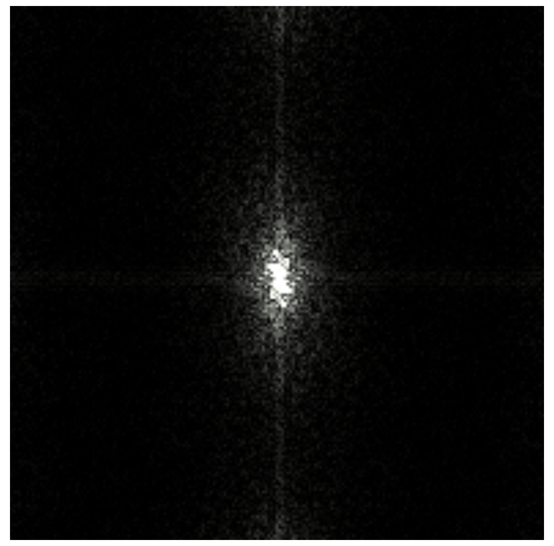




Variance Formulation: For Homogenized Samples

$$\mathrm{Var}(I_N) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^* \mathbf{f}_m \langle \mathbf{S}_m^* \mathbf{S}_m \rangle$$

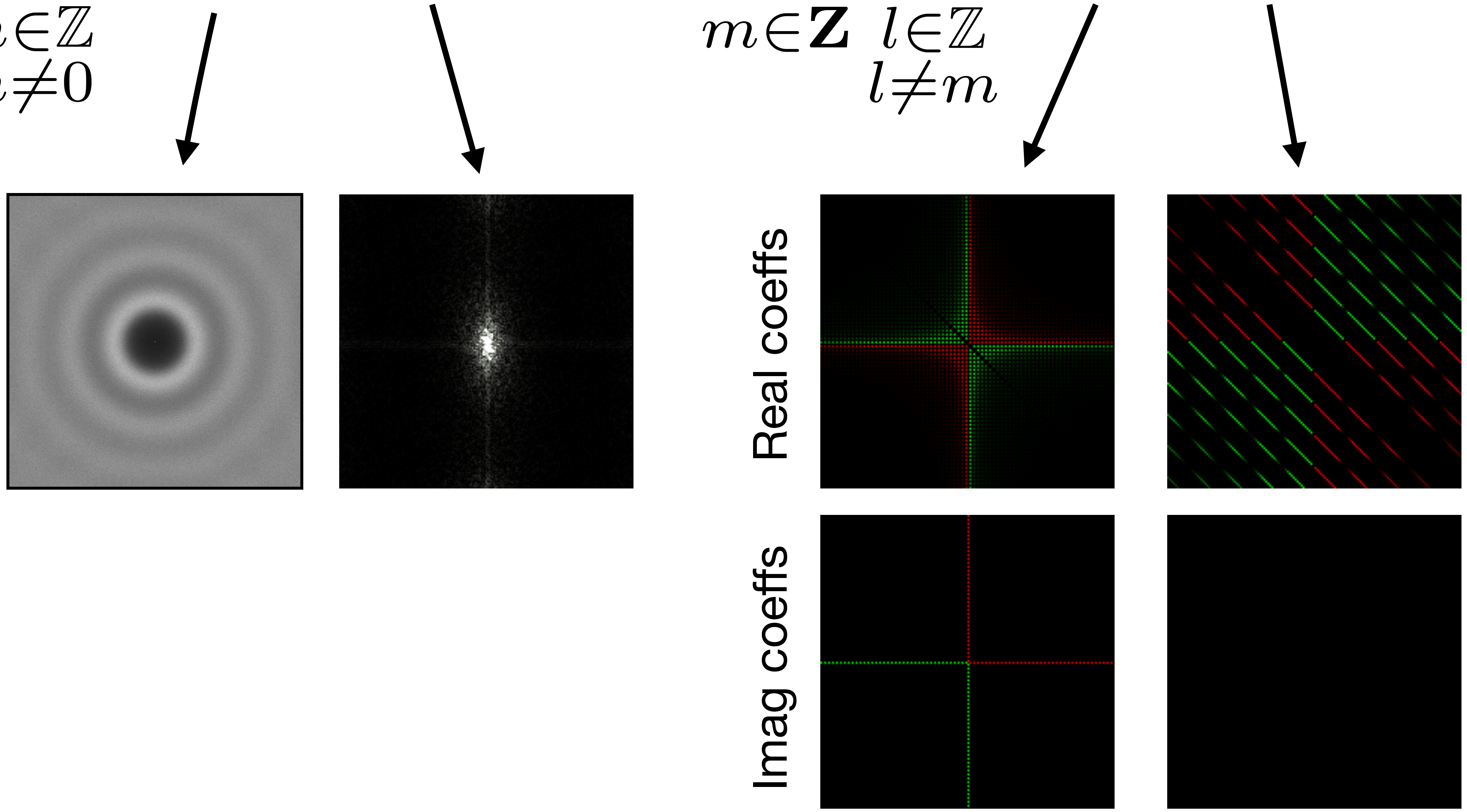




Generalized Variance Formulation

Third term

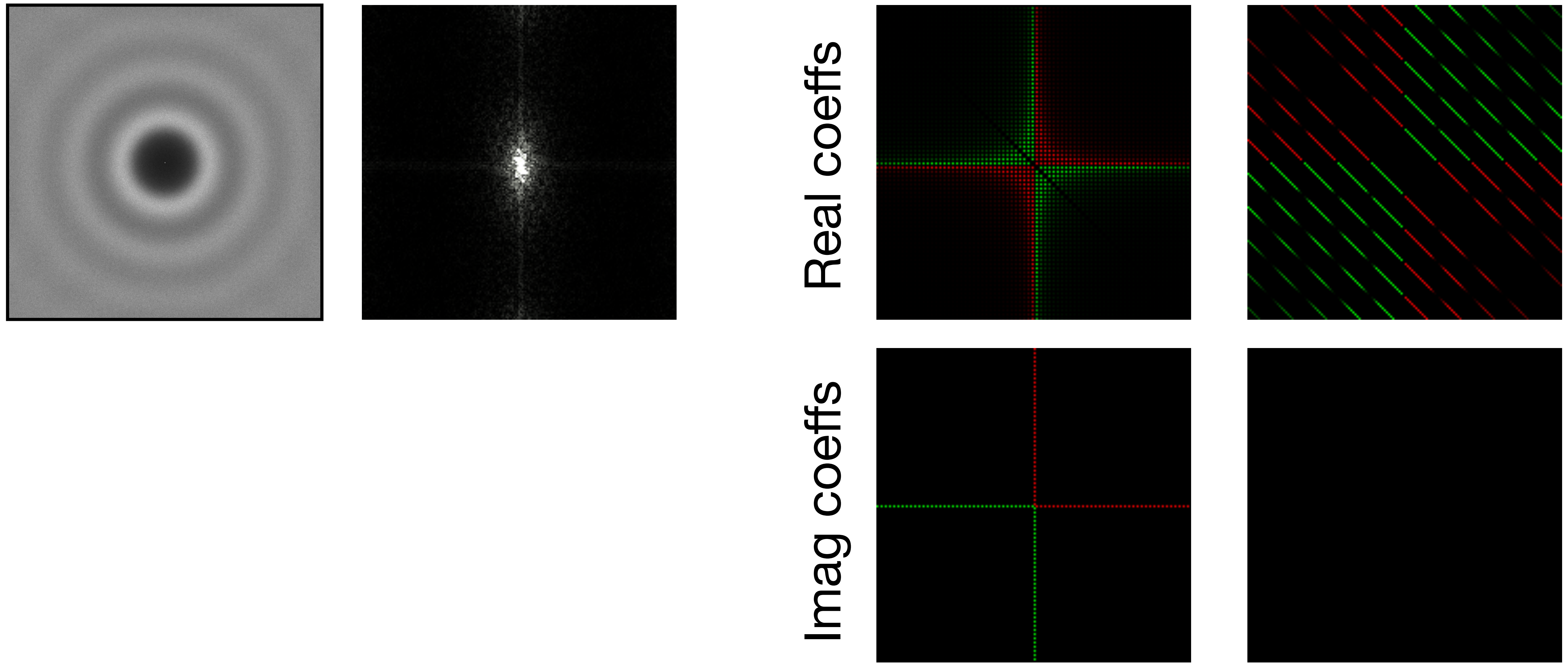
$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle\mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle\mathbf{S}_m^*\mathbf{S}_l\rangle$$



Generalized Variance Formulation

Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle \mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle \mathbf{S}_m^*\mathbf{S}_l\rangle$$



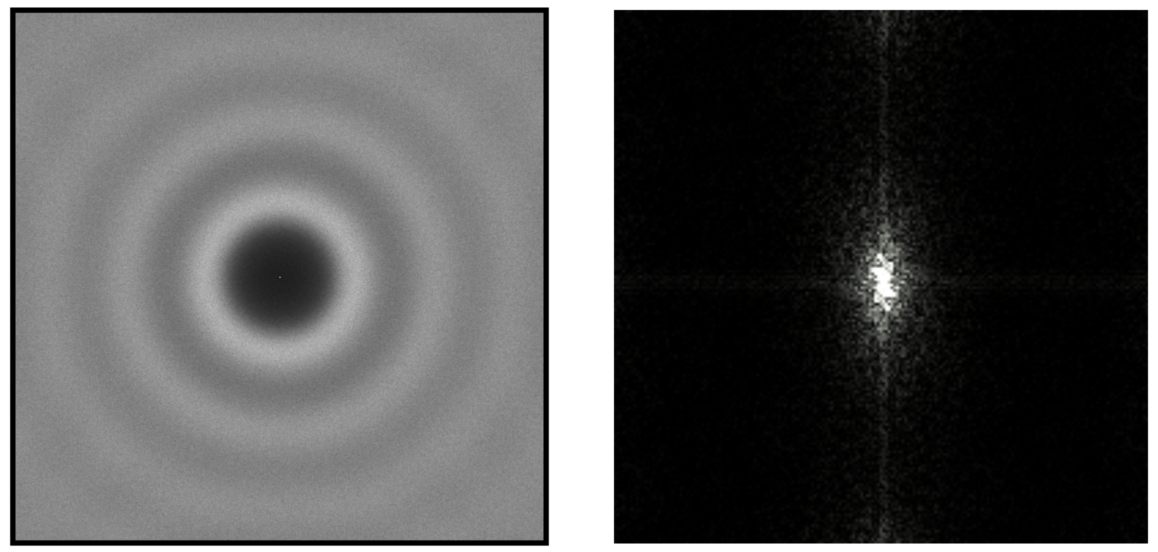
Covariance Matrix Form

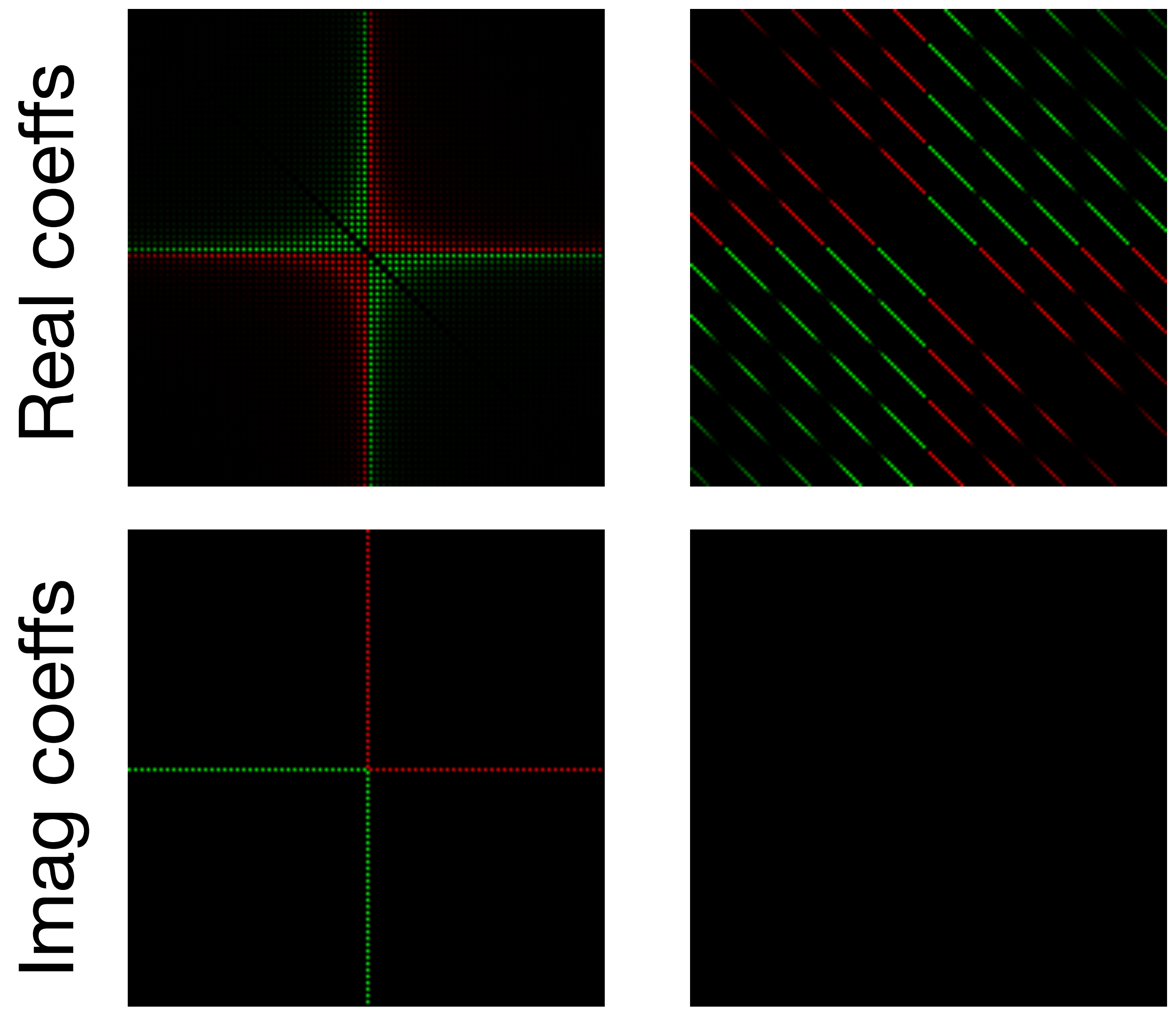
$$\begin{pmatrix} I^2\mathrm{Var}(\mathbf{S}_0) & & \\ & \ddots & \mathbf{f}_m^*\mathbf{f}_l \left\langle \mathbf{S}_m^*\mathbf{S}_l \right\rangle \\ & \mathbf{f}_m^*\mathbf{f}_m \langle \mathbf{S}_m^*\mathbf{S}_m \rangle & \\ \mathbf{f}_m^*\mathbf{f}_l \left\langle \mathbf{S}_m^*\mathbf{S}_l \right\rangle & & \ddots \end{pmatrix}$$

Generalized Variance Formulation

Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle \mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle \mathbf{S}_m^*\mathbf{S}_l\rangle$$





Generalized Variance Formulation

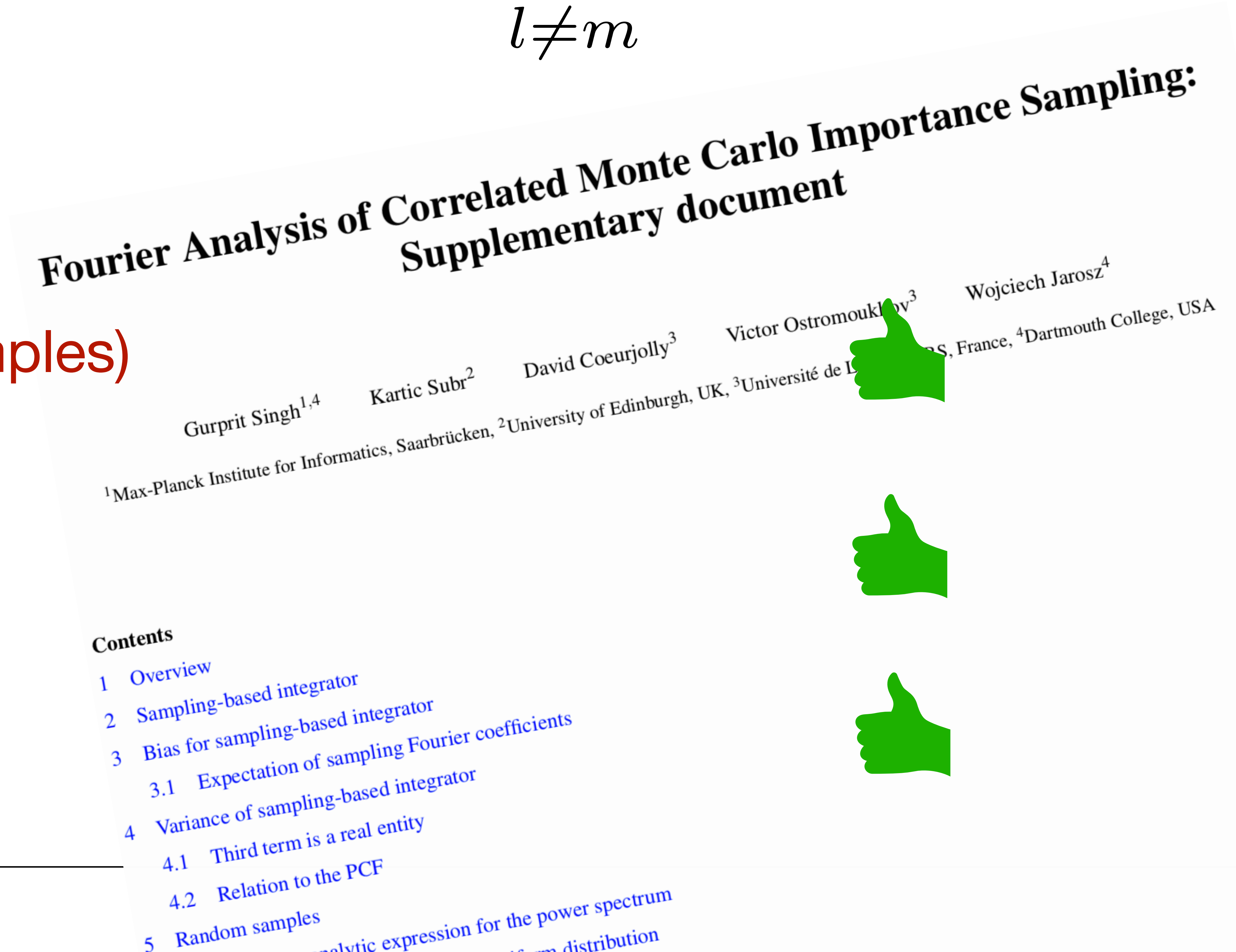
Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle \mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle \mathbf{S}_m^*\mathbf{S}_l\rangle$$

Valid for non-uniform PDFs (importance samples)

No Homogenization (CPr) performed

Finite sampling domain is properly handled


Fourier Analysis of Correlated Monte Carlo Importance Sampling:
Supplementary document

Gurprit Singh[1,4] Kartic Subr[2] David Coeurjolly[3] Victor Ostromoukhov[3] Wojciech Jarosz[4]

[1]Max-Planck Institute for Informatics, Saarbrücken, [2]University of Edinburgh, UK, [3]Université de L... ...S, France, [4]Dartmouth College, USA

Contents

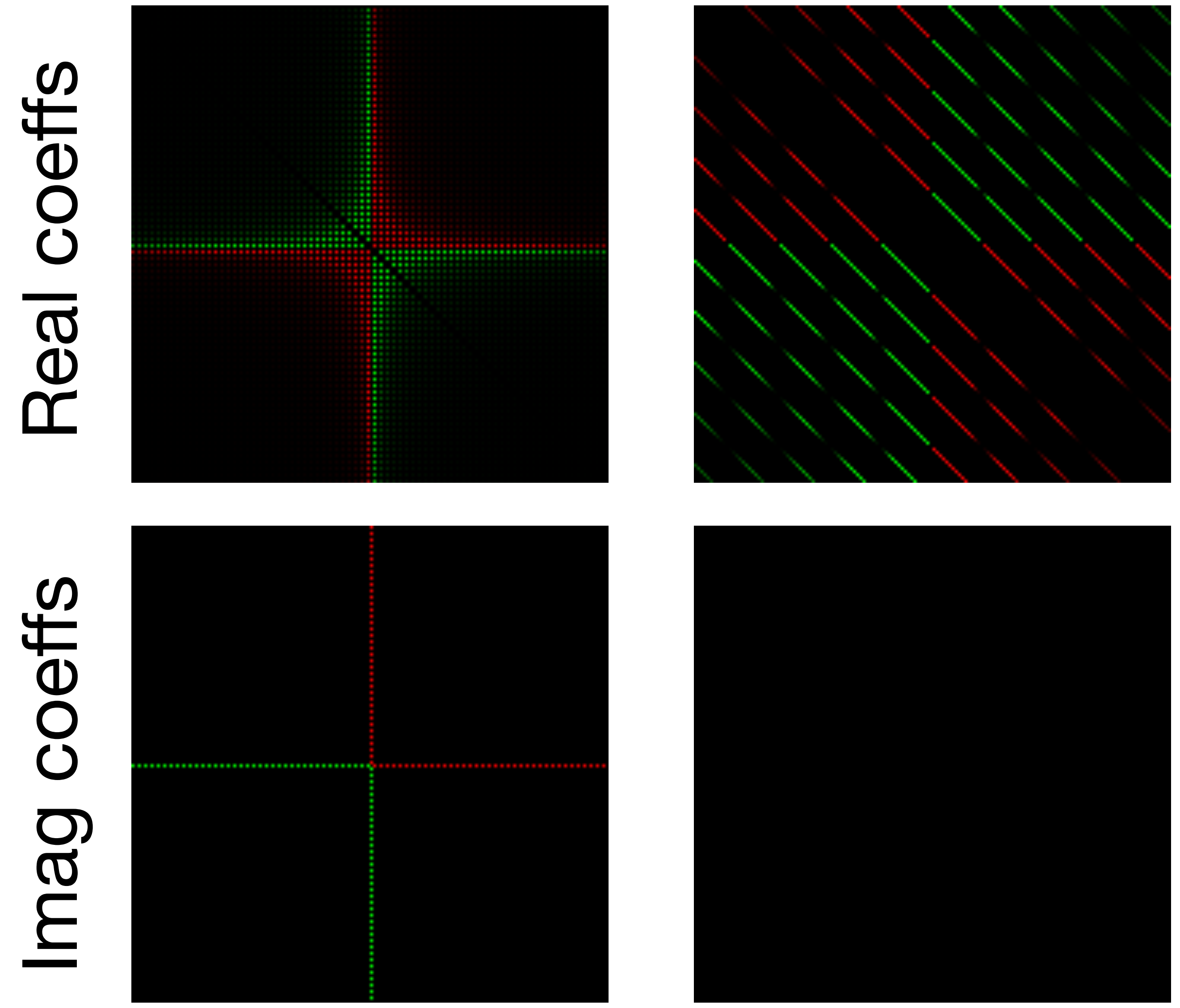
1 Overview
2 Sampling-based integrator
3 Bias for sampling-based integrator
3.1 Expectation of sampling Fourier coefficients
4 Variance of sampling-based integrator
4.1 Third term is a real entity
4.2 Relation to the PCF
5 Random samples

Third Term is Crucial

Generalized Variance Formulation: Third Term Crucial

Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle \mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle \mathbf{S}_m^*\mathbf{S}_l\rangle$$



First term cannot be ignored for IS variance prediction

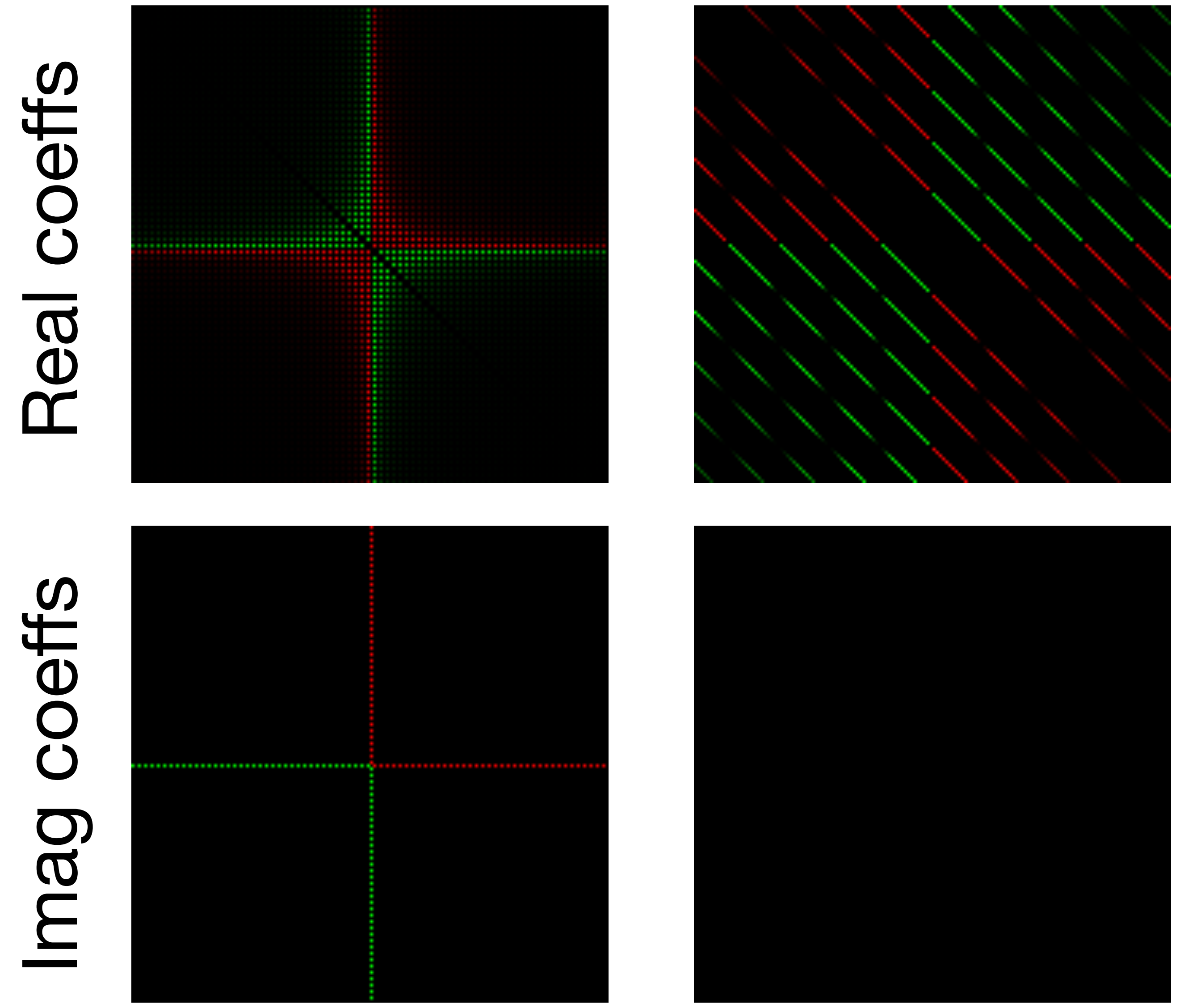
Third term allows correct prediction of variance:

- when samples and integrand have correlations
- during importance sampling

Generalized Variance Formulation: Third Term Crucial

Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle\mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle\mathbf{S}_m^*\mathbf{S}_l\rangle$$



Second term is always positive

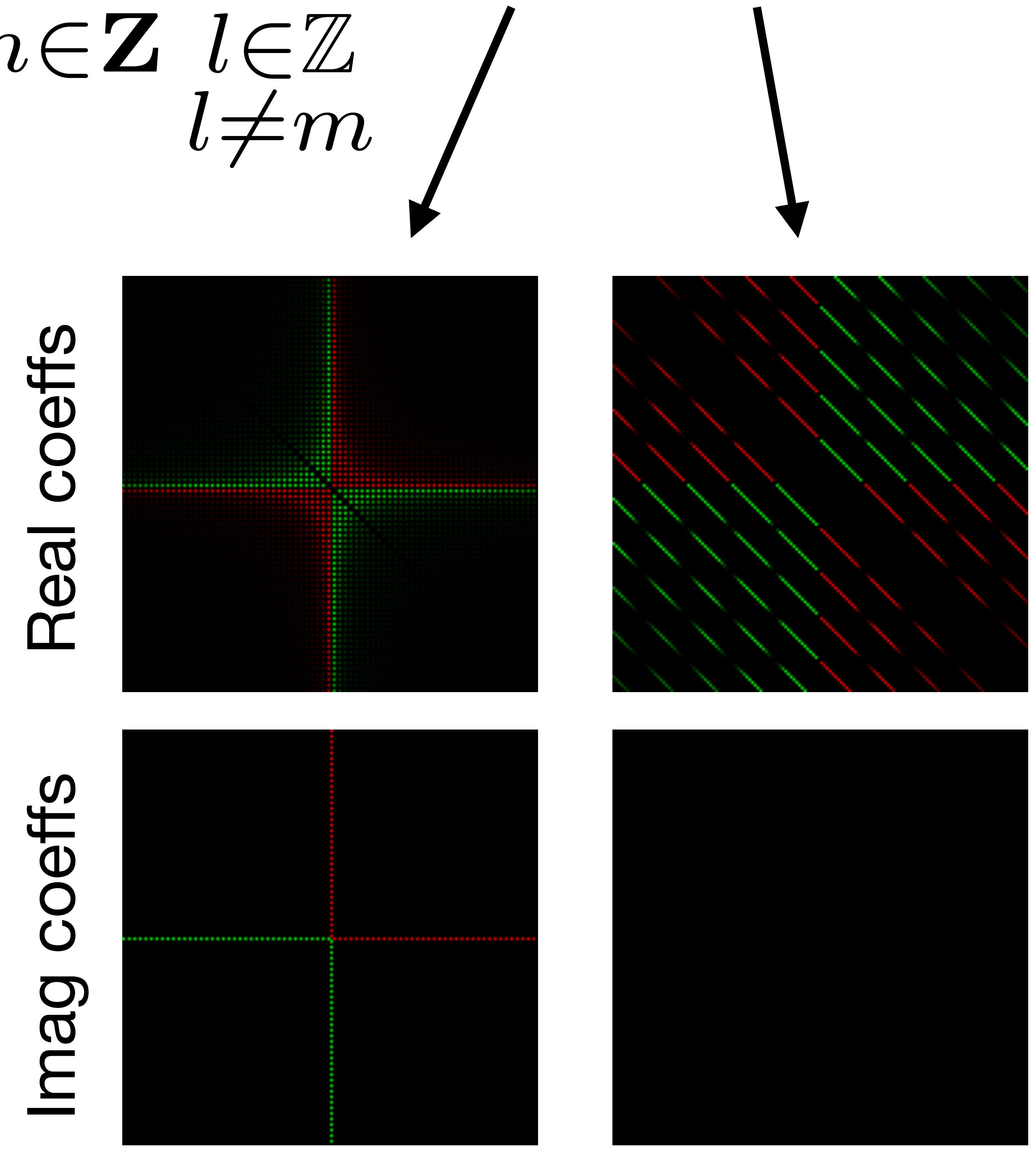
For constant PDF, first term is zero, therefore, third term is negative and reduces variance

With IS, both the first and the third term reduces variance

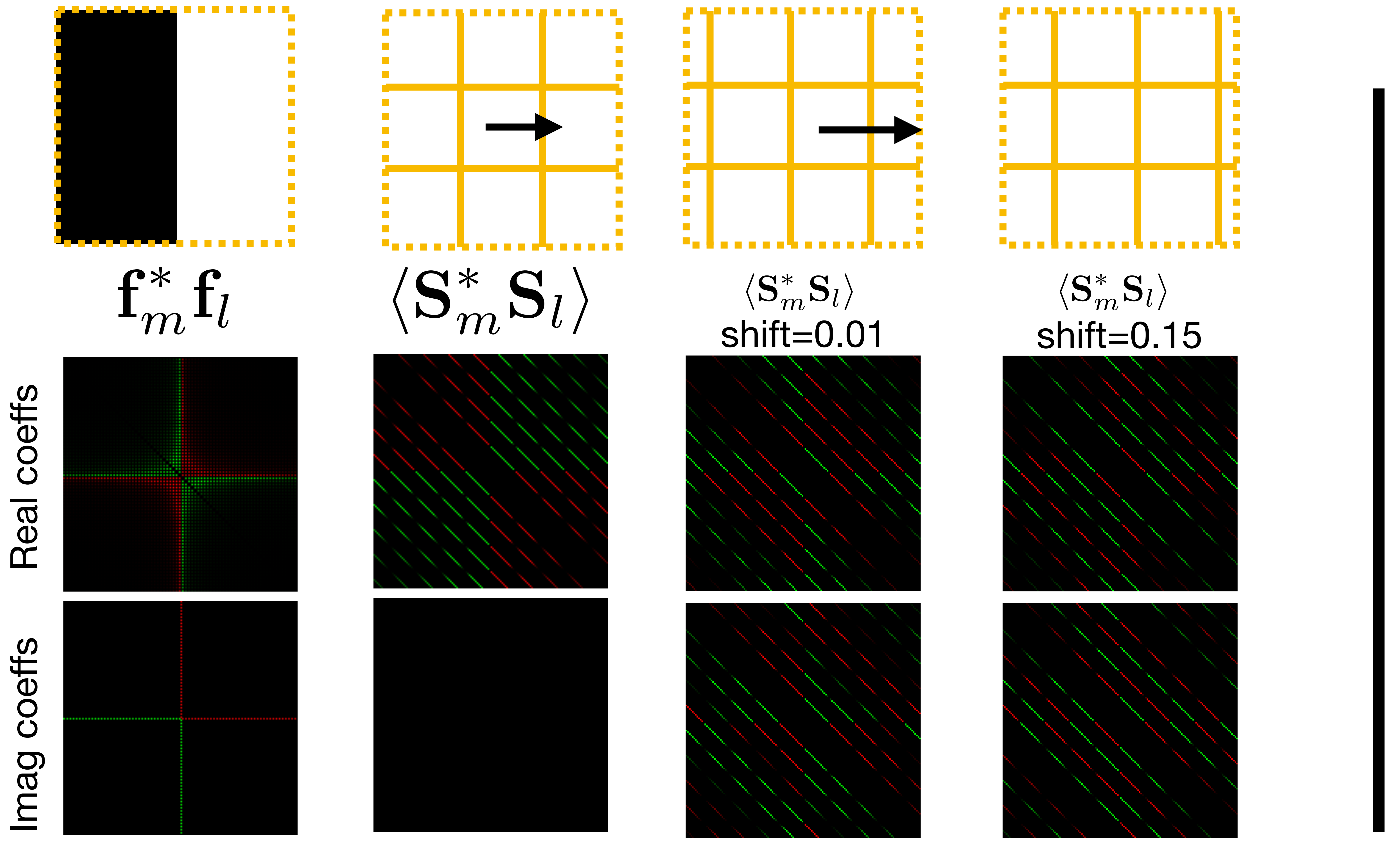
Third term is difficult to analyze

Third term

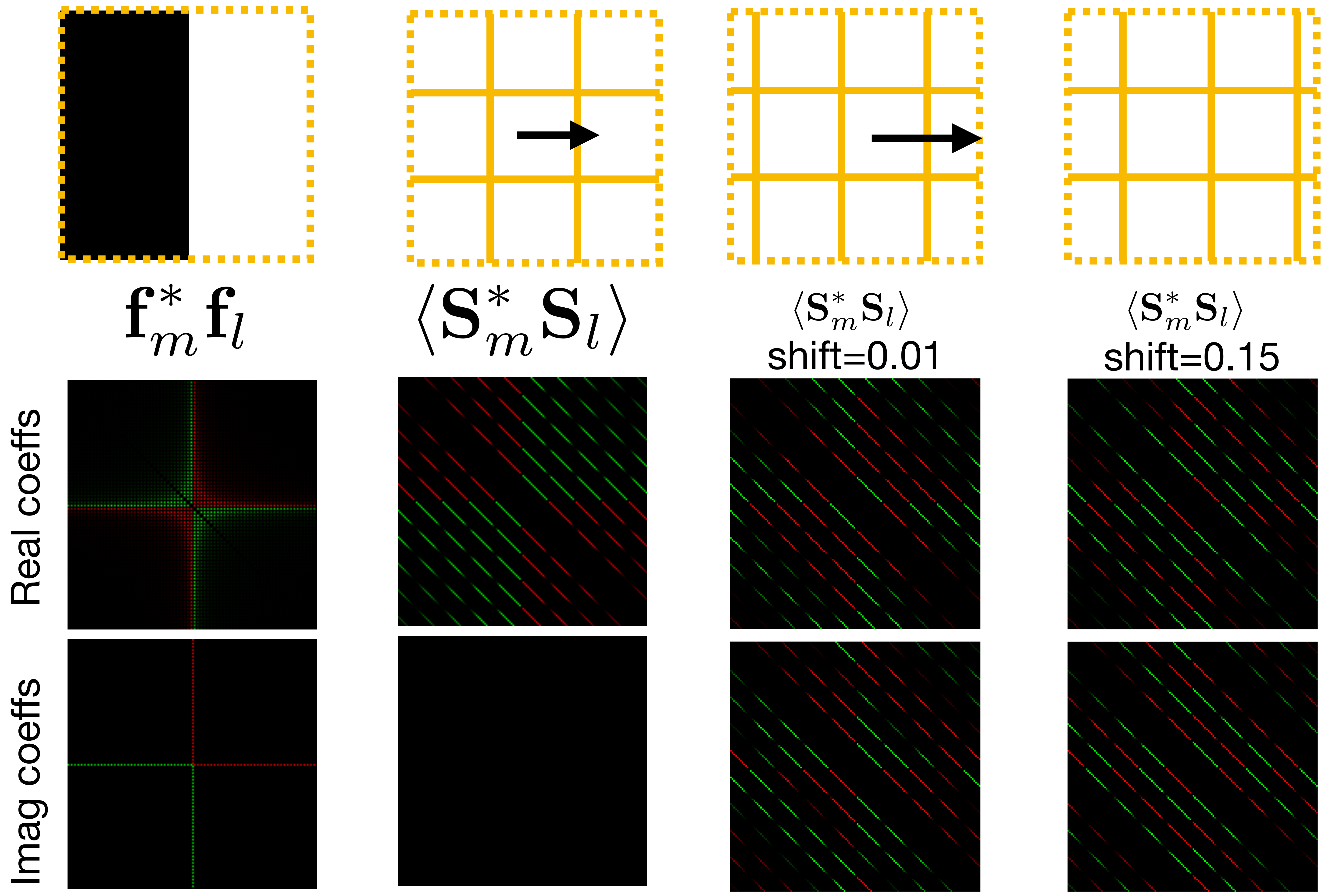
$$\sum_{m \in \mathbf{Z}} \sum_{\substack{l \in \mathbb{Z} \\ l \neq m}} \mathbf{f}_m^* \mathbf{f}_l \langle \mathbf{S}_m^* \mathbf{S}_l \rangle$$

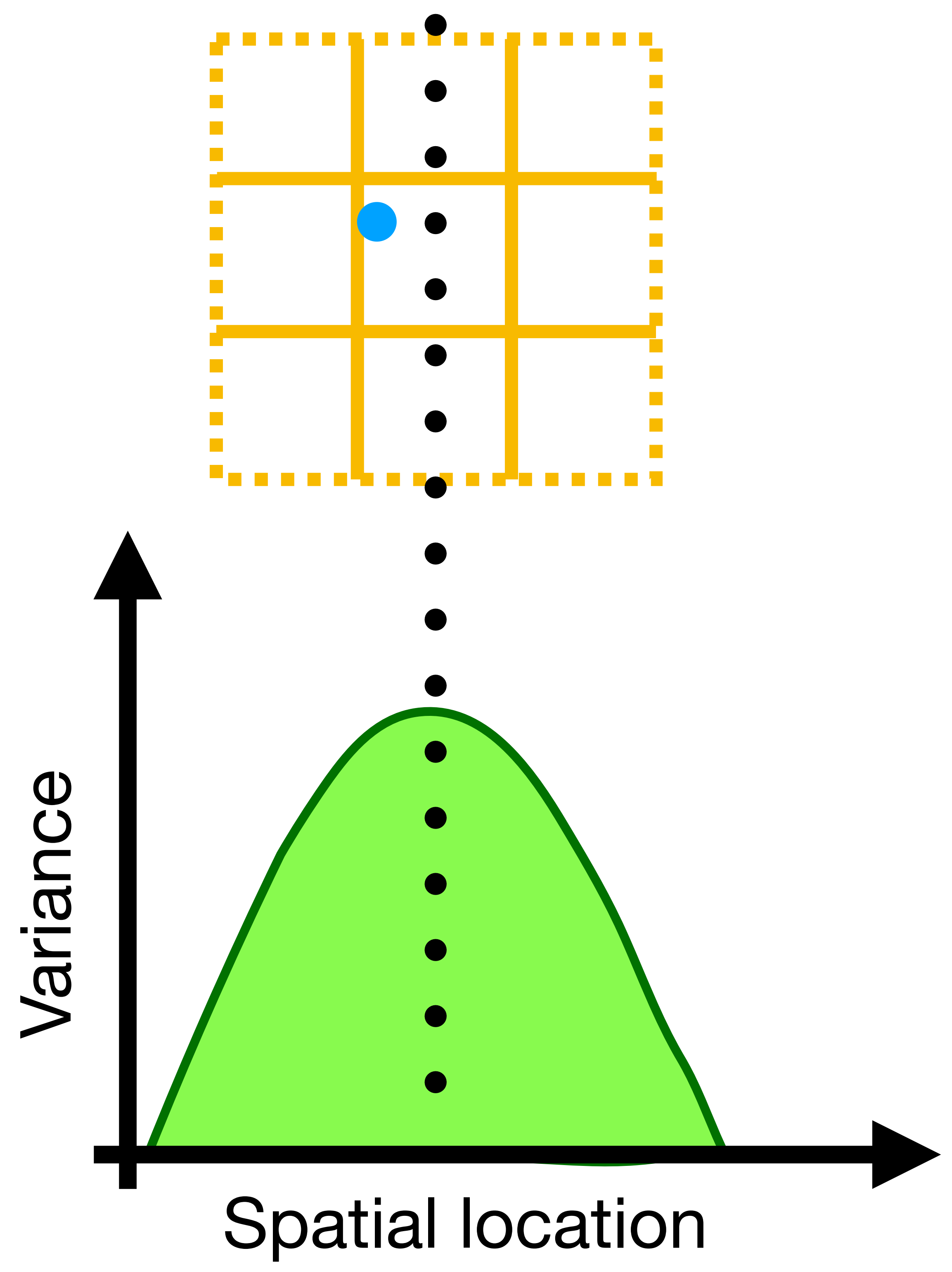


Third Term: Encodes phase



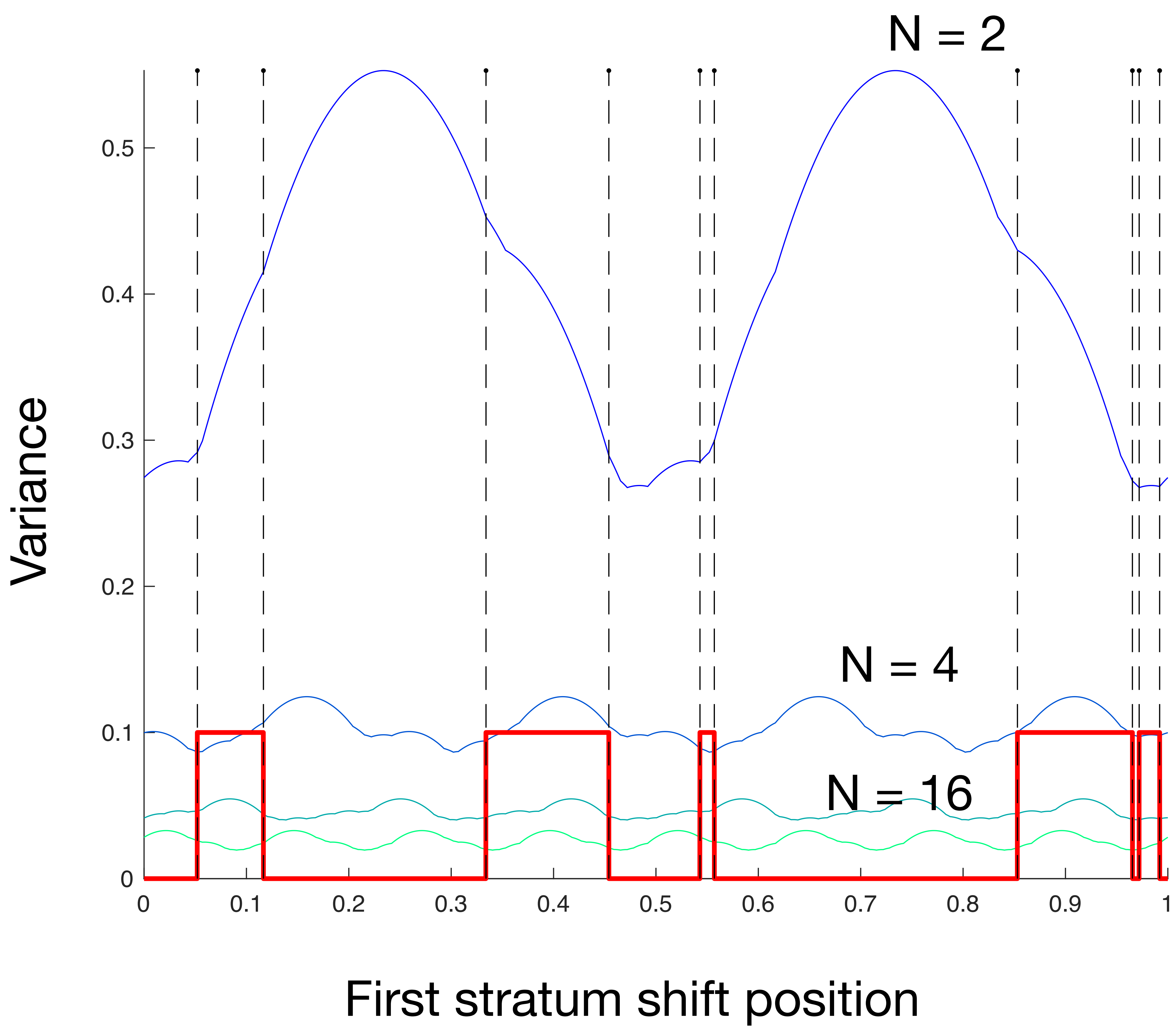
Third Term: Encodes phase

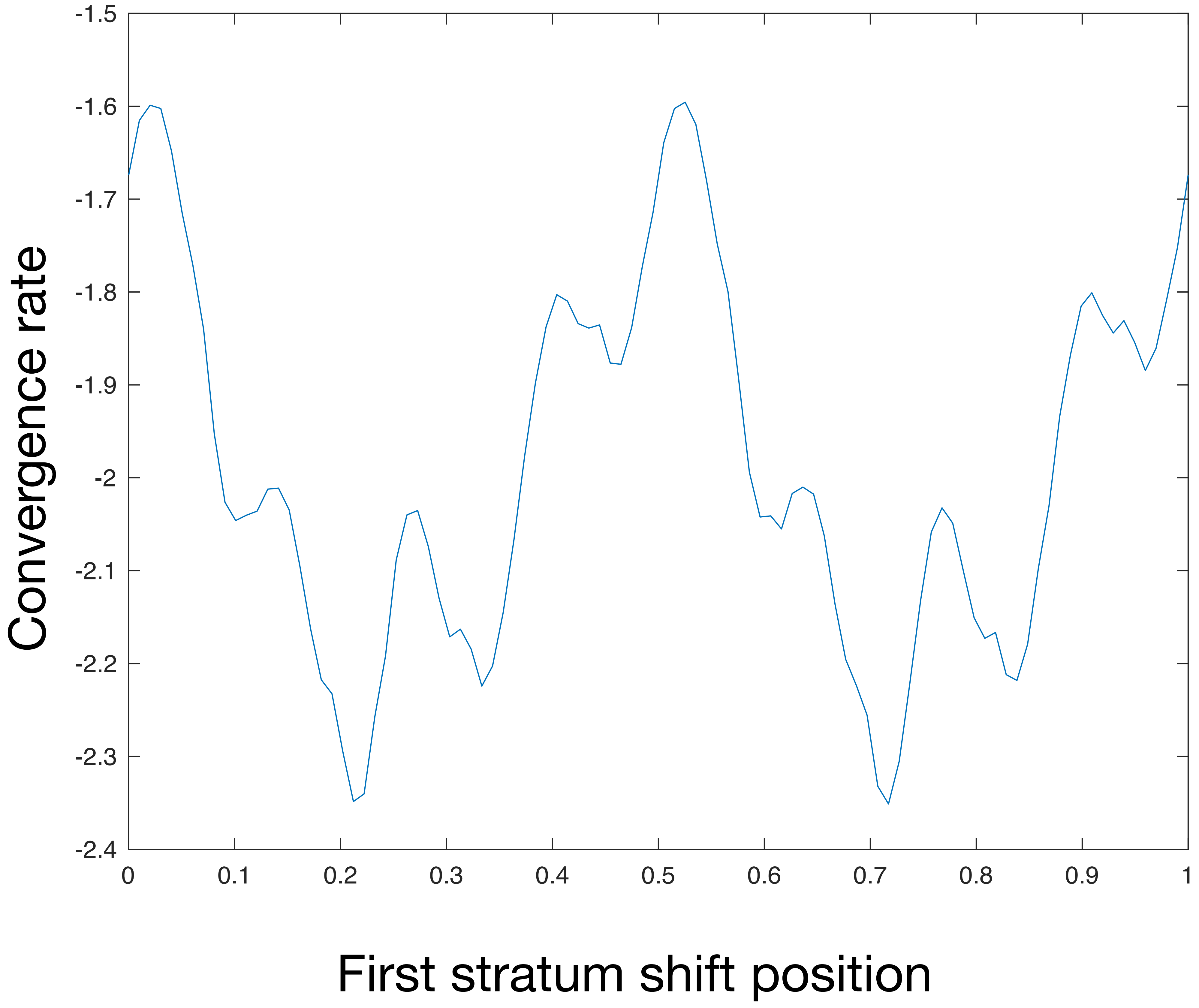




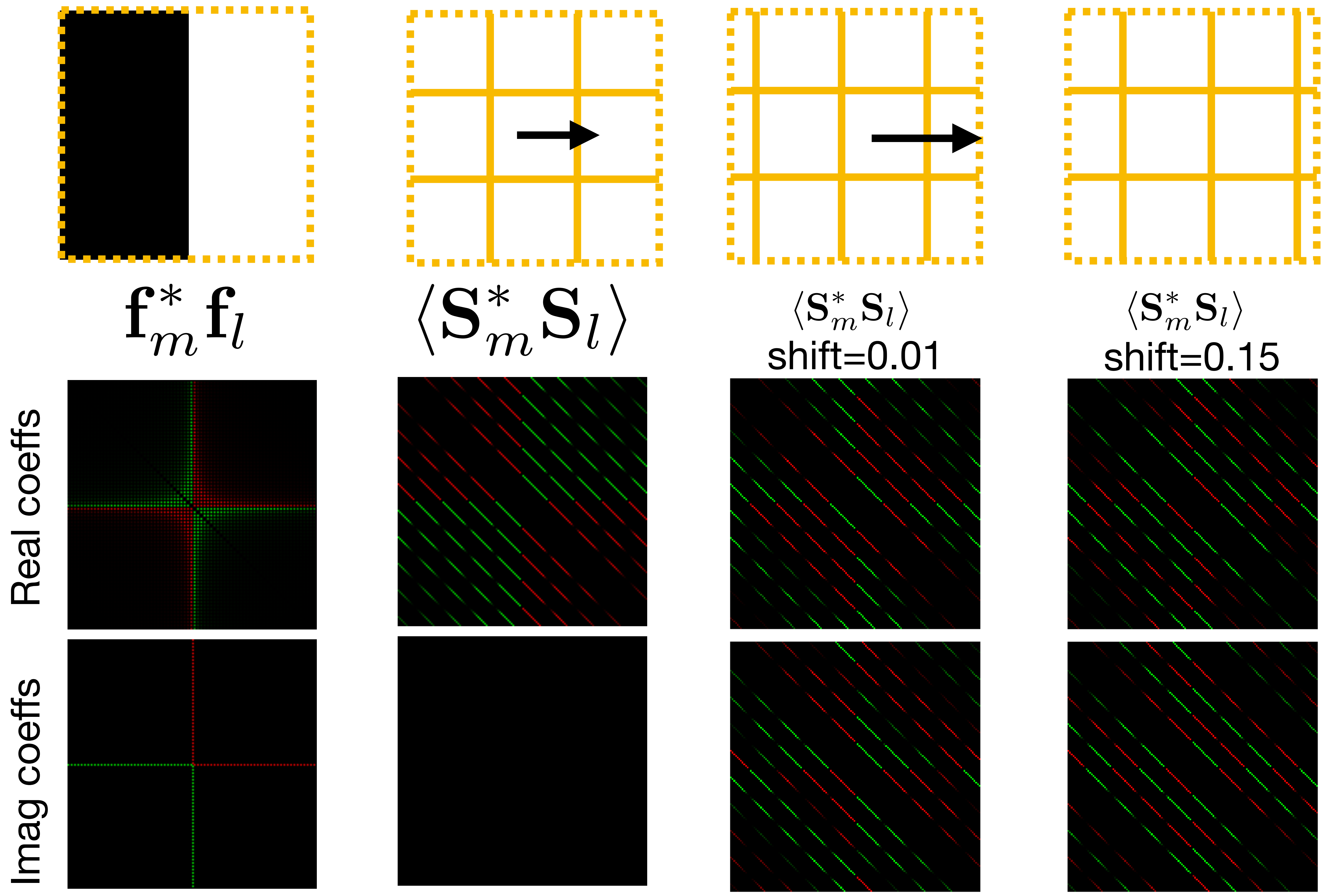
Ramamoorthi et al.[2012]

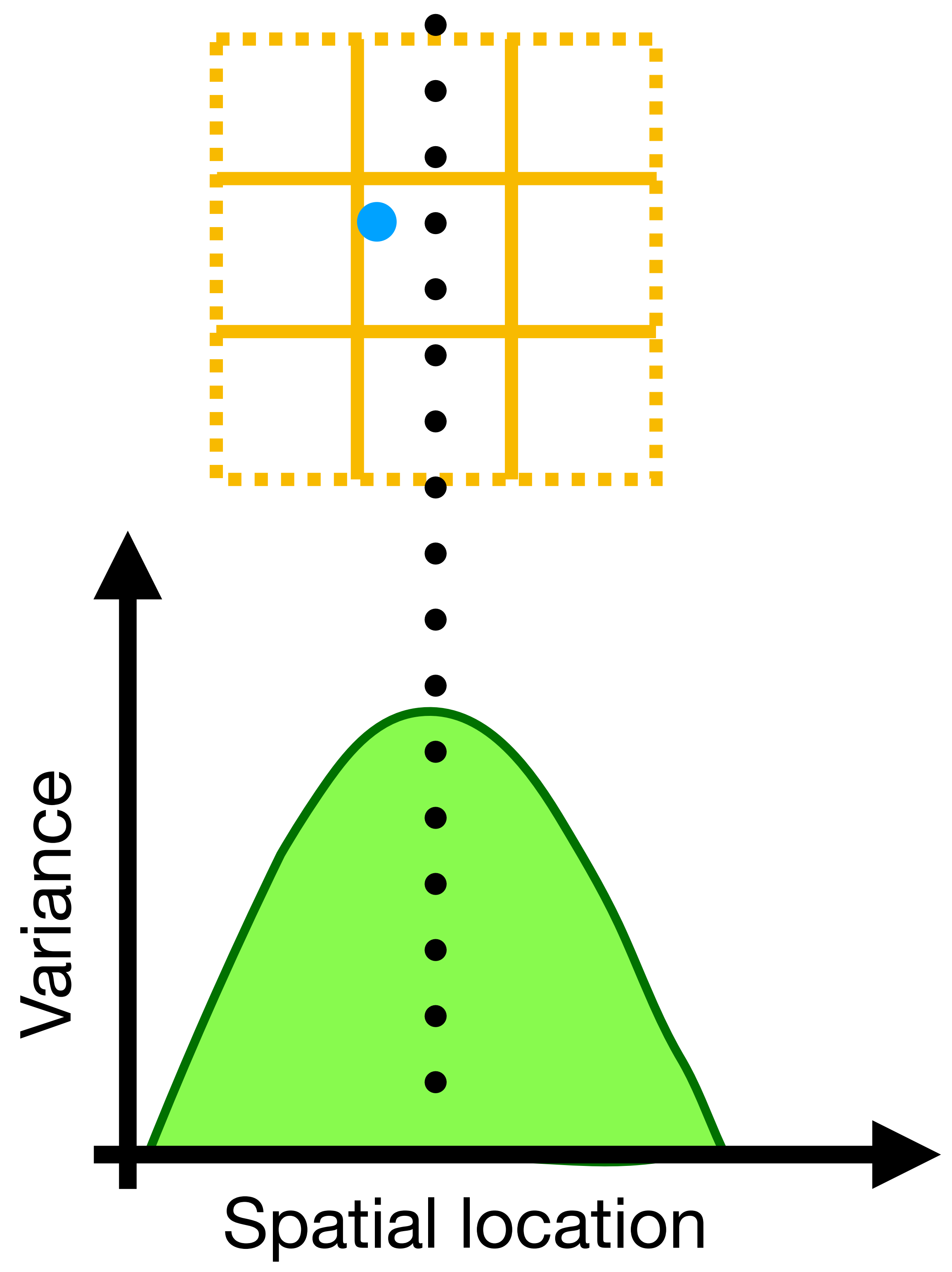
Strata shifting affects convergence





Third Term: Encodes phase





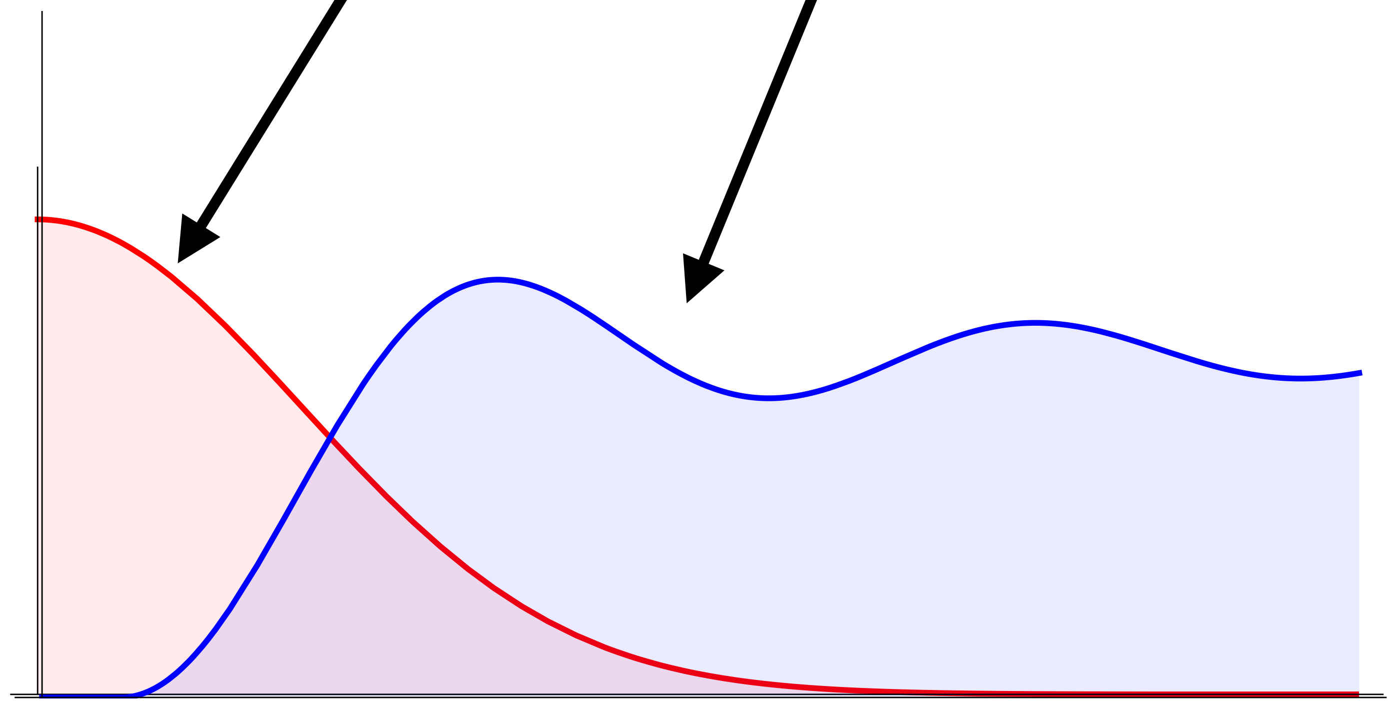
Ramamoorthi et al.[2012]

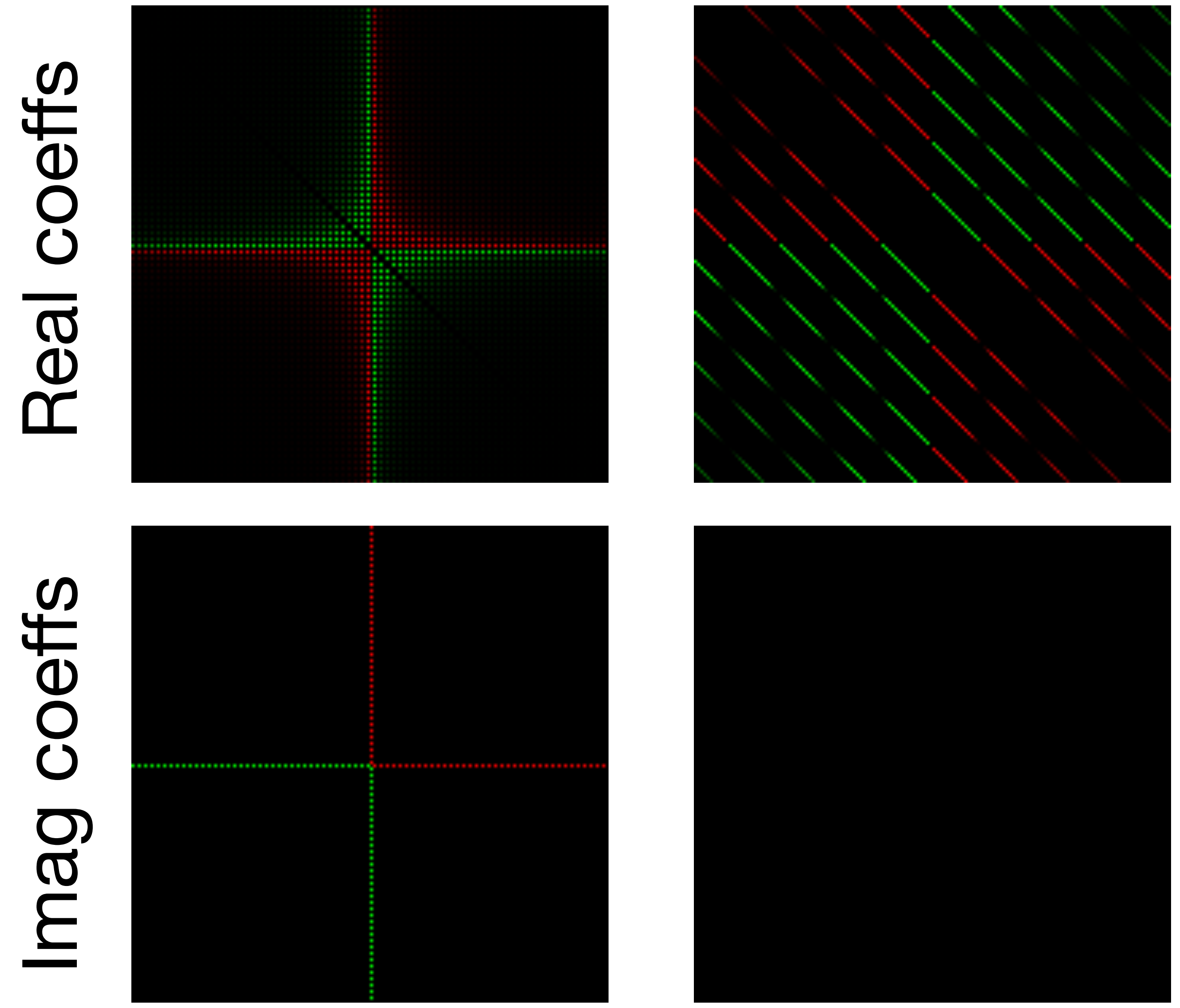
Third Term: Dimensionality grows fast

Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^* \mathbf{f}_m \langle \mathbf{S}_m^* \mathbf{S}_m \rangle + \sum_{m\in\mathbf{Z}} \sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^* \mathbf{f}_l \langle \mathbf{S}_m^* \mathbf{S}_l \rangle$$

For one-dimensional problem:





Correlated Importance Sampling

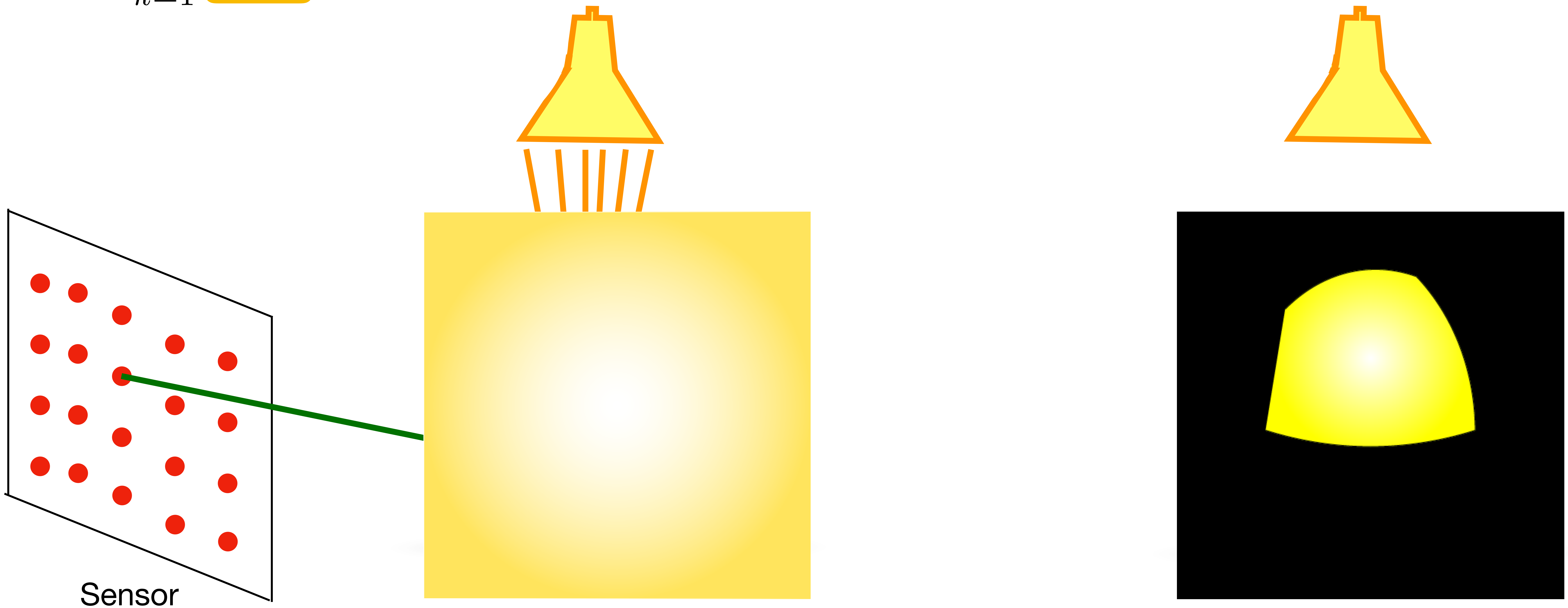
affects convergence rate

Direct Illumination Integral

$$I_N = \frac{1}{N} \sum_{k=1}^{N} \frac{f(\vec{x}_k)}{p(\vec{x}_k)}$$

Light PDF Sampling

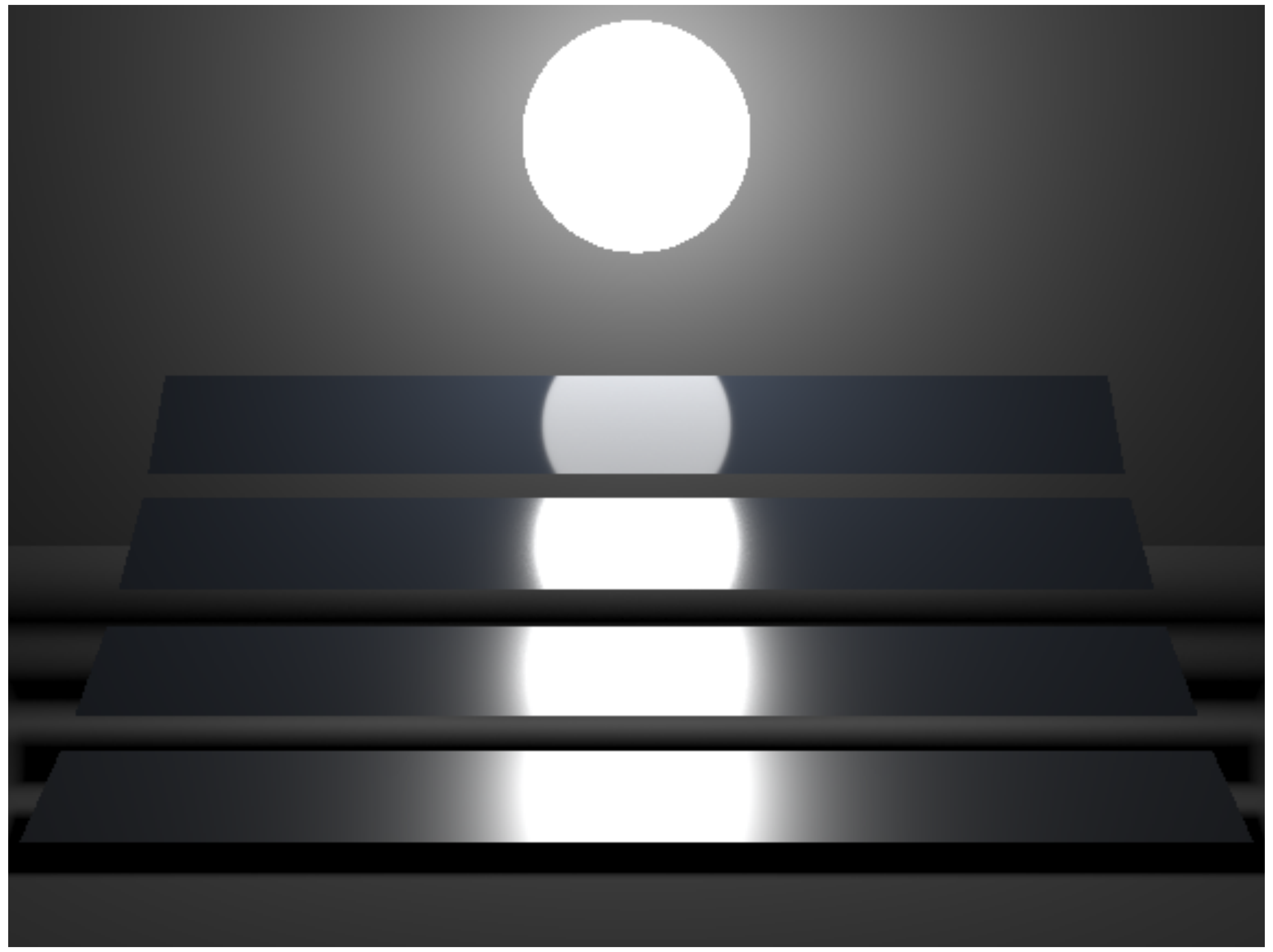
BSDF PDF Sampling



Light IS

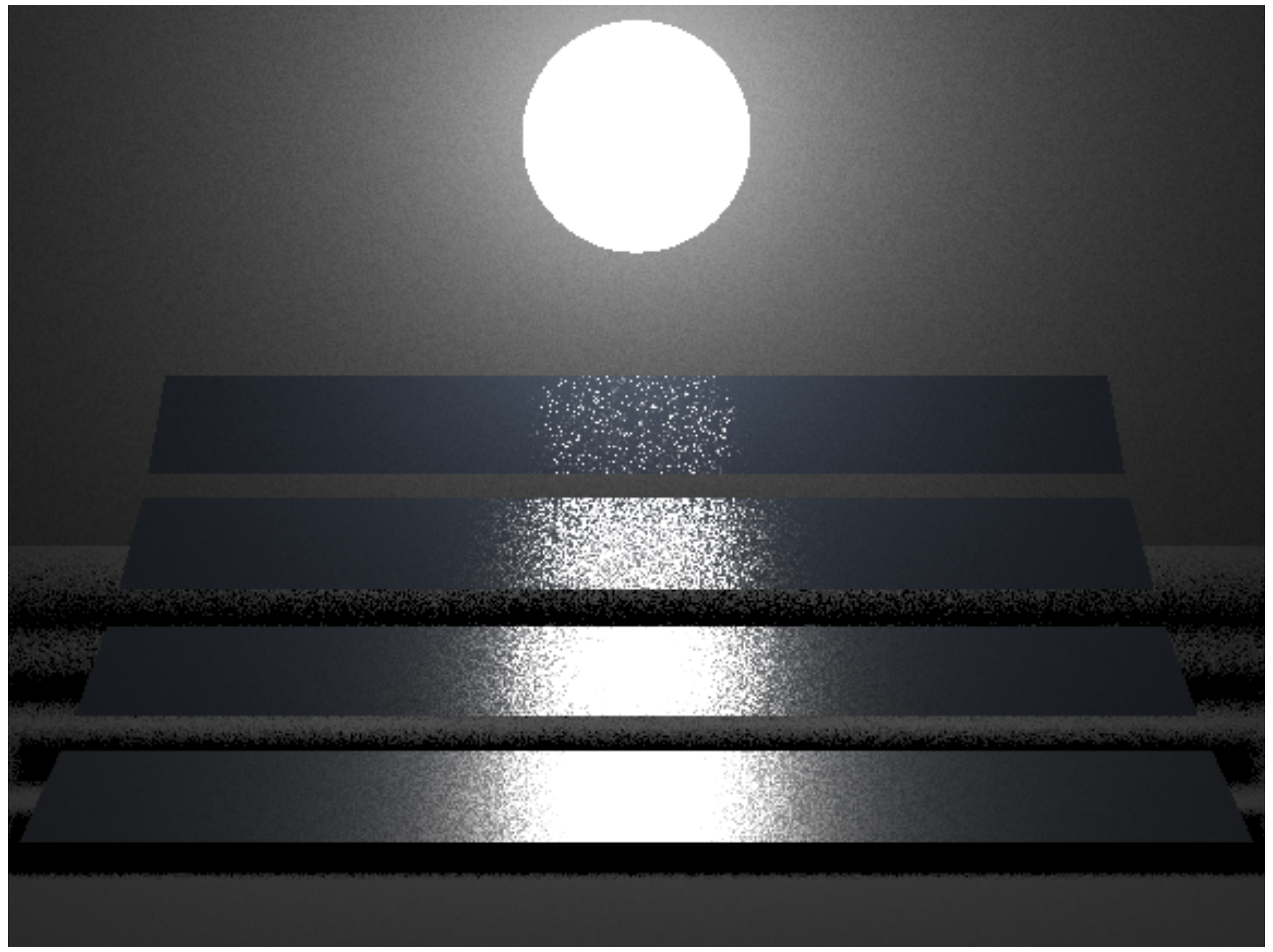
BSDF IS

Veach Scene: Multiple Importance Sampling



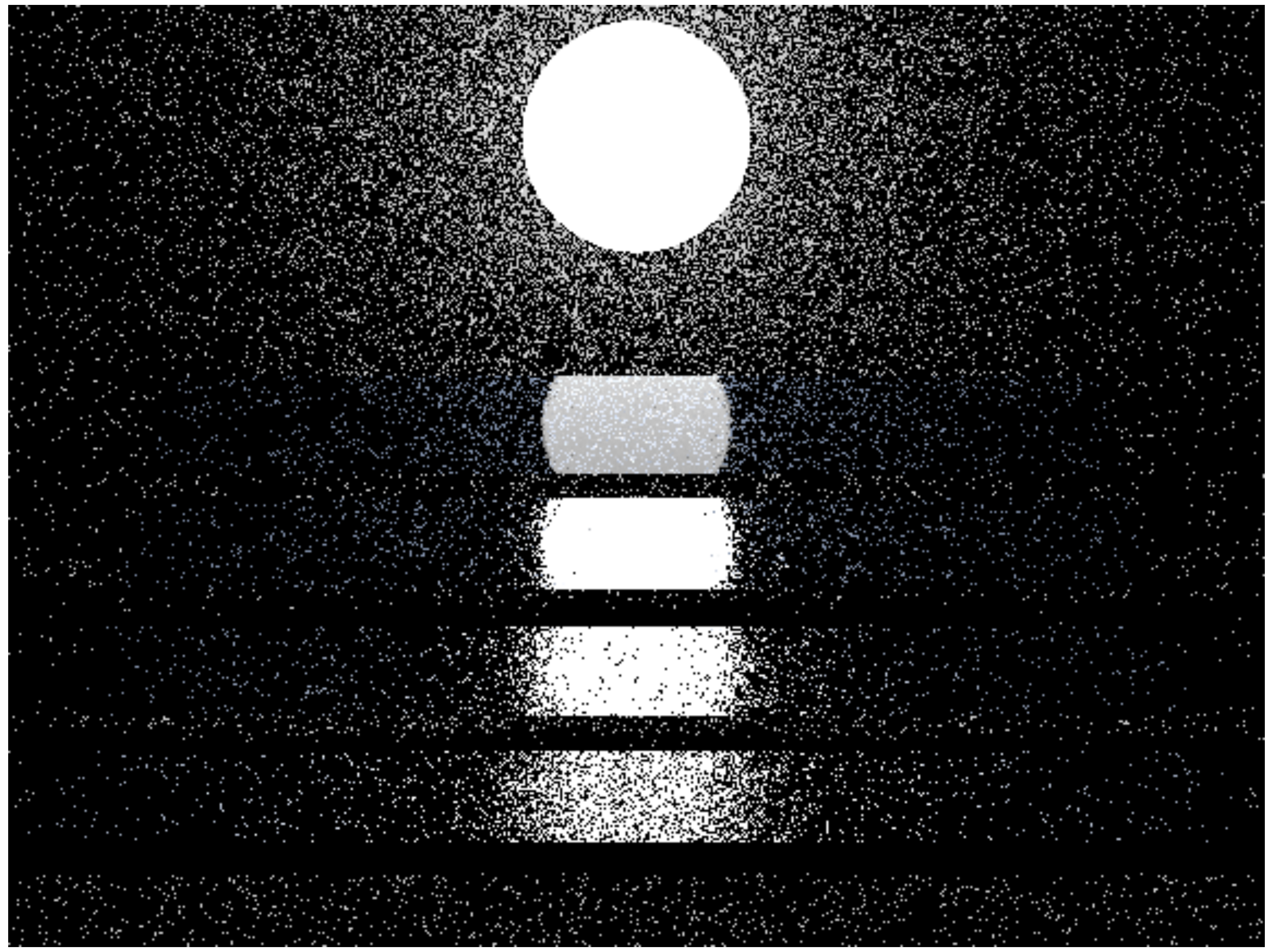
Reference image

N = 1024 spp



Light importance sampling

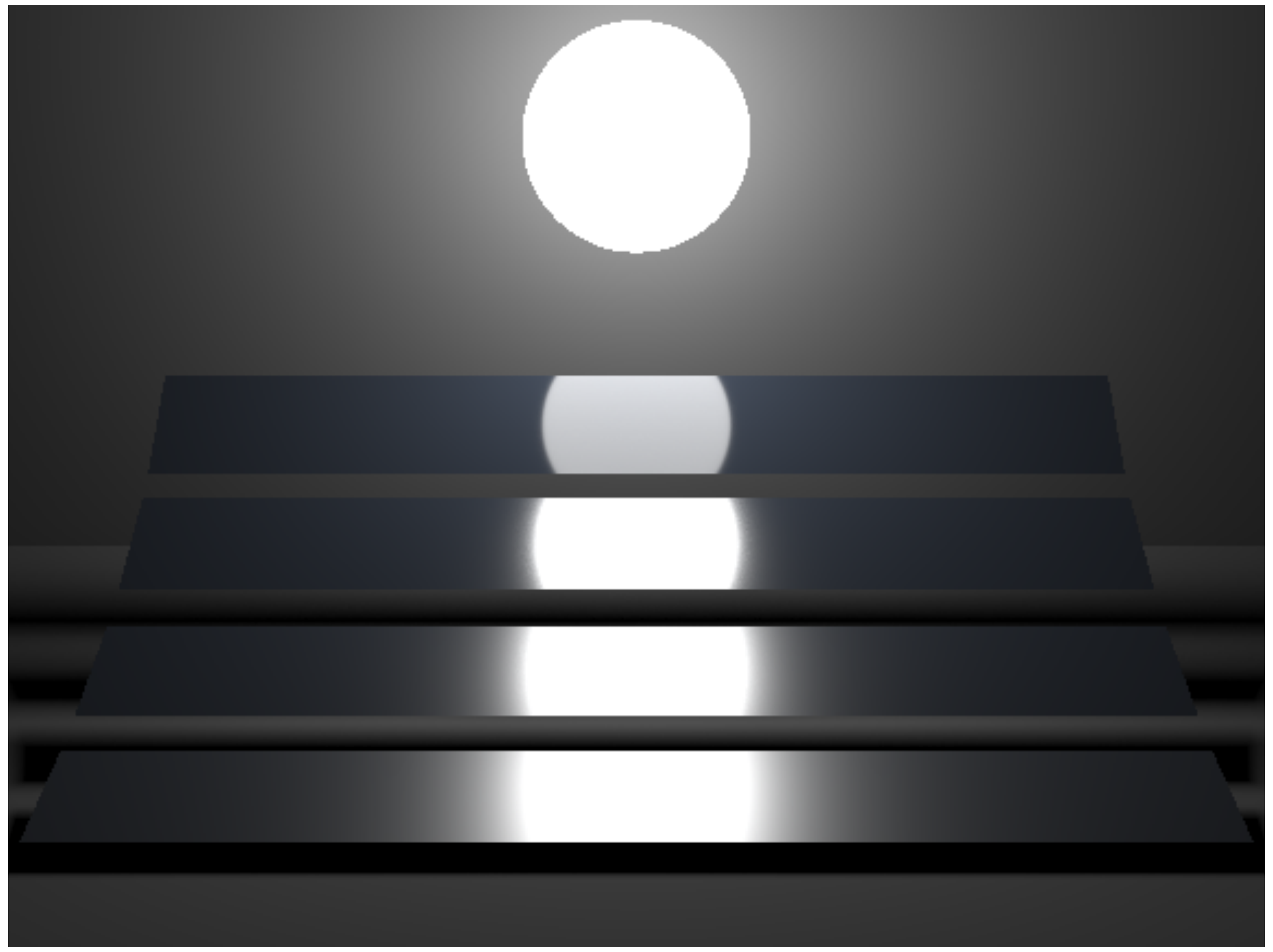
N = 4 spp



BSDF importance sampling

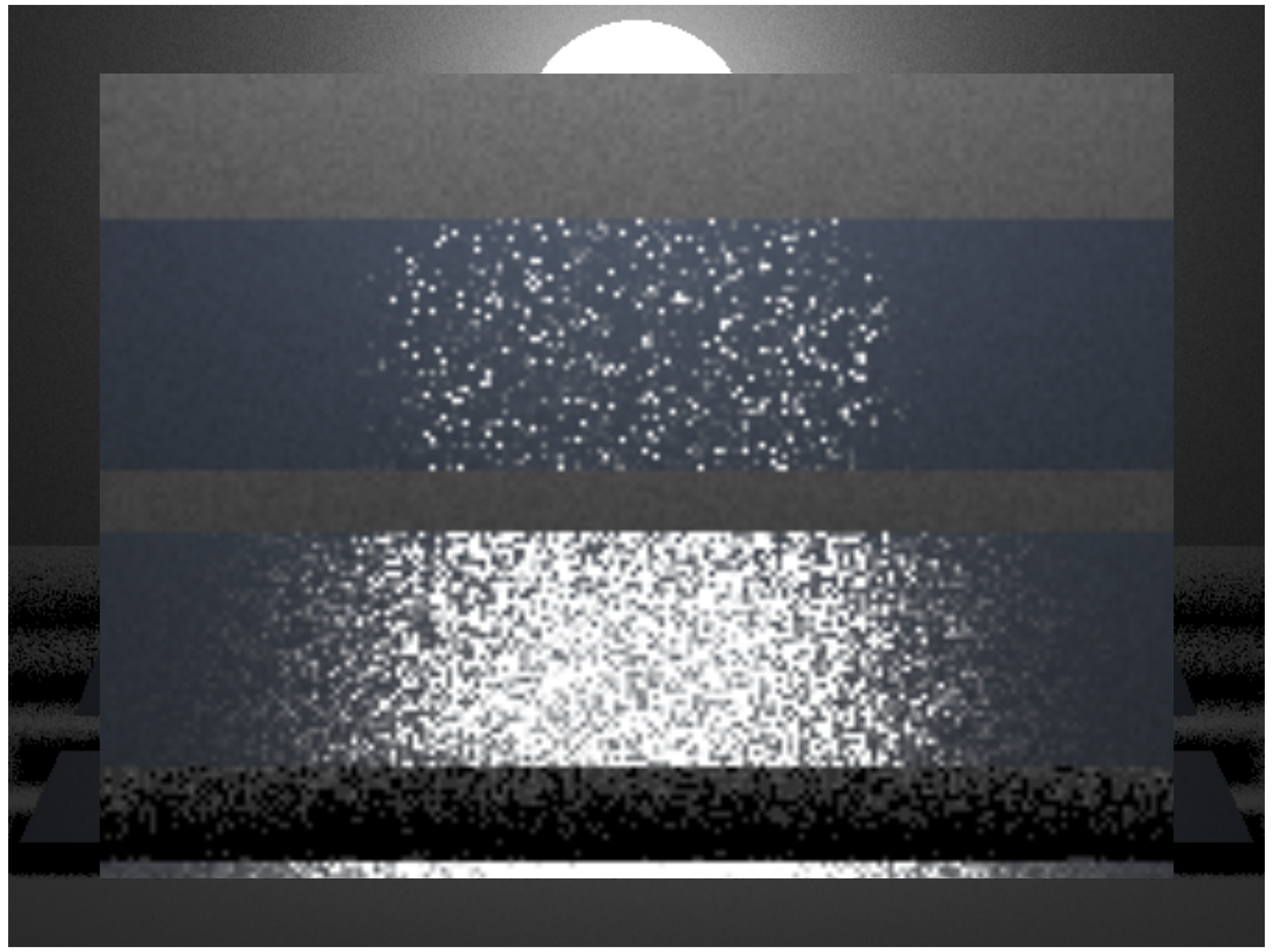
N = 4 spp

Veach Scene: Multiple Importance Sampling



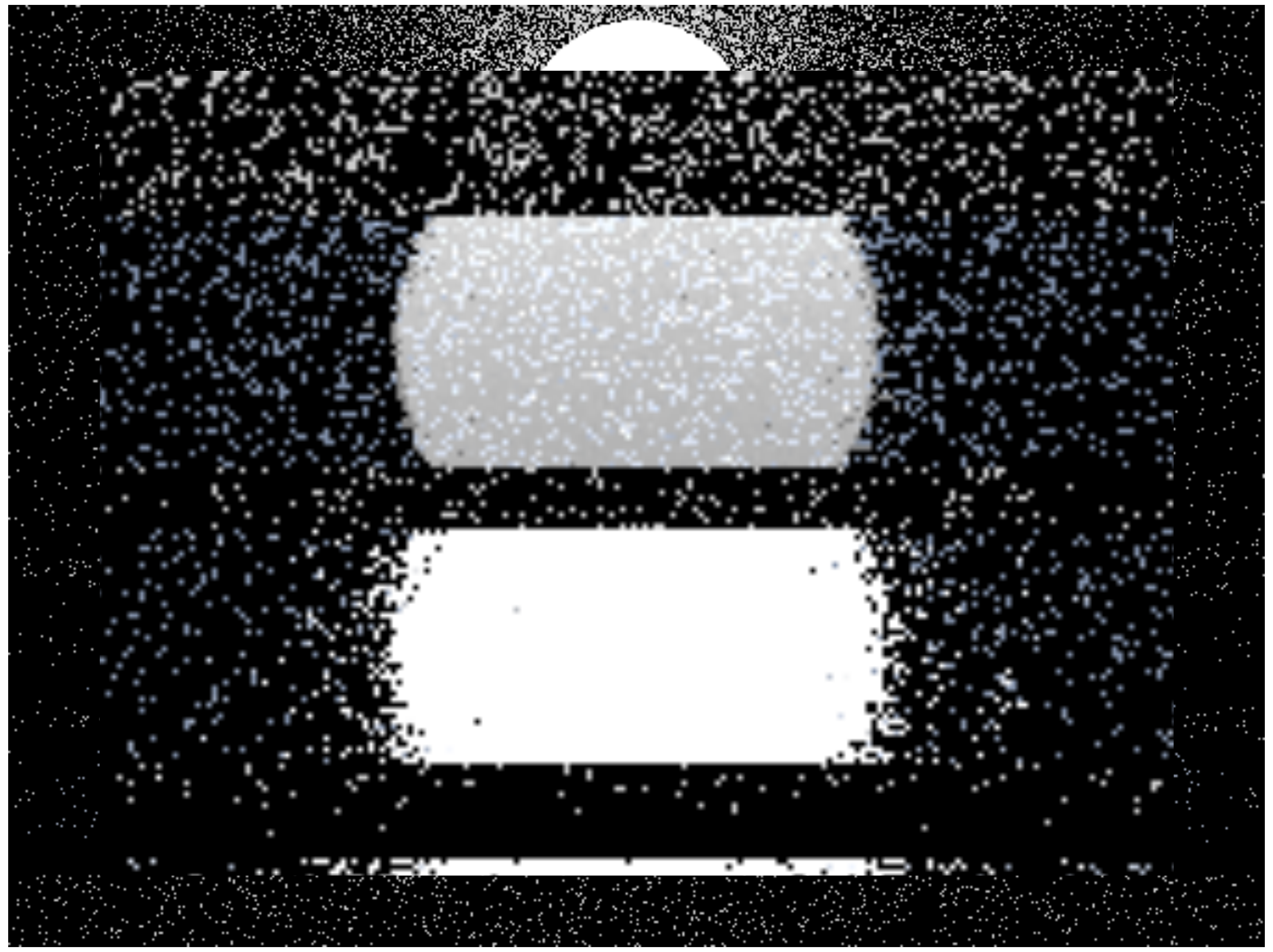
Reference image

N = 1024 spp



Light importance sampling

N = 4 spp

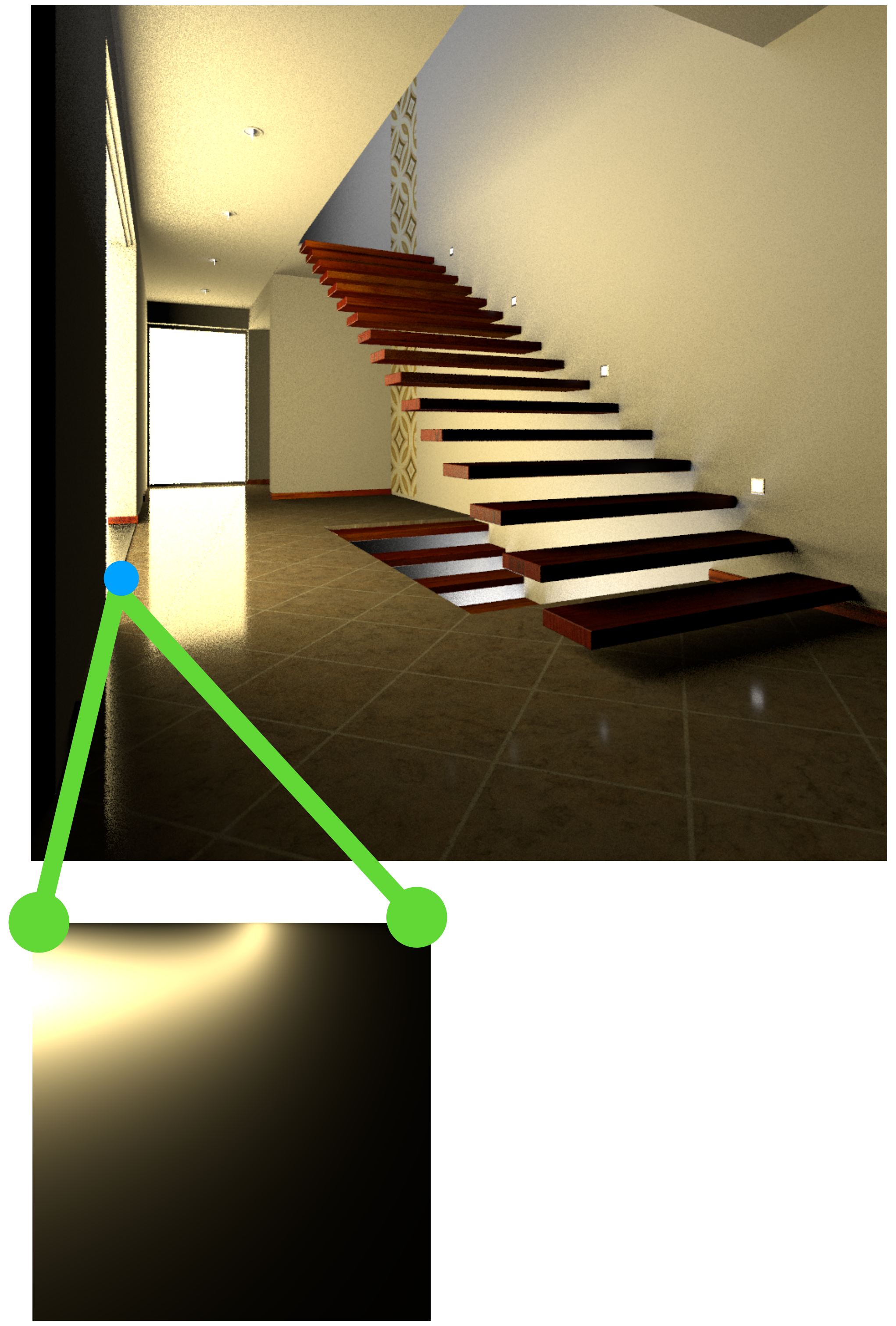


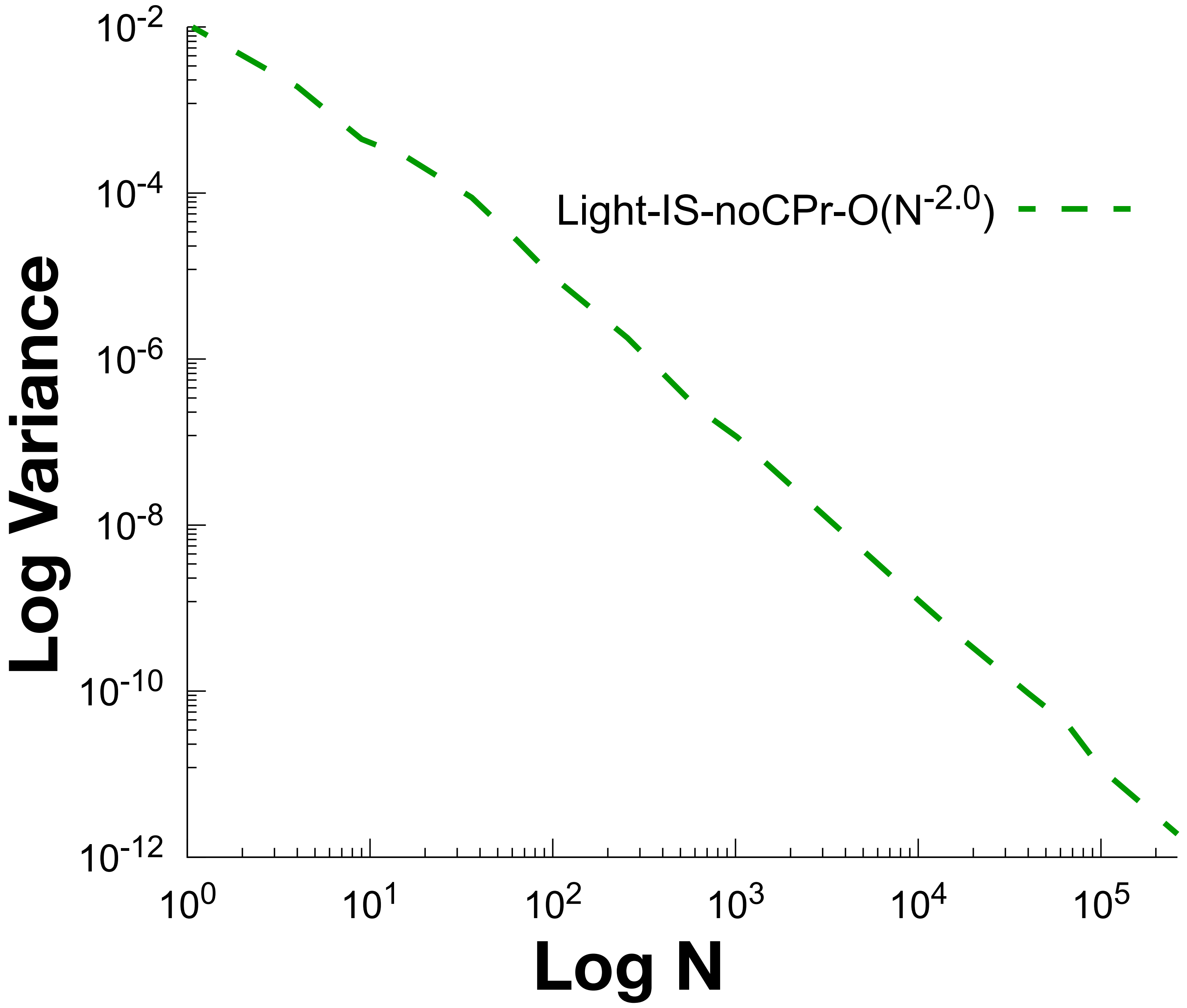
BSDF importance sampling

N = 4 spp

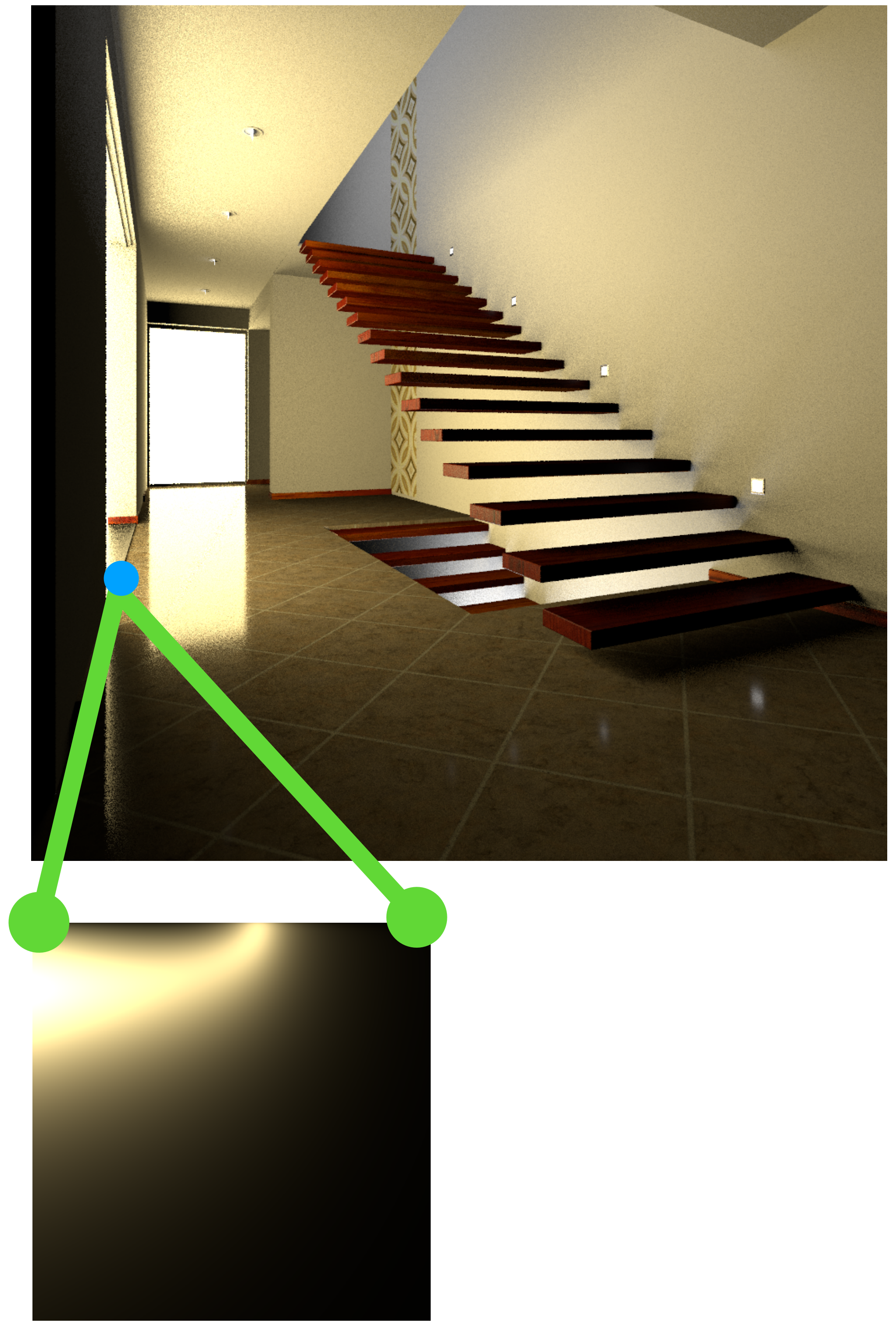
Variance Convergence: Importance Sampling

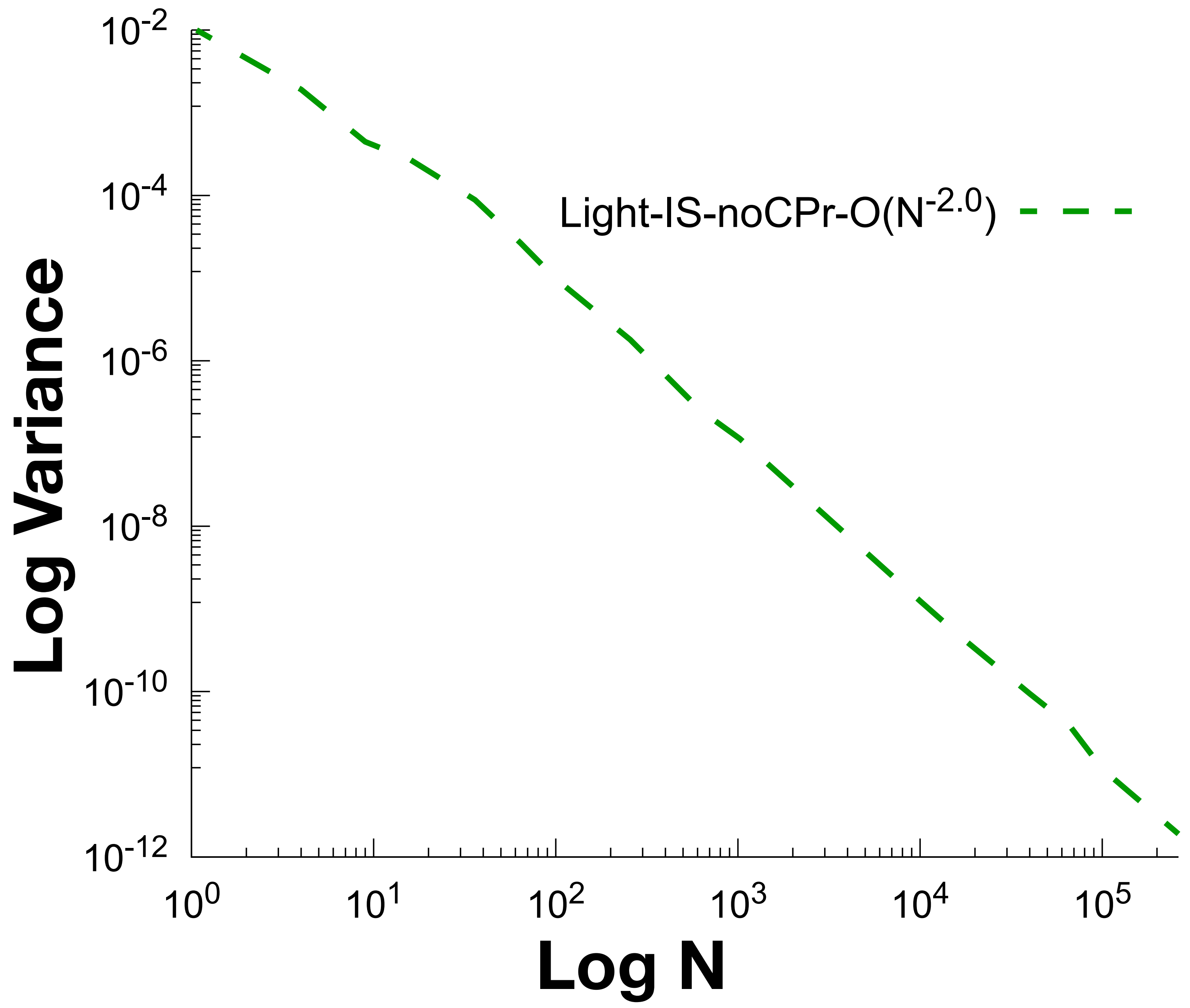
Variance Convergence: Importance Sampling



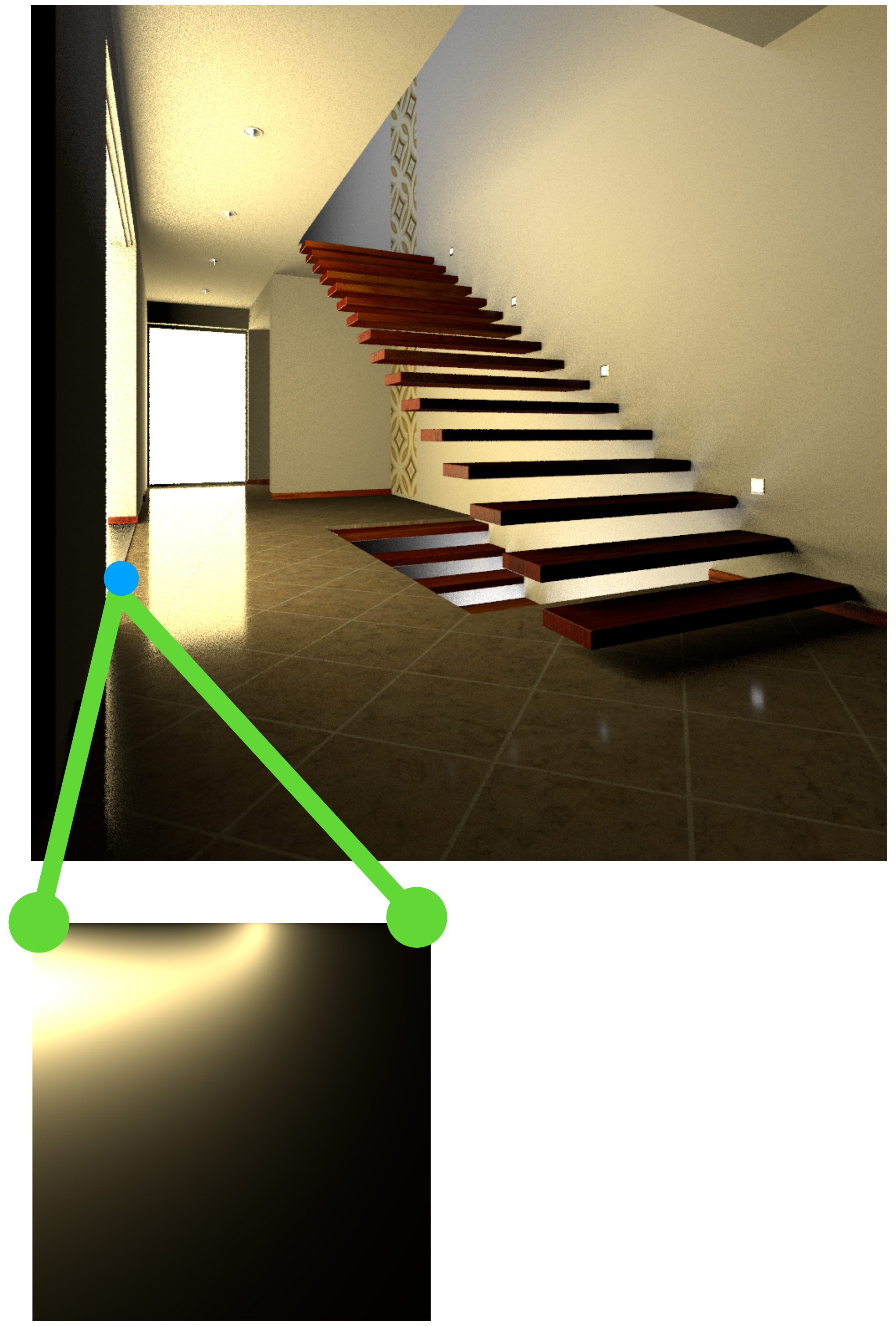


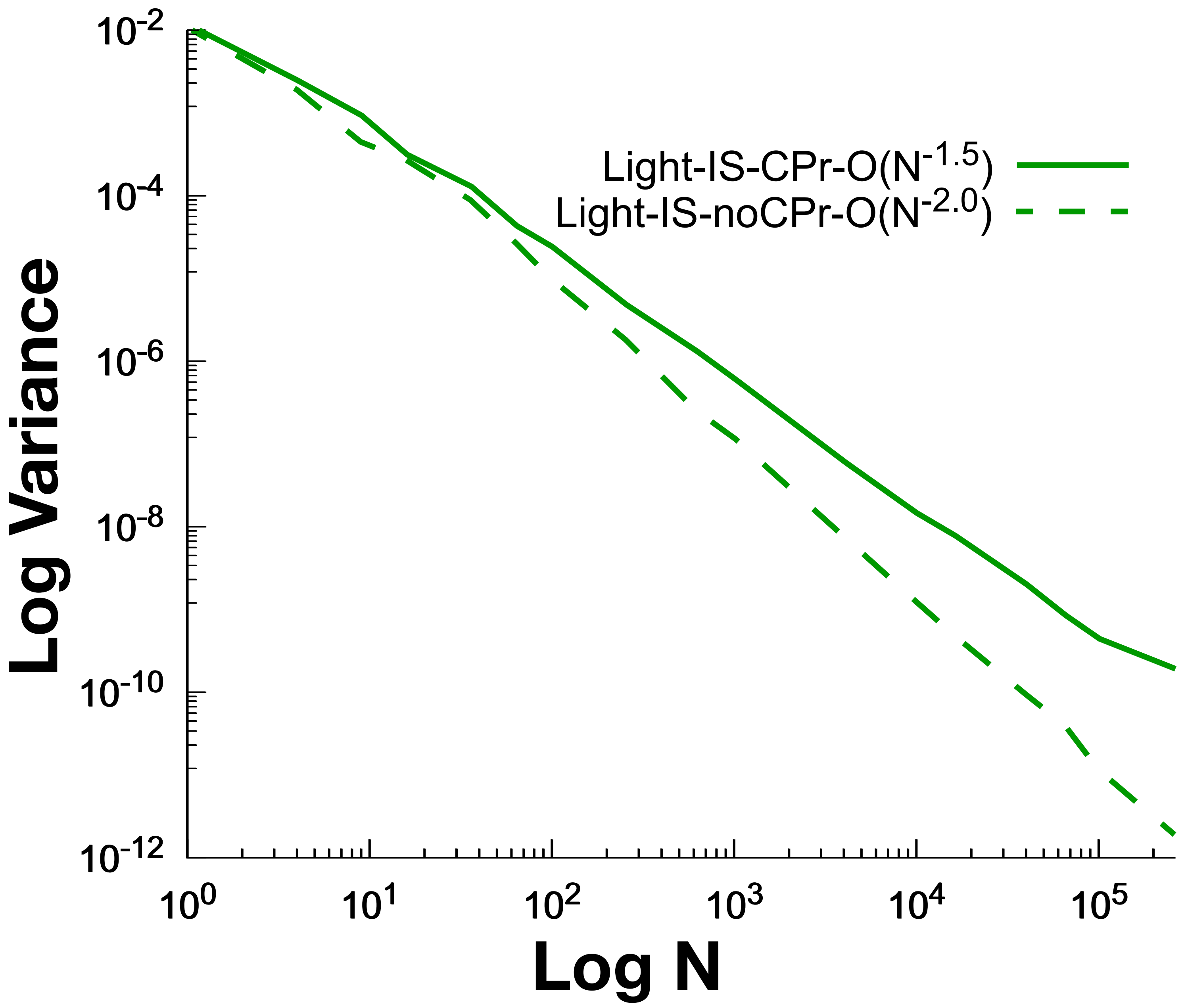
Variance Convergence: Importance Sampling



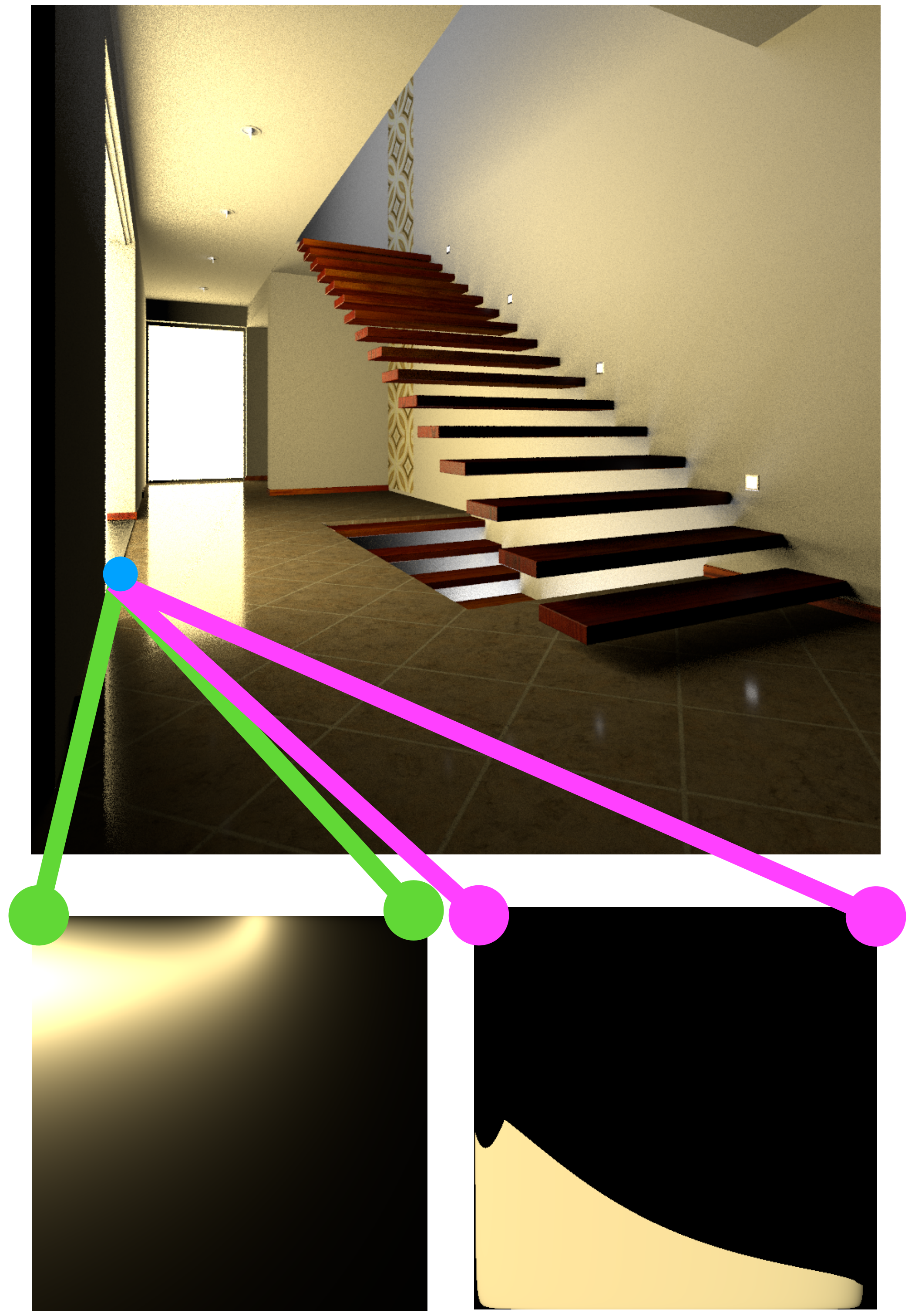


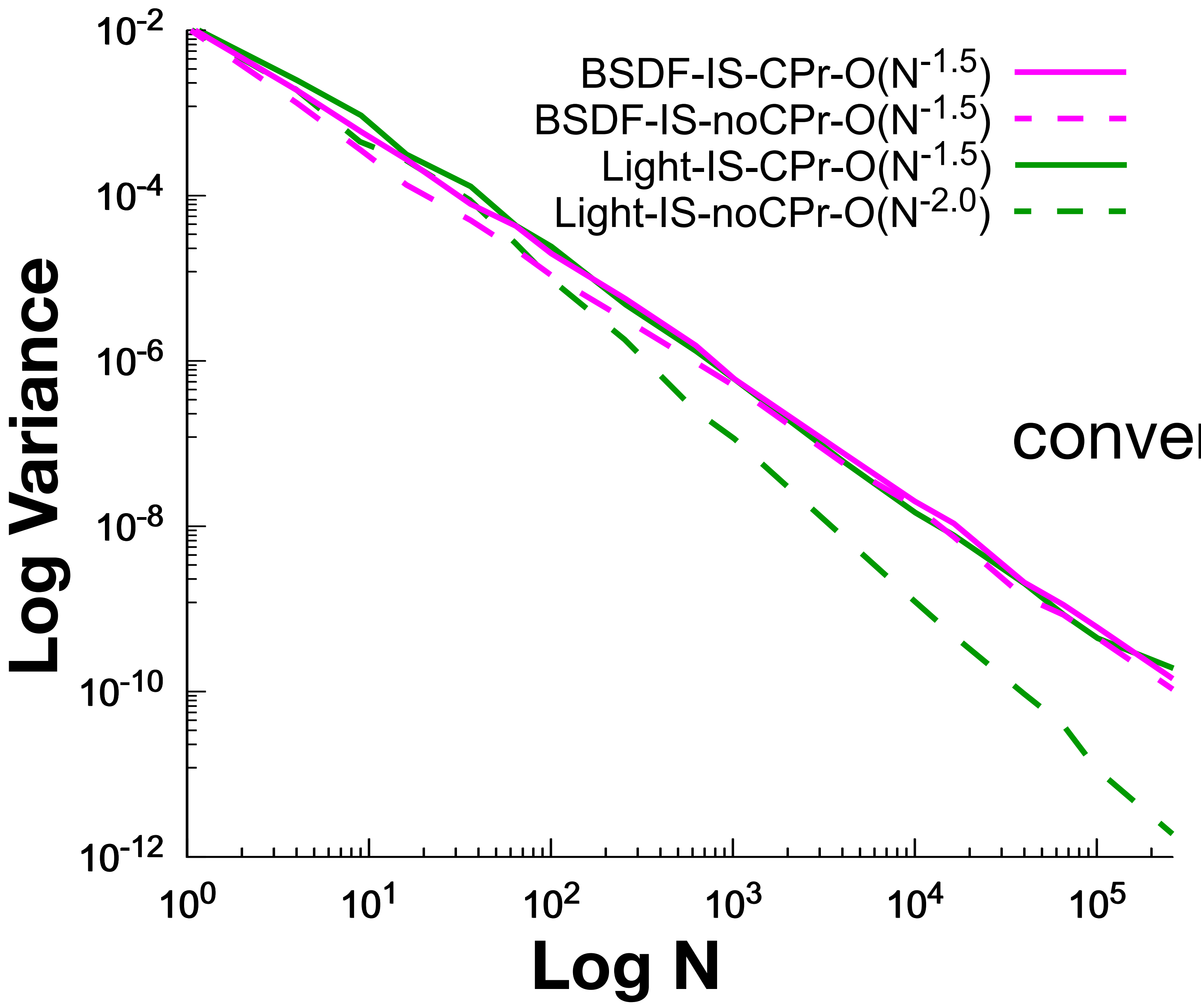
Variance Convergence: Importance Sampling





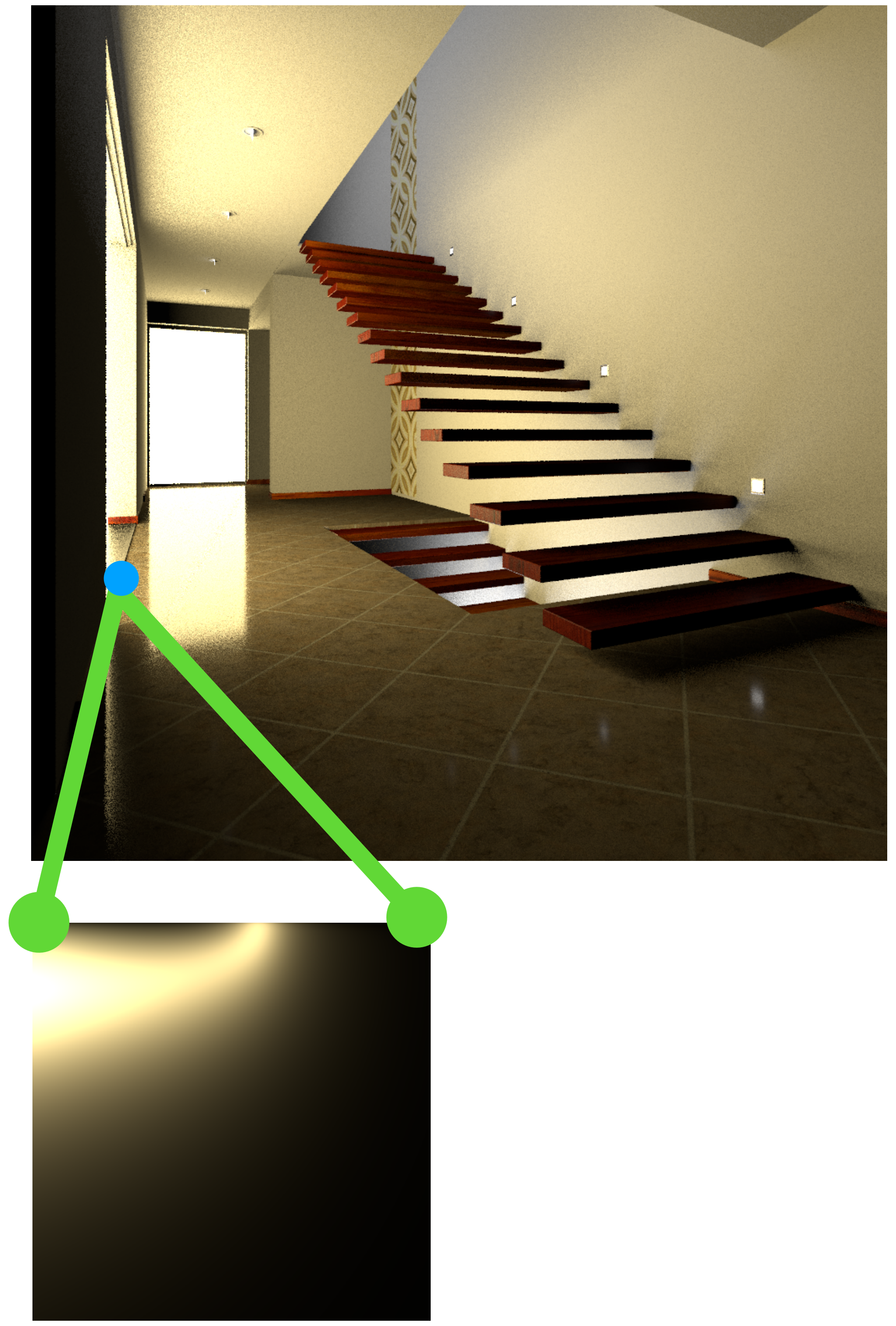
Variance Convergence: Importance Sampling

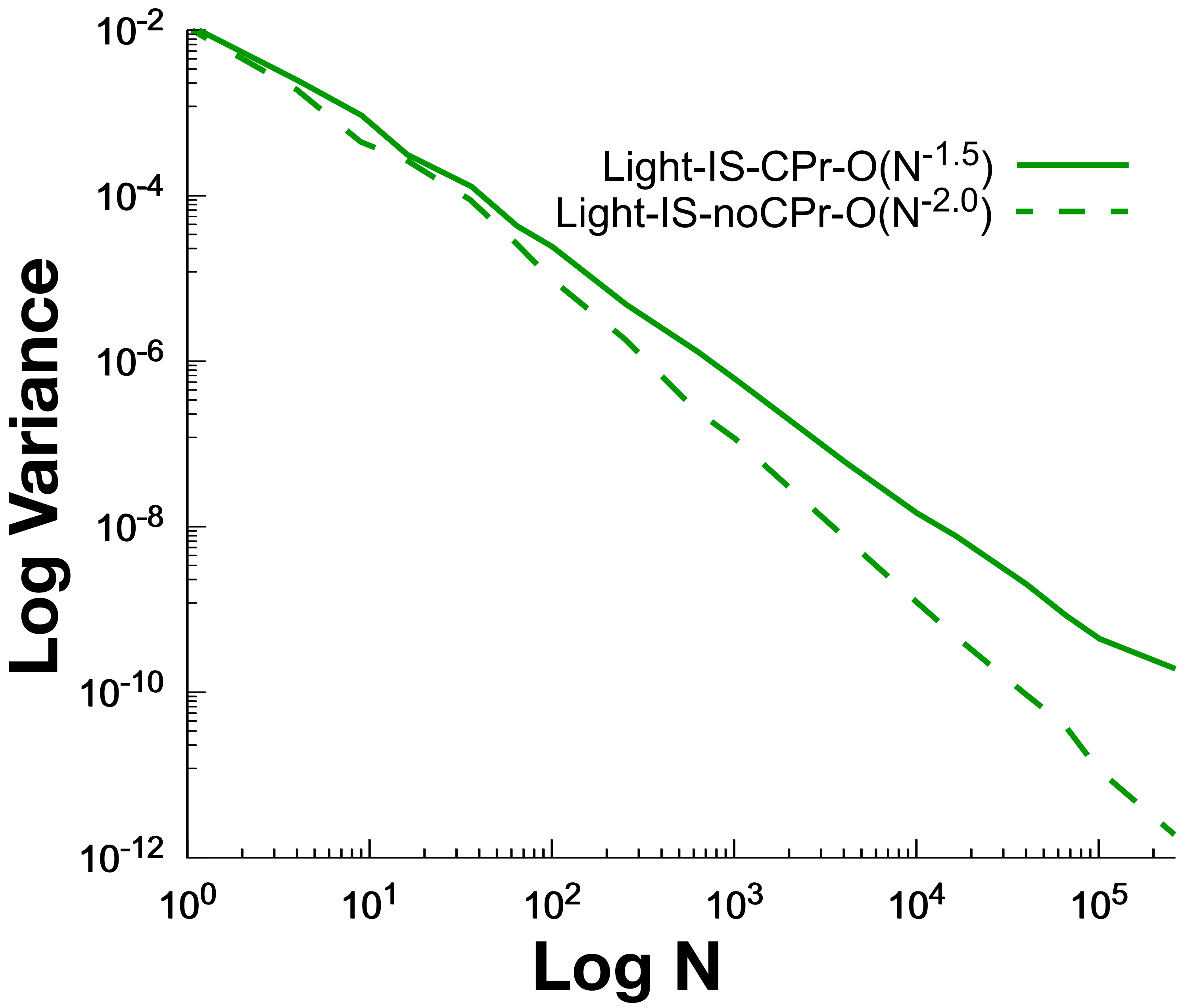


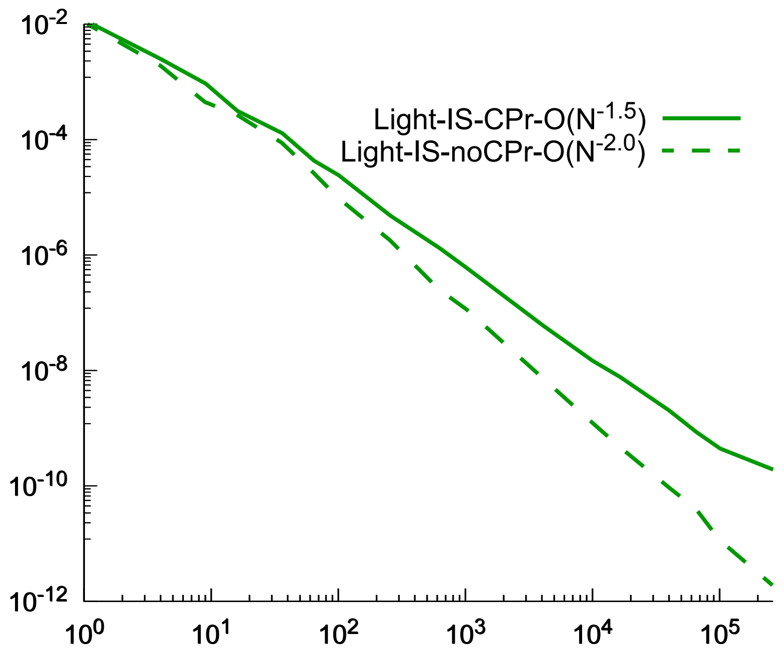


For multiple importance sampling (MIS), convergence is determined by the BSDF sampling strategy.

Variance Convergence: Importance Sampling

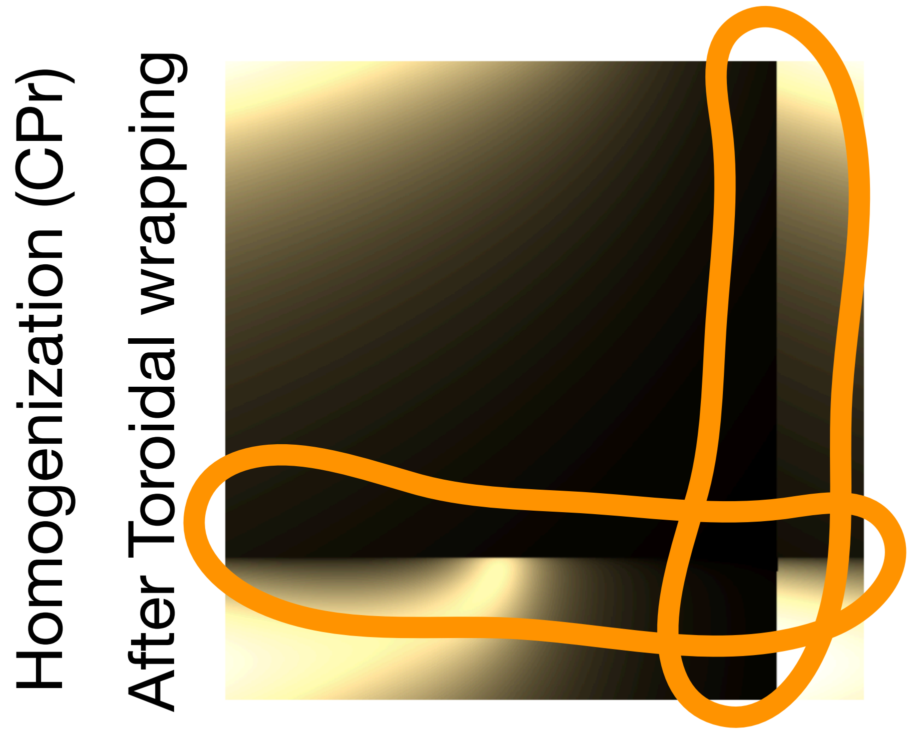


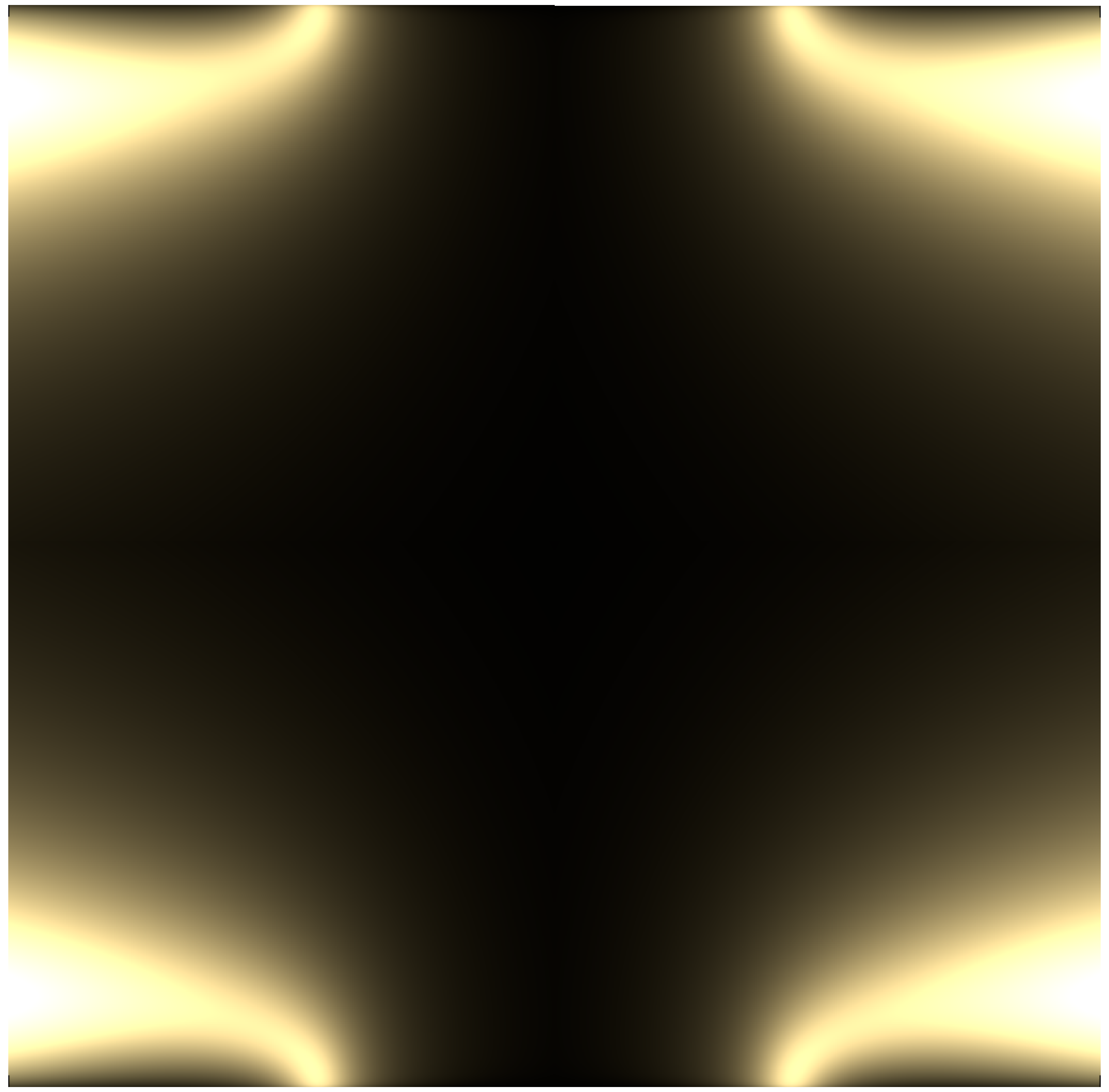


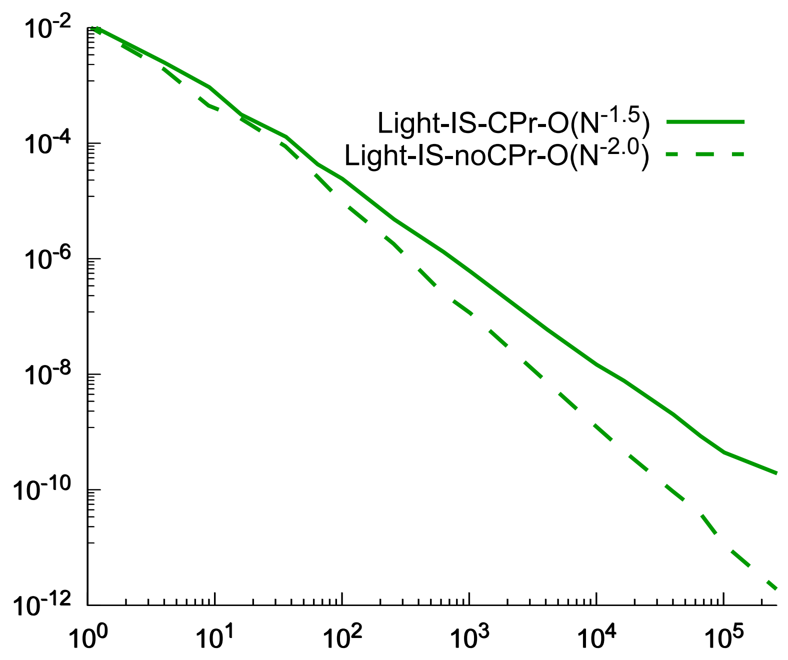


Integrand Mirroring

Original → Mirrored



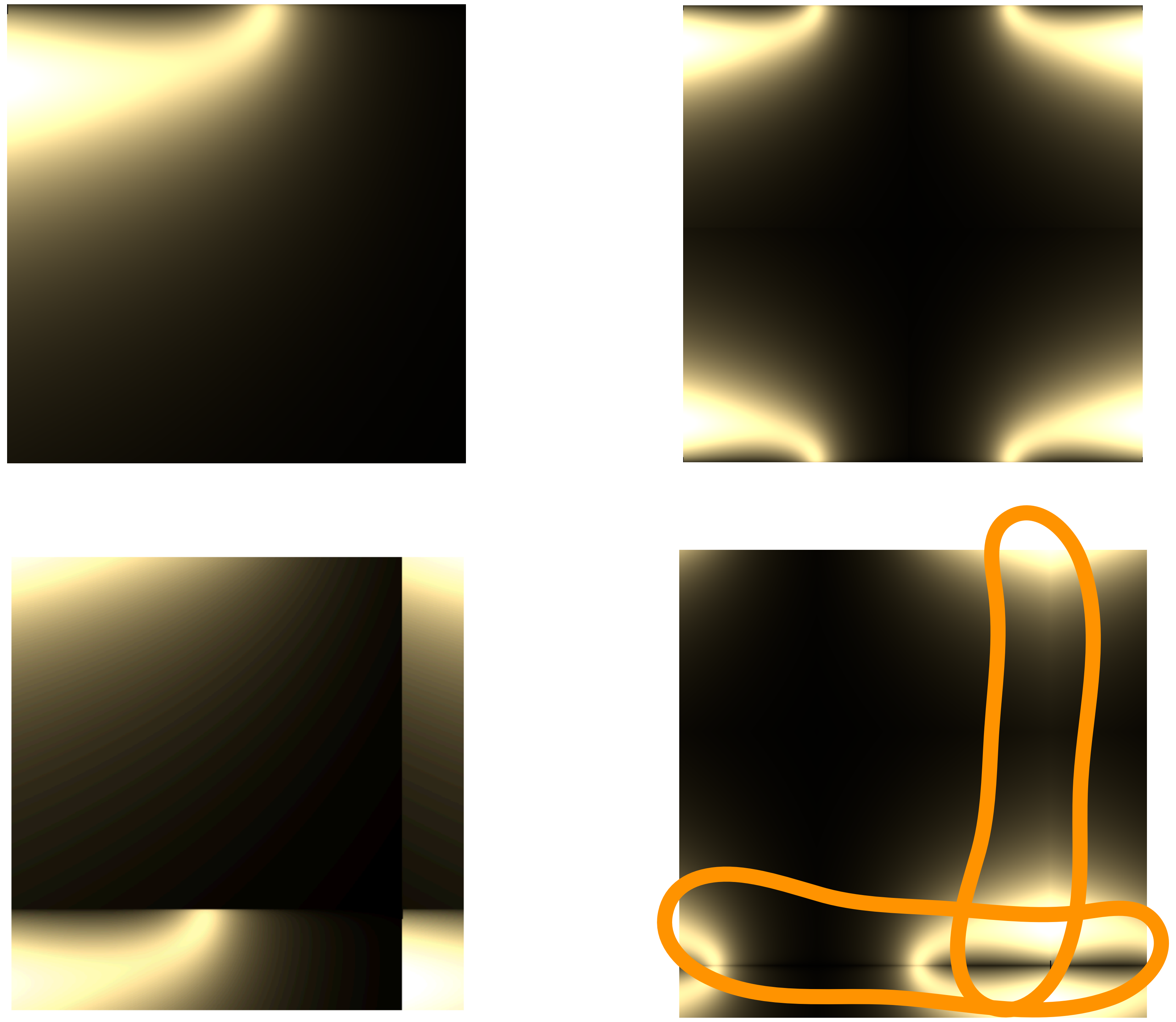




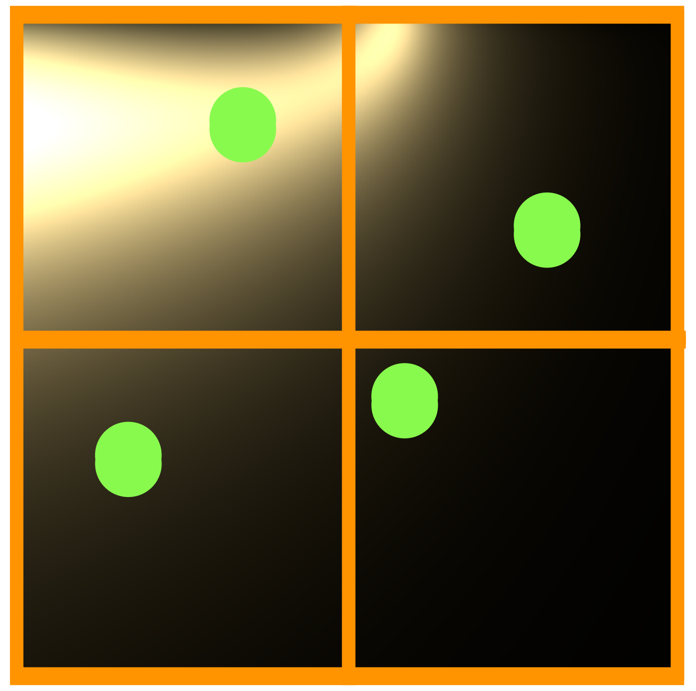
Integrand Mirroring

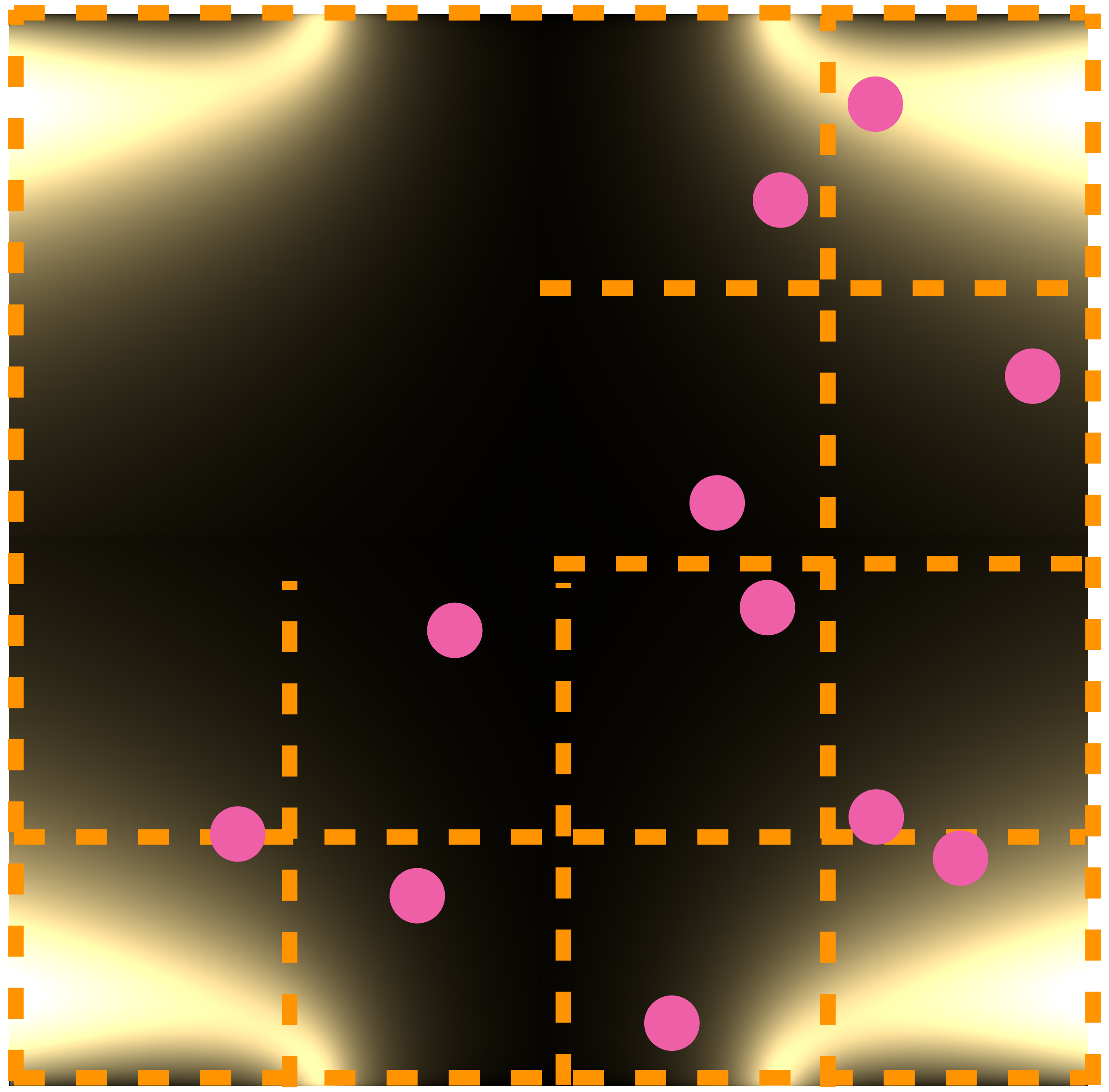
Original → Mirrored

Homogenization (CPr)
After Toroidal wrapping



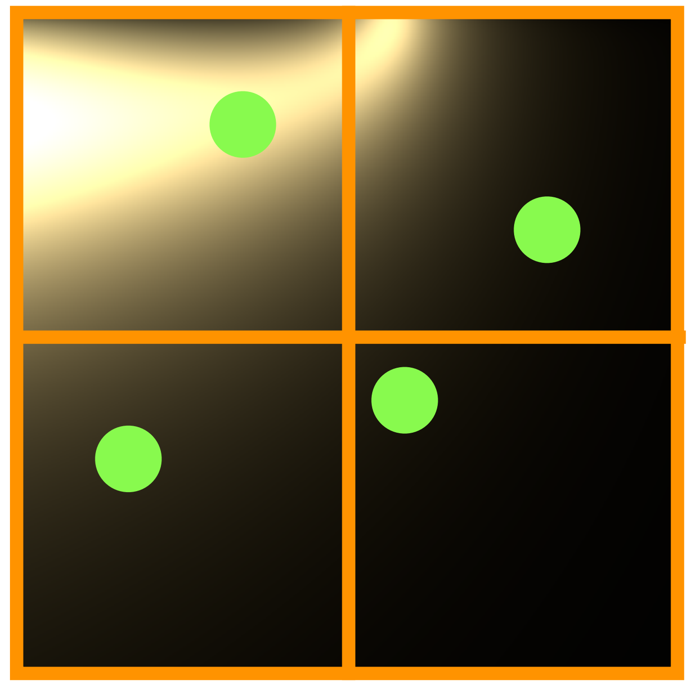
Sampling Integrand Mirroring



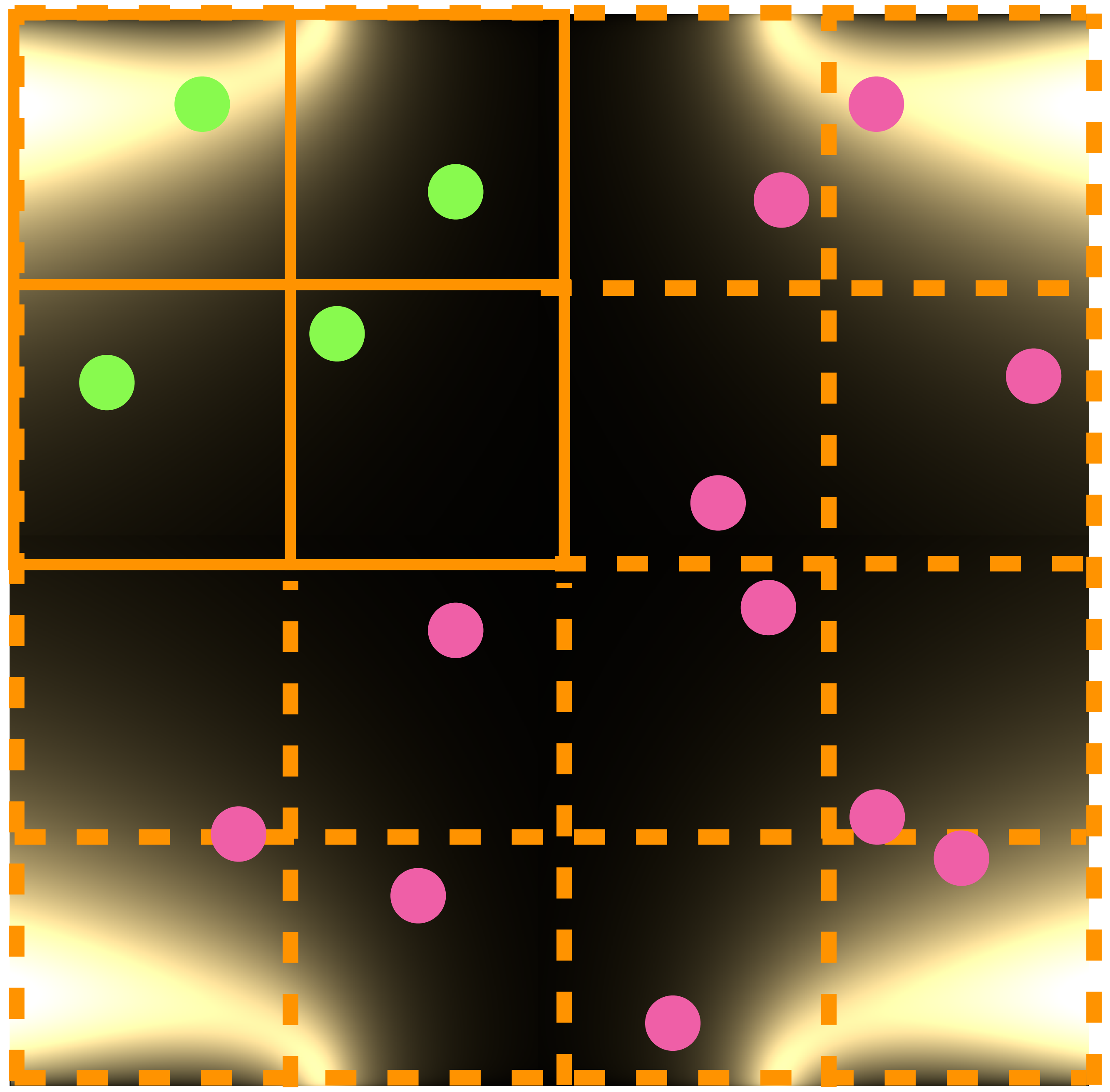


Sampling Integrand Mirroring

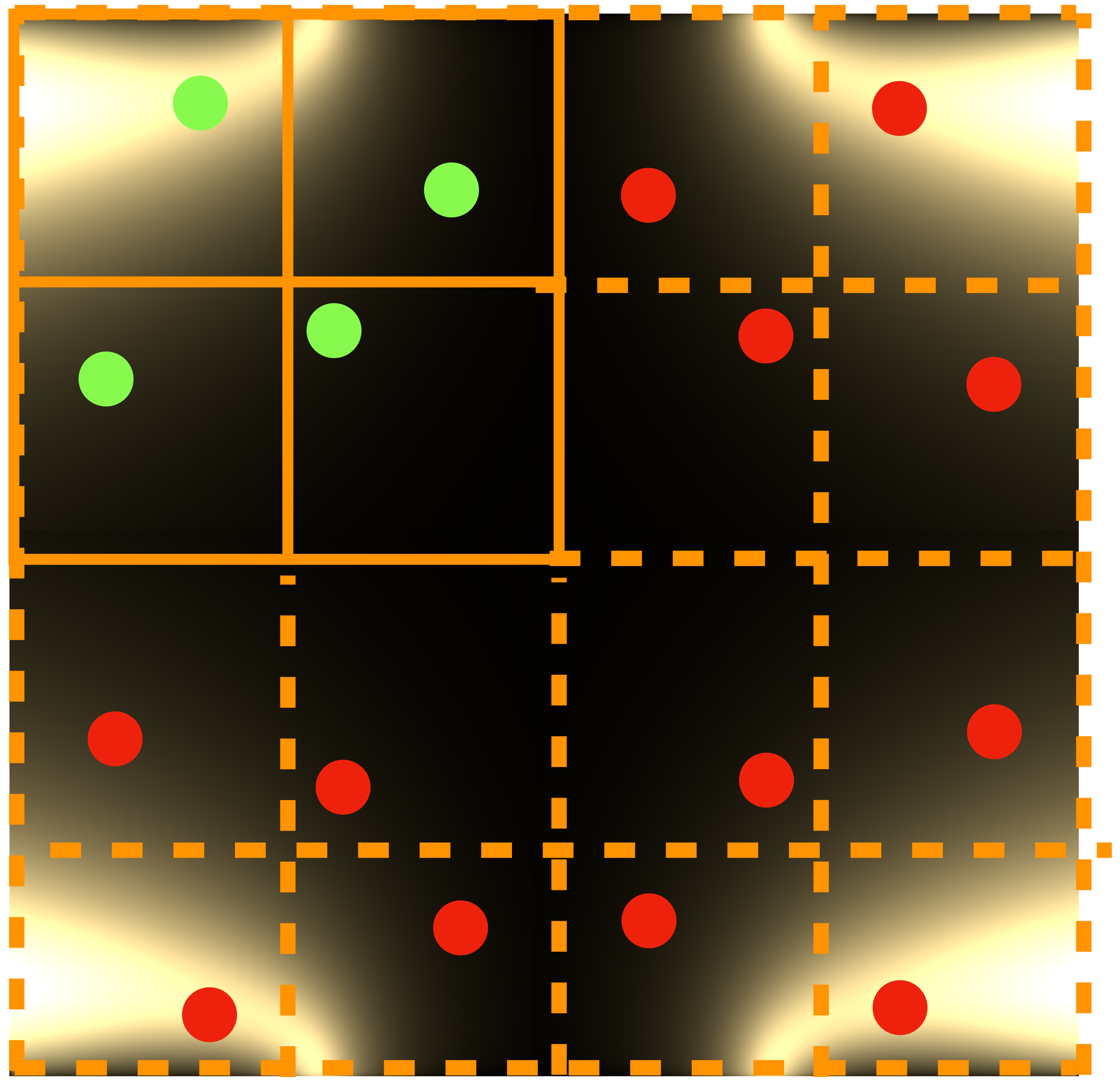
Original



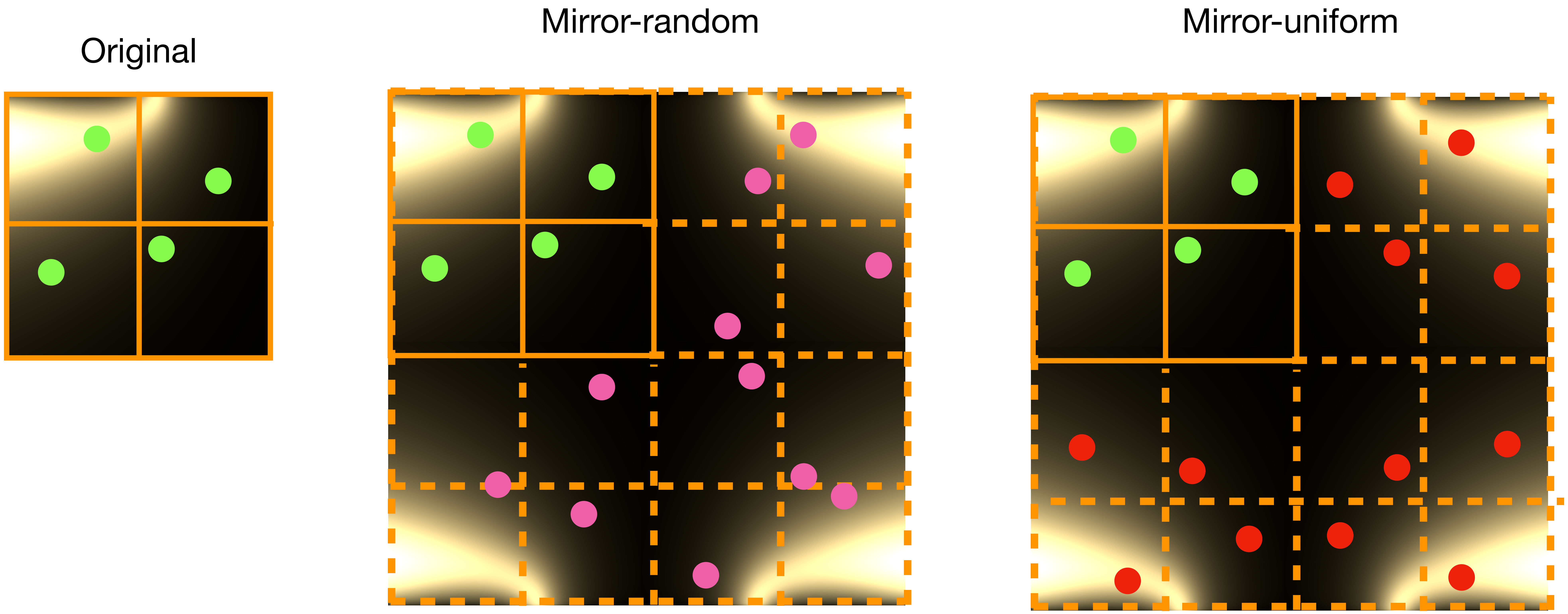
Mirror-random



Mirror-uniform

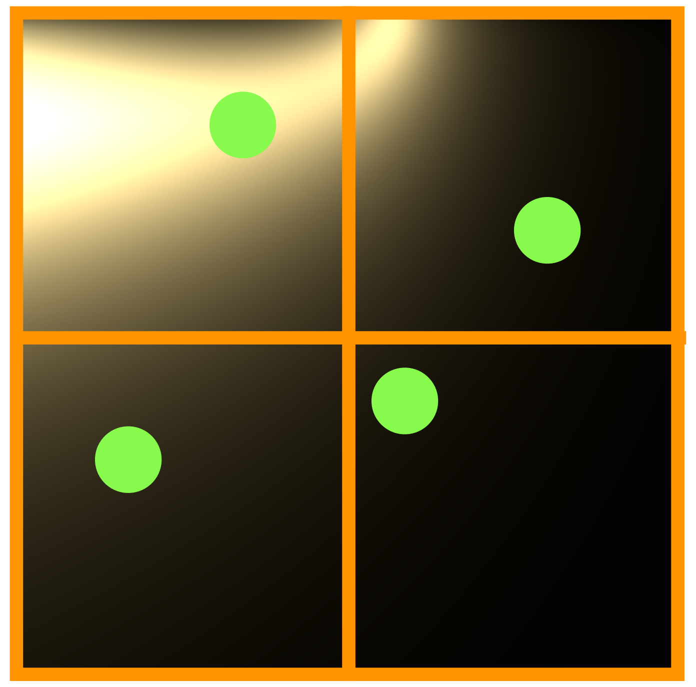


Sampling Integrand Mirroring

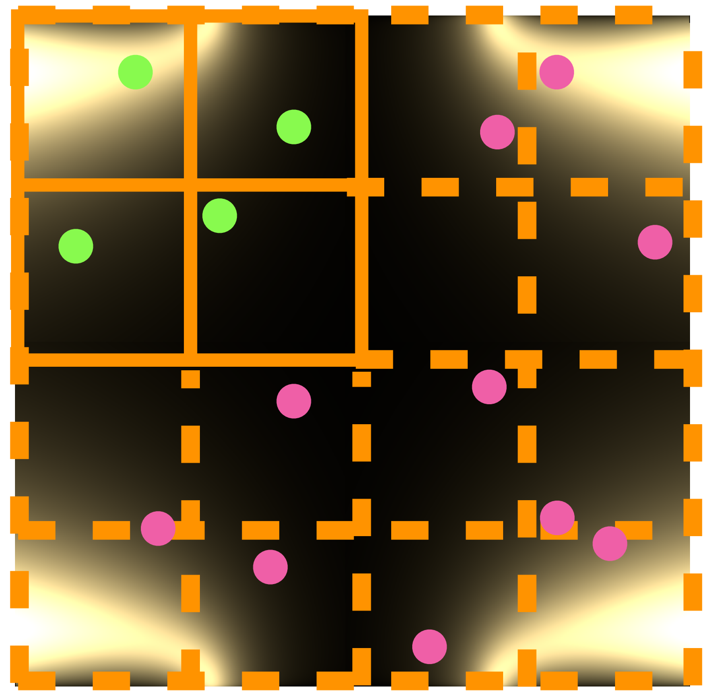


Convergence: Homogenized not good

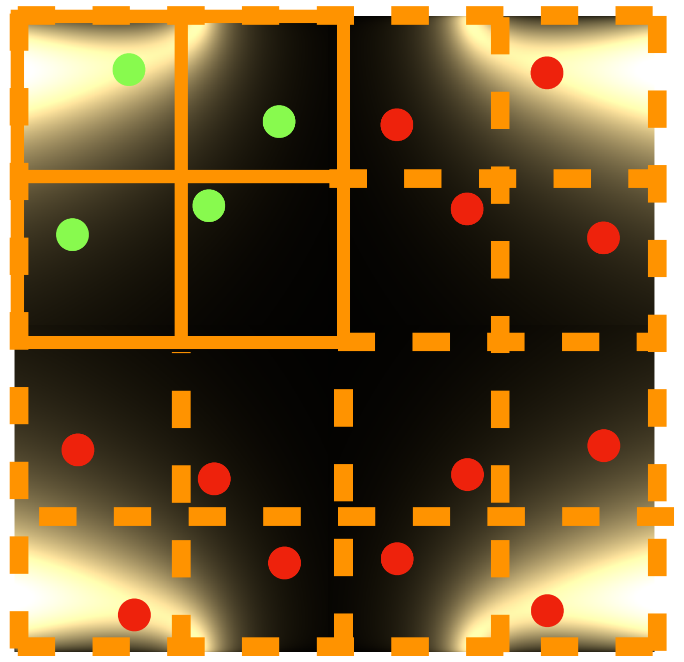
Original

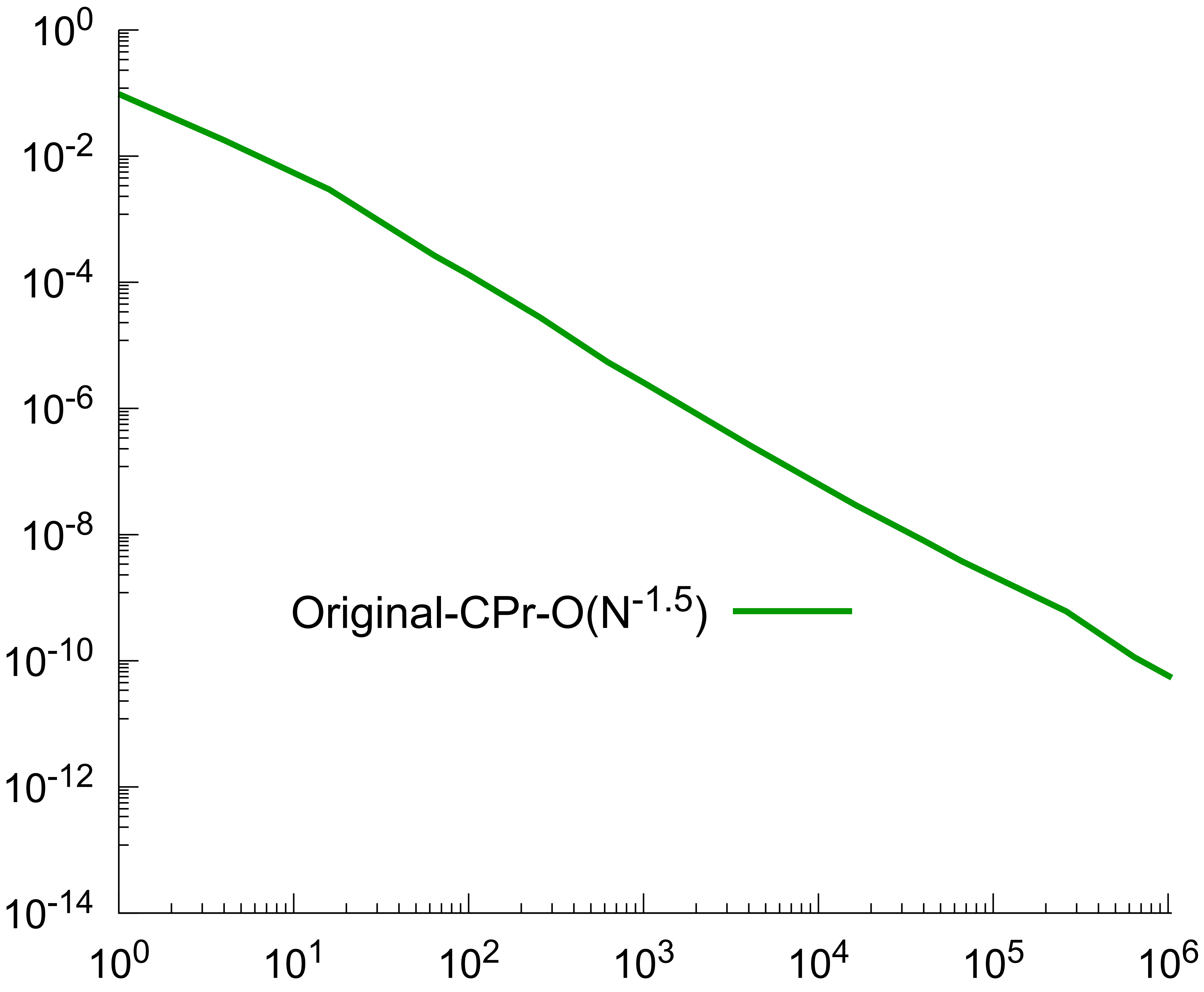


Mirror-random

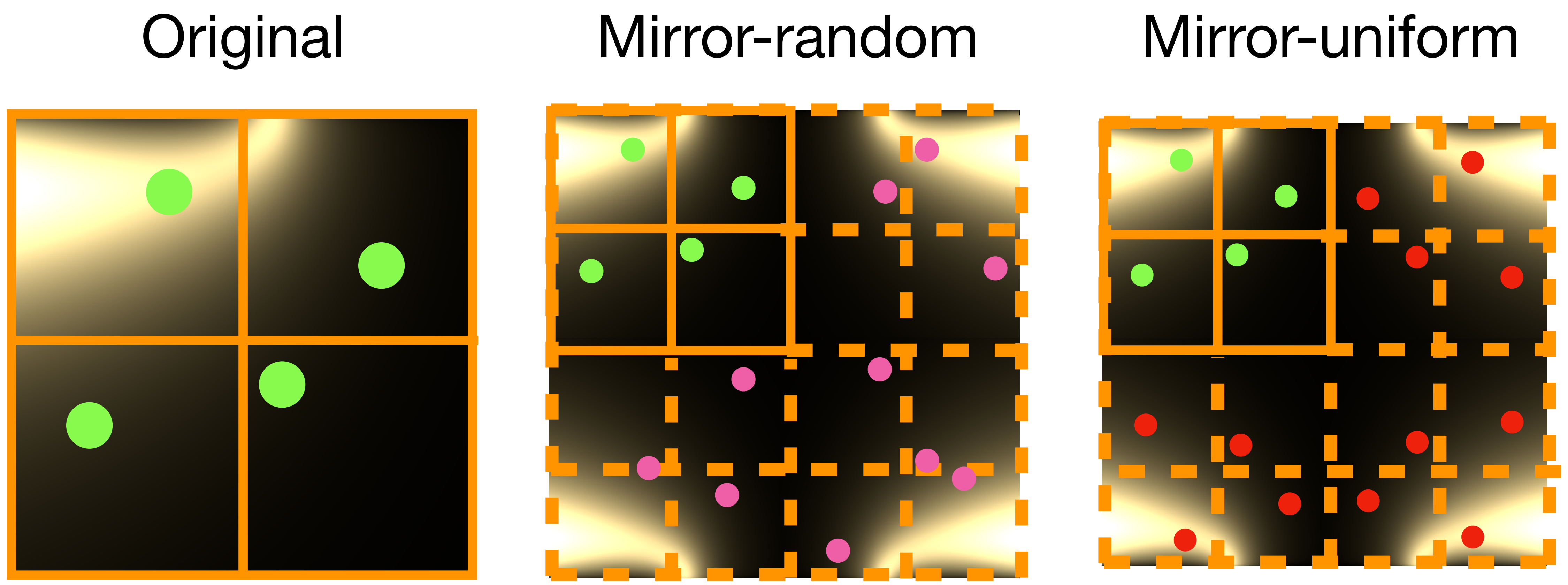


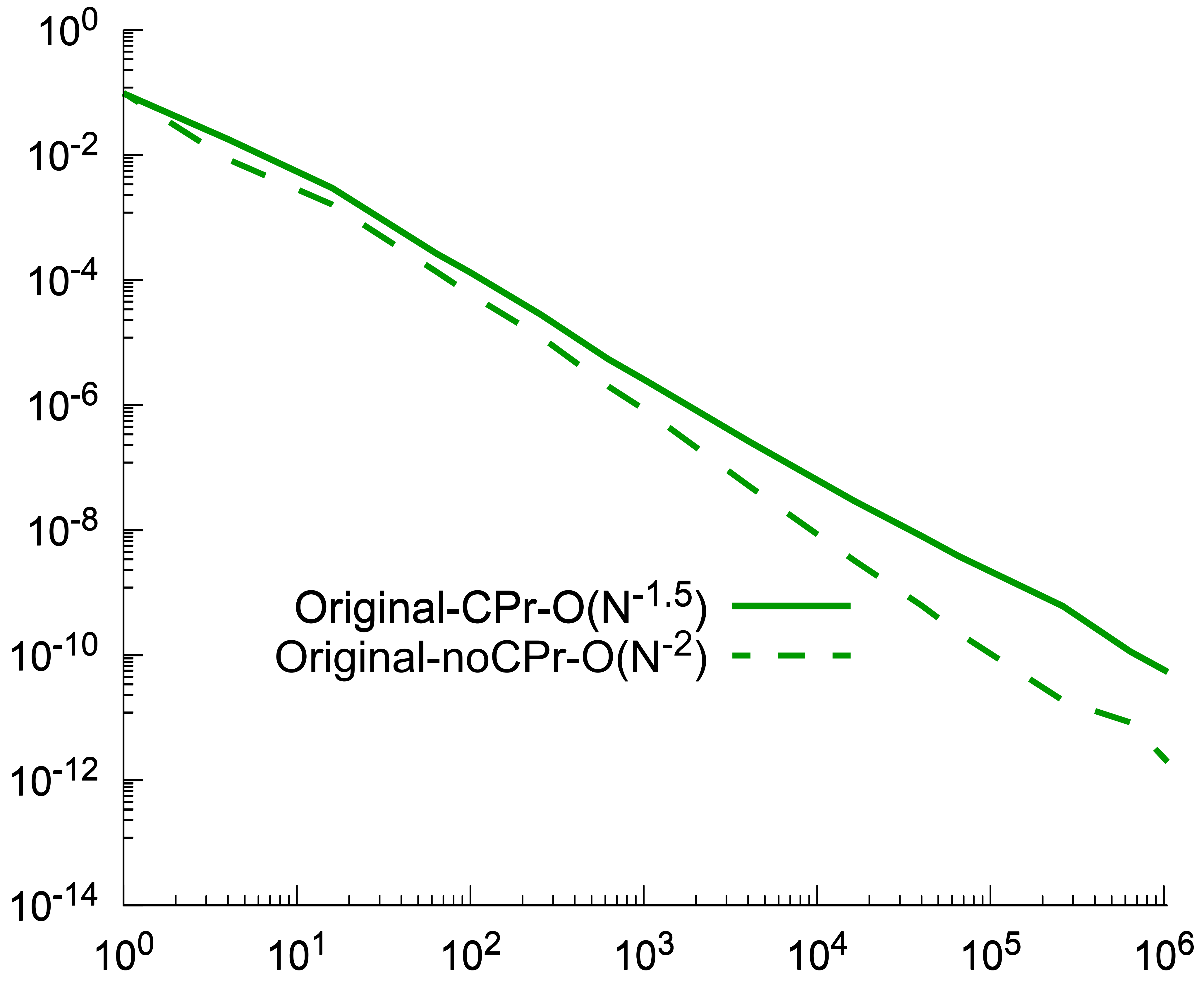
Mirror-uniform



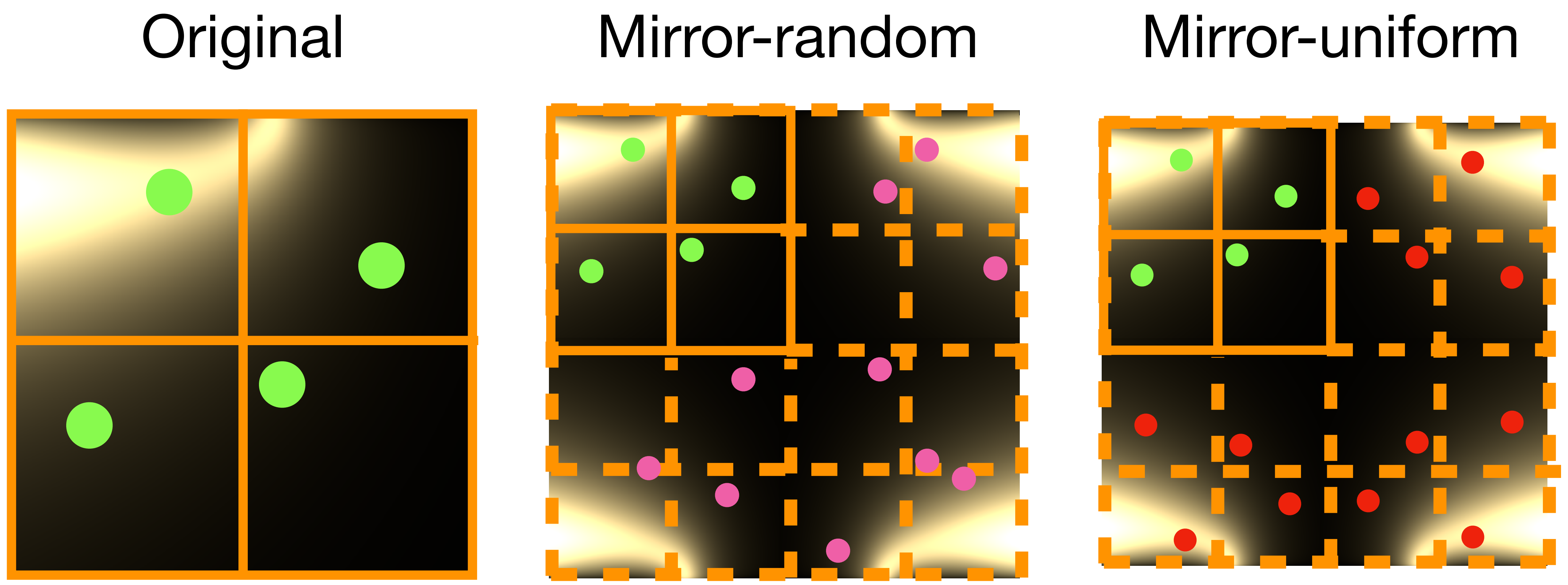


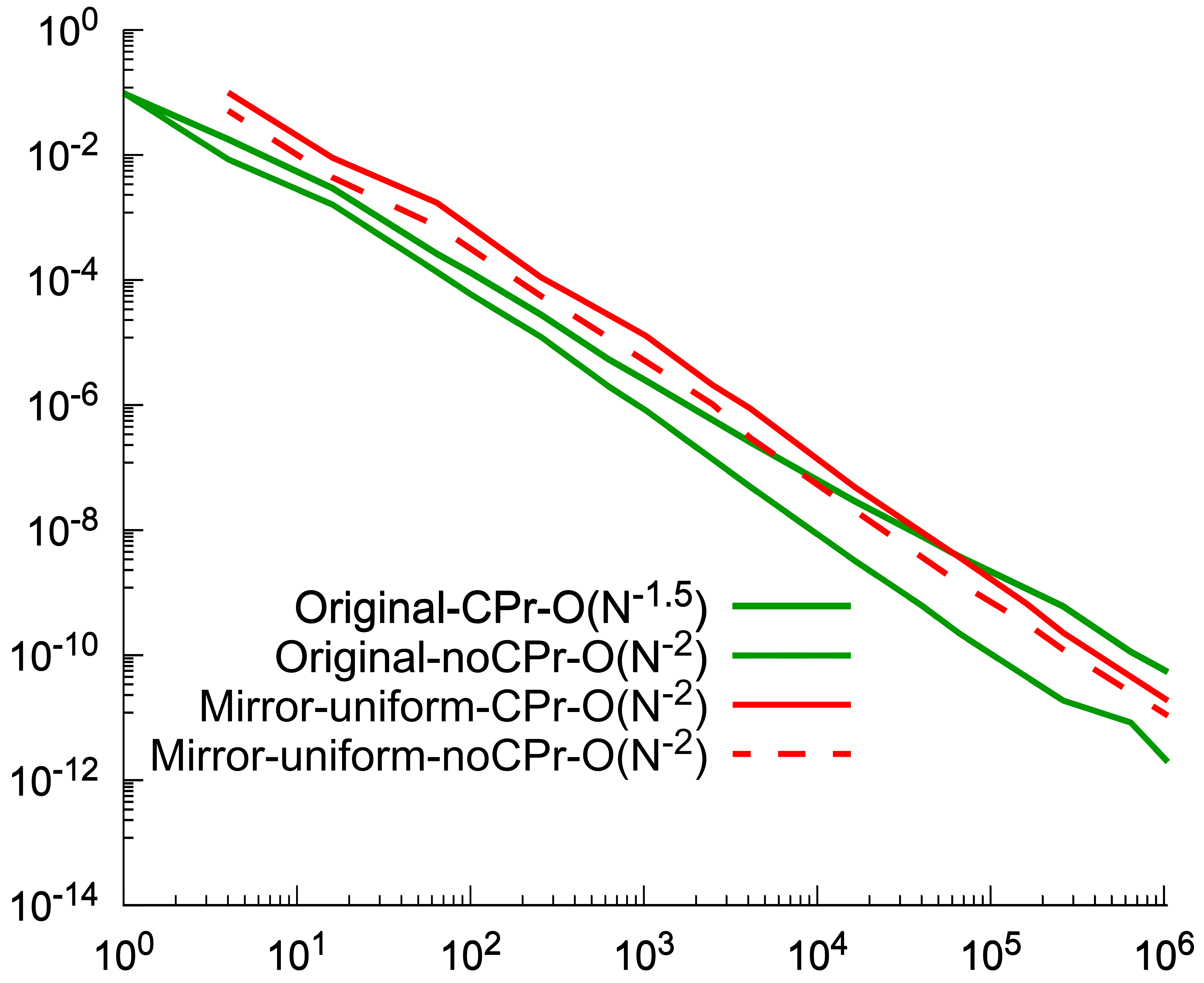
Convergence: No homogenized good



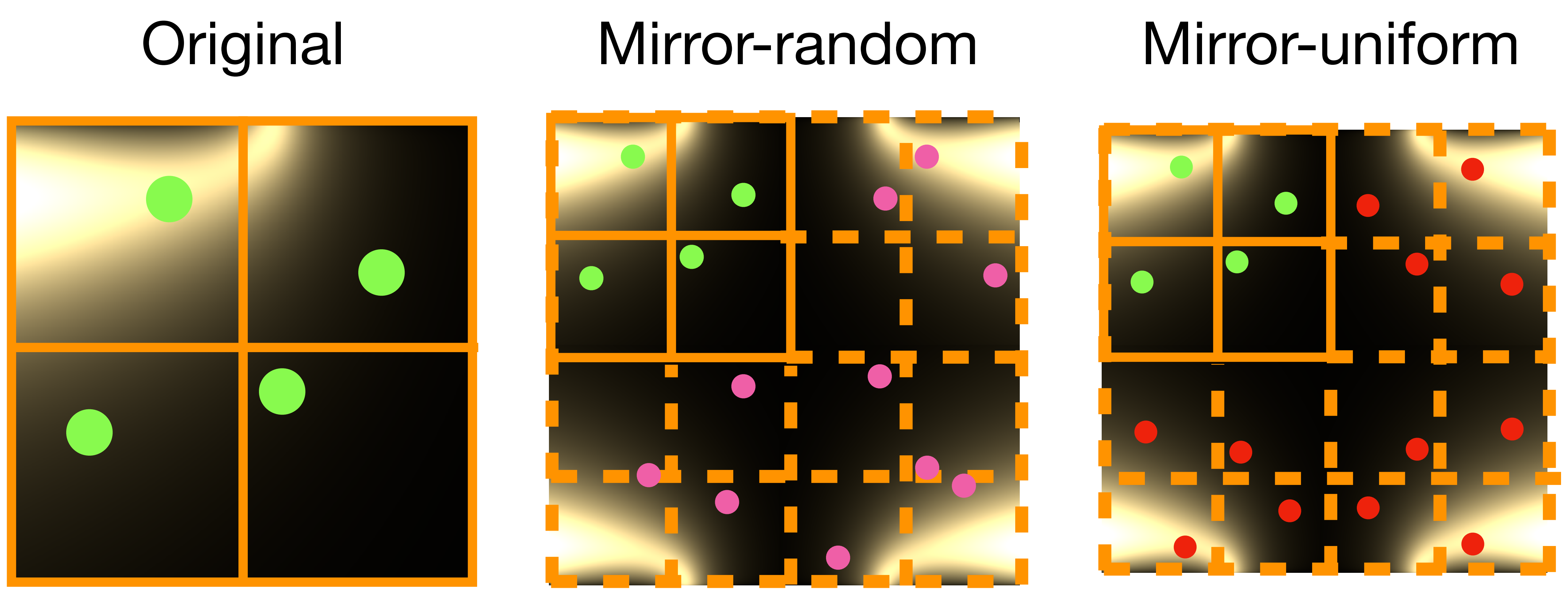


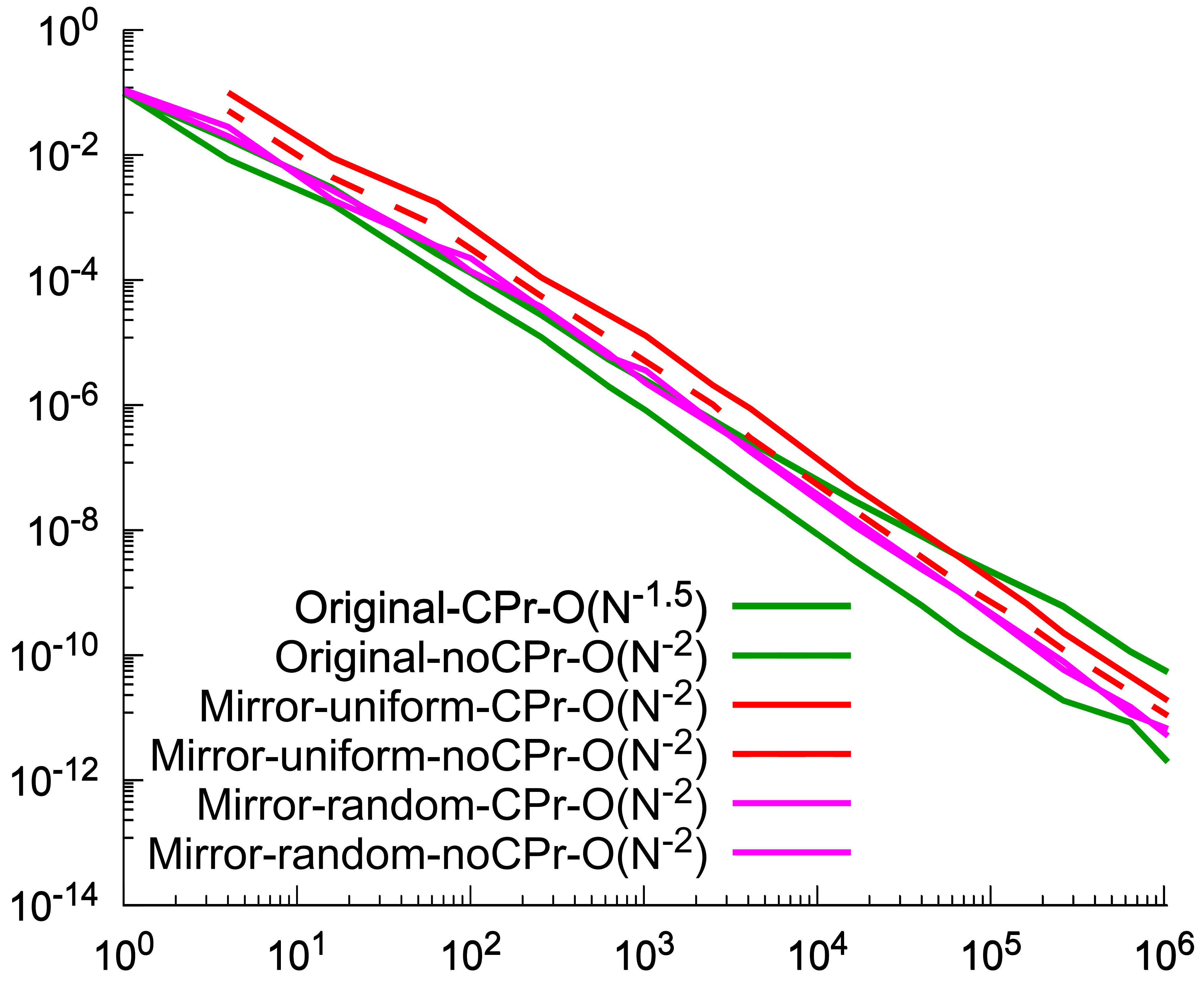
Convergence: Mirroring variance convergence



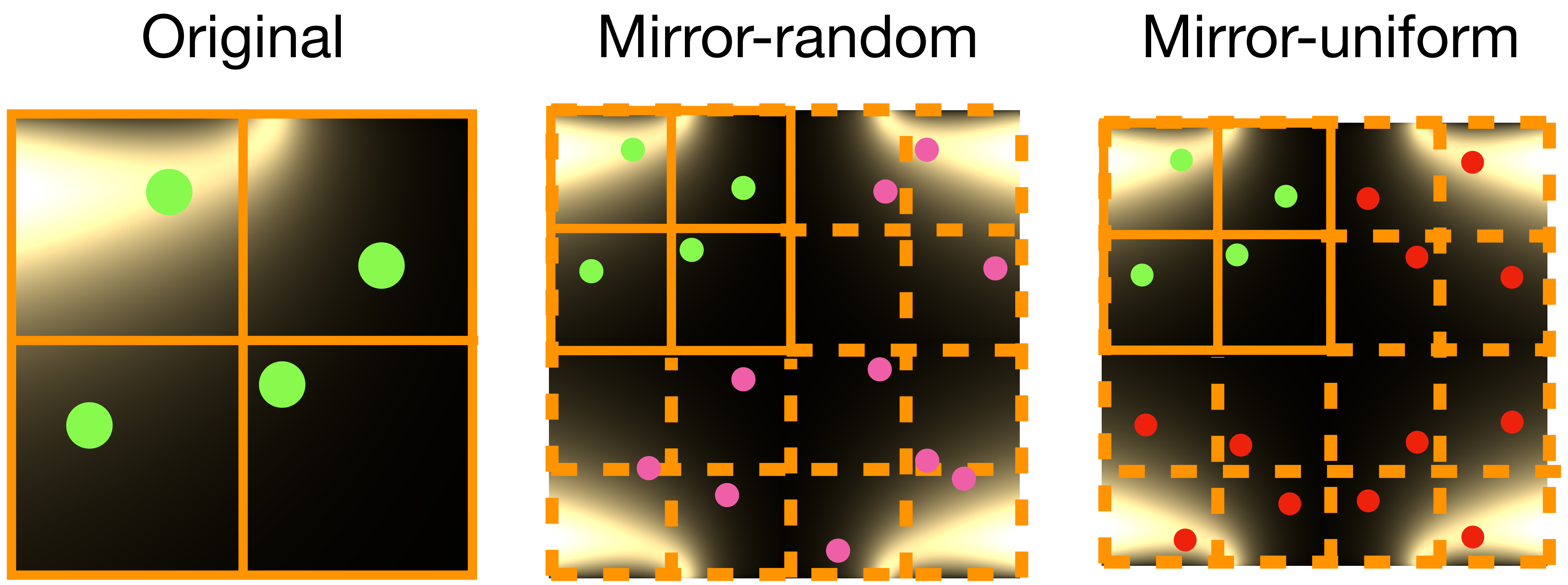


Convergence: Mirroring variance convergence





Convergence: Take away



Homogenization introduces boundary discontinuities

Integrand Mirroring helps avoid these discontinuities

But, Integrand mirroring quadruples the sampling domain in 2D

Theory side

Third term

$$\mathrm{Var}(I_N) = I^2\mathrm{Var}(\mathbf{S}_0) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathbf{f}_m^*\mathbf{f}_m\langle\mathbf{S}_m^*\mathbf{S}_m\rangle + \sum_{m\in\mathbf{Z}}\sum_{\substack{l\in\mathbb{Z}\\ l\neq m}} \mathbf{f}_m^*\mathbf{f}_l\langle\mathbf{S}_m^*\mathbf{S}_l\rangle$$

Third term is crucial and must not be missed

- consider correlations within samples w.r.t the integrand

The formulation handles Importance Sampling

Difficult to gain insights in 2D (and beyond) due to high-dimensional nature of the third term.

Practical side

In **MIS**, the worst of the two strategies would determine the overall convergence rate.

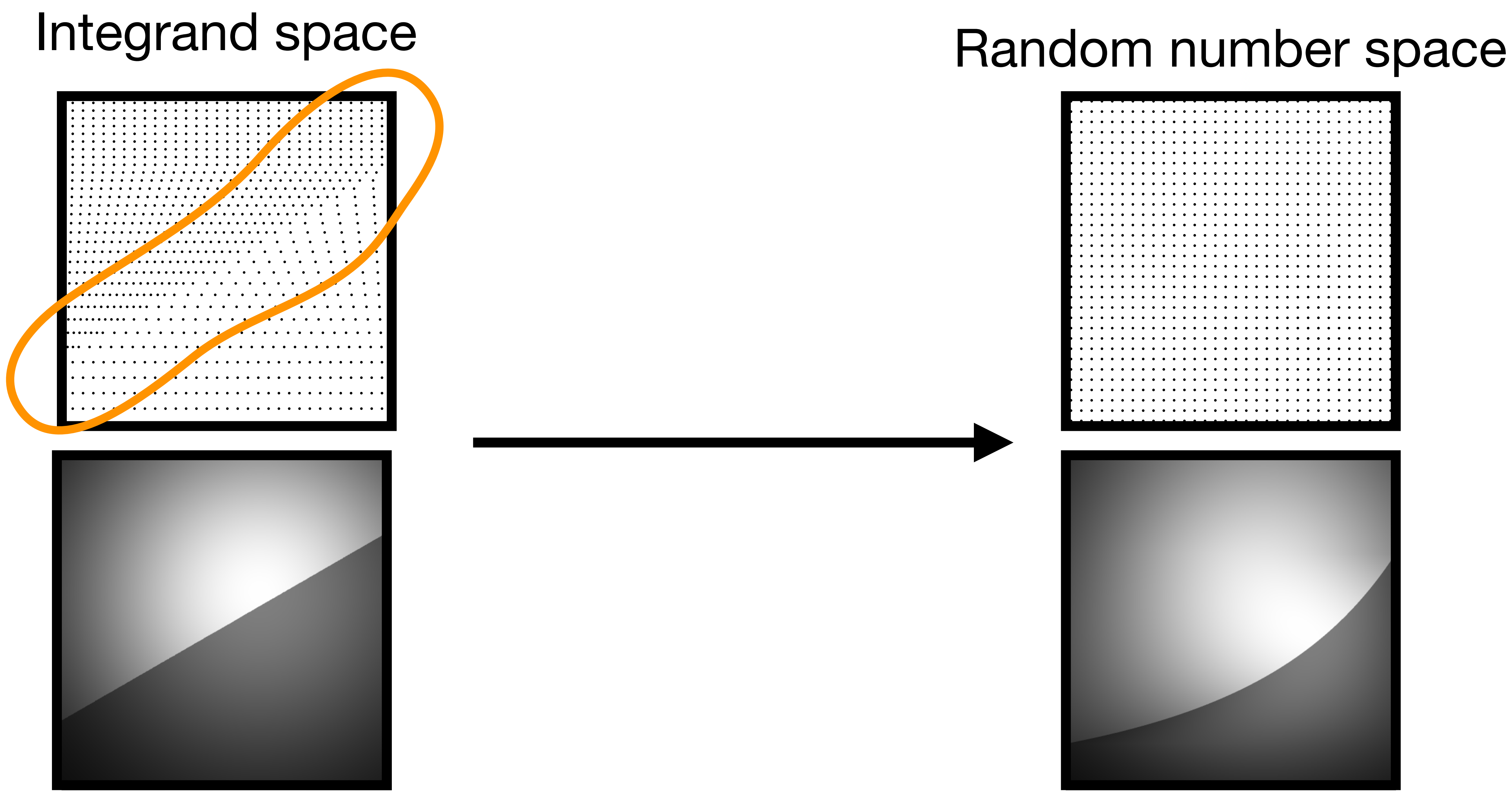
In **environment map sampling**, simply importance sampling w.r.t. the gray channel introduces discontinuities. IS all the channels.

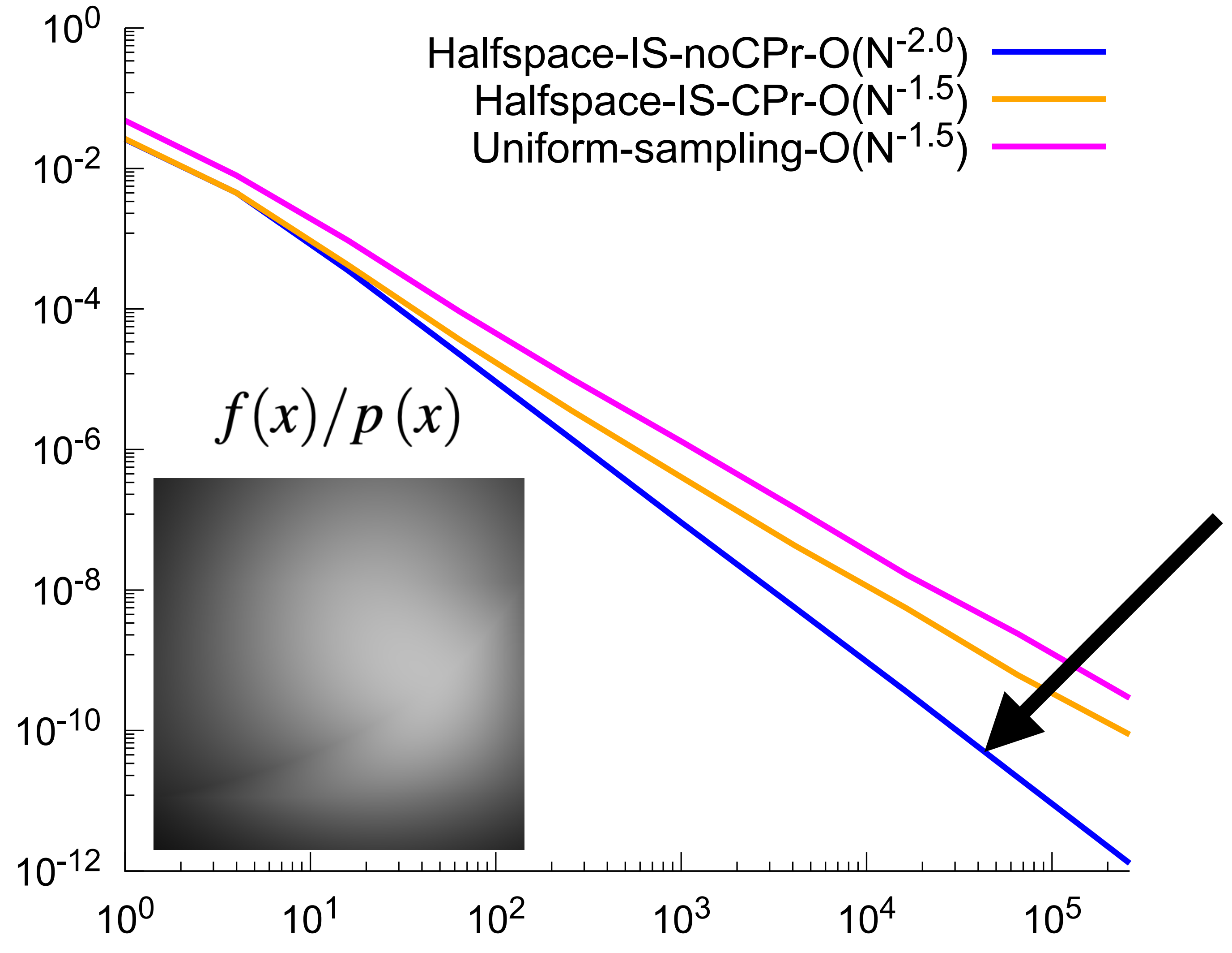
Future Directions

How can we leverage more insights from this formulation?

How we can use other statistical tools to represent variance?   PCF is there but what else?

Can we do better than traditional Importance Sampling?



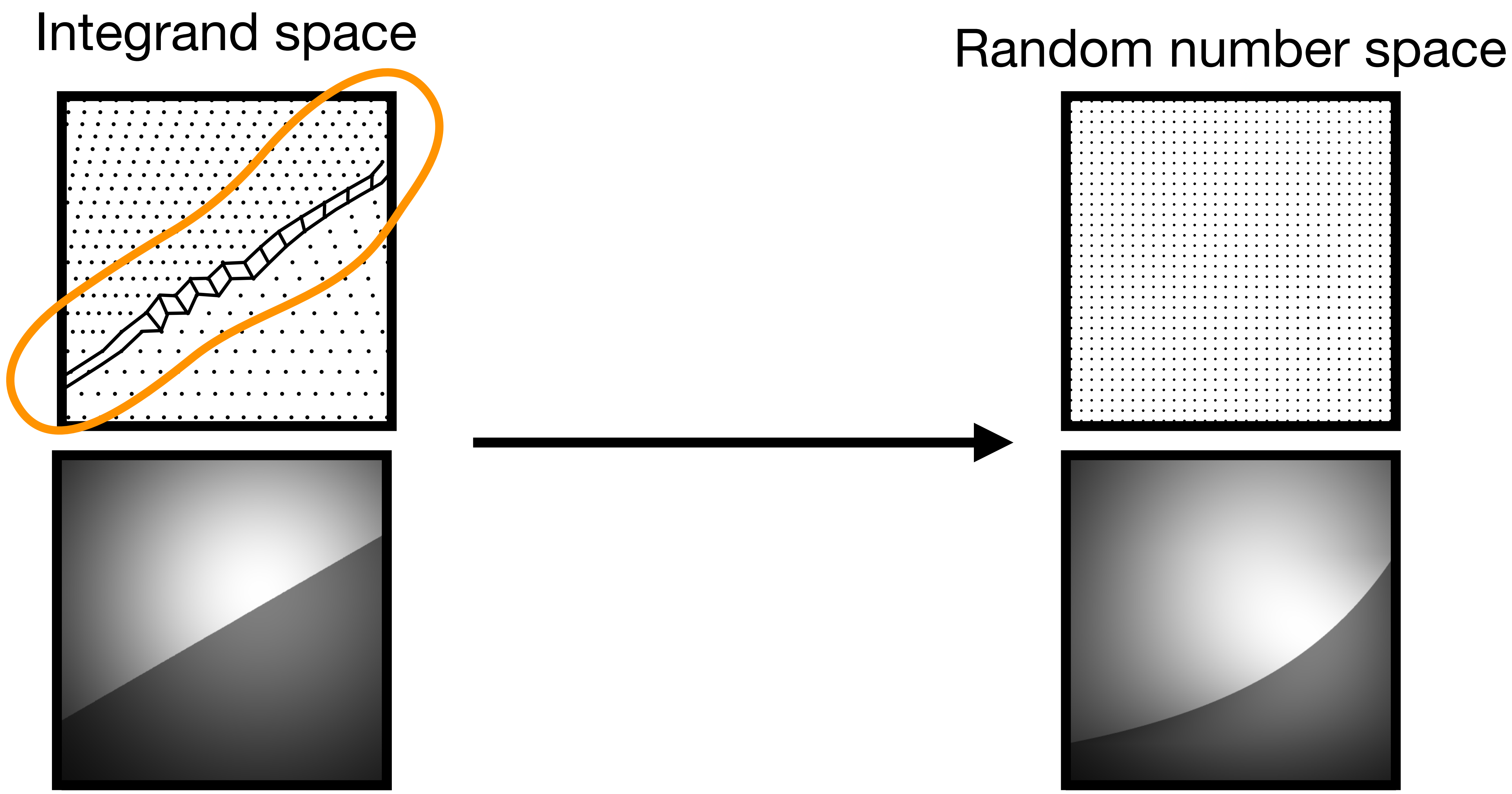


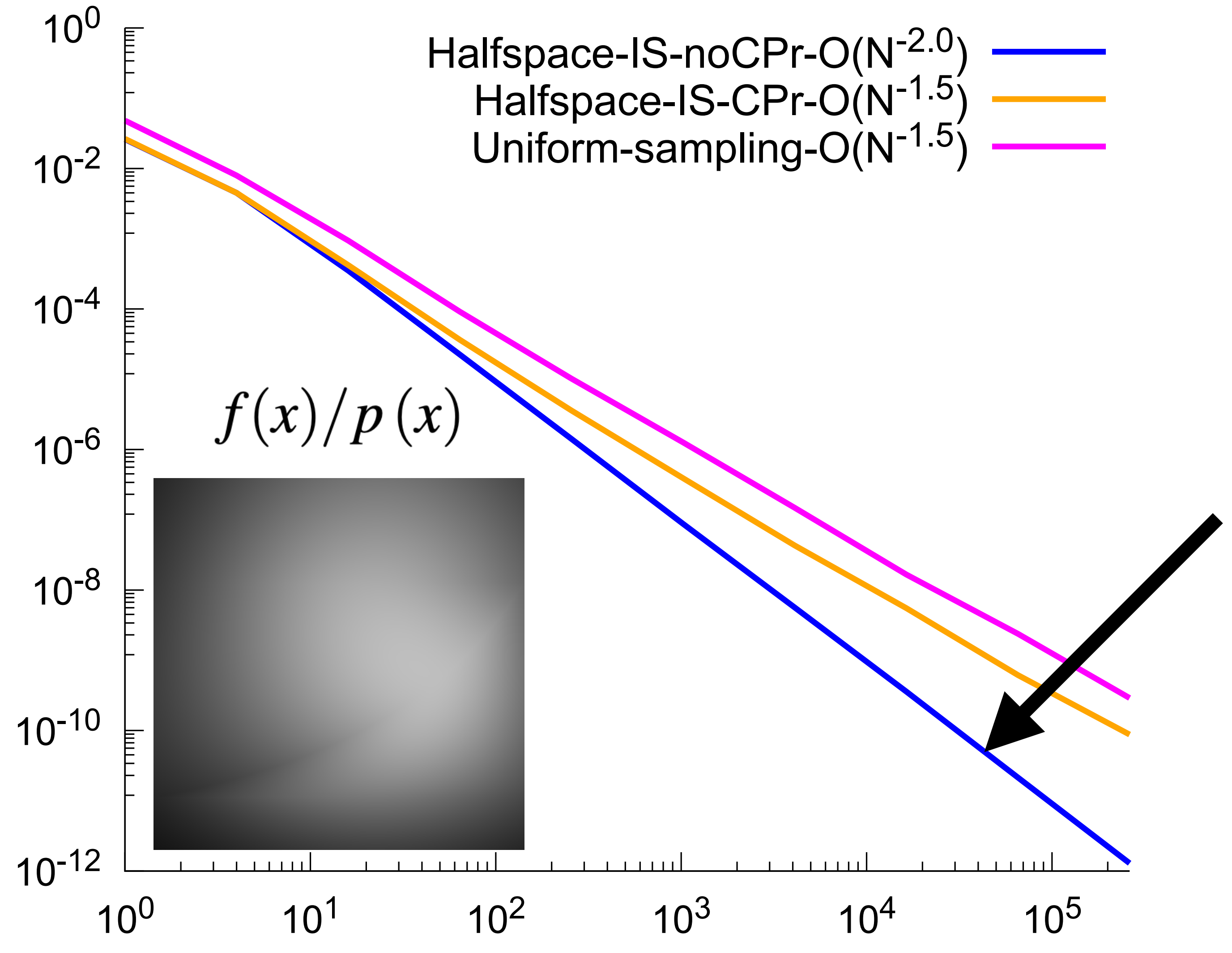
Future Directions

How can we leverage more insights from this formulation?

How we can use other statistical tools to represent variance? PCF is there but what else?

Can we do better than traditional Importance Sampling? e.g., more for strata alignment

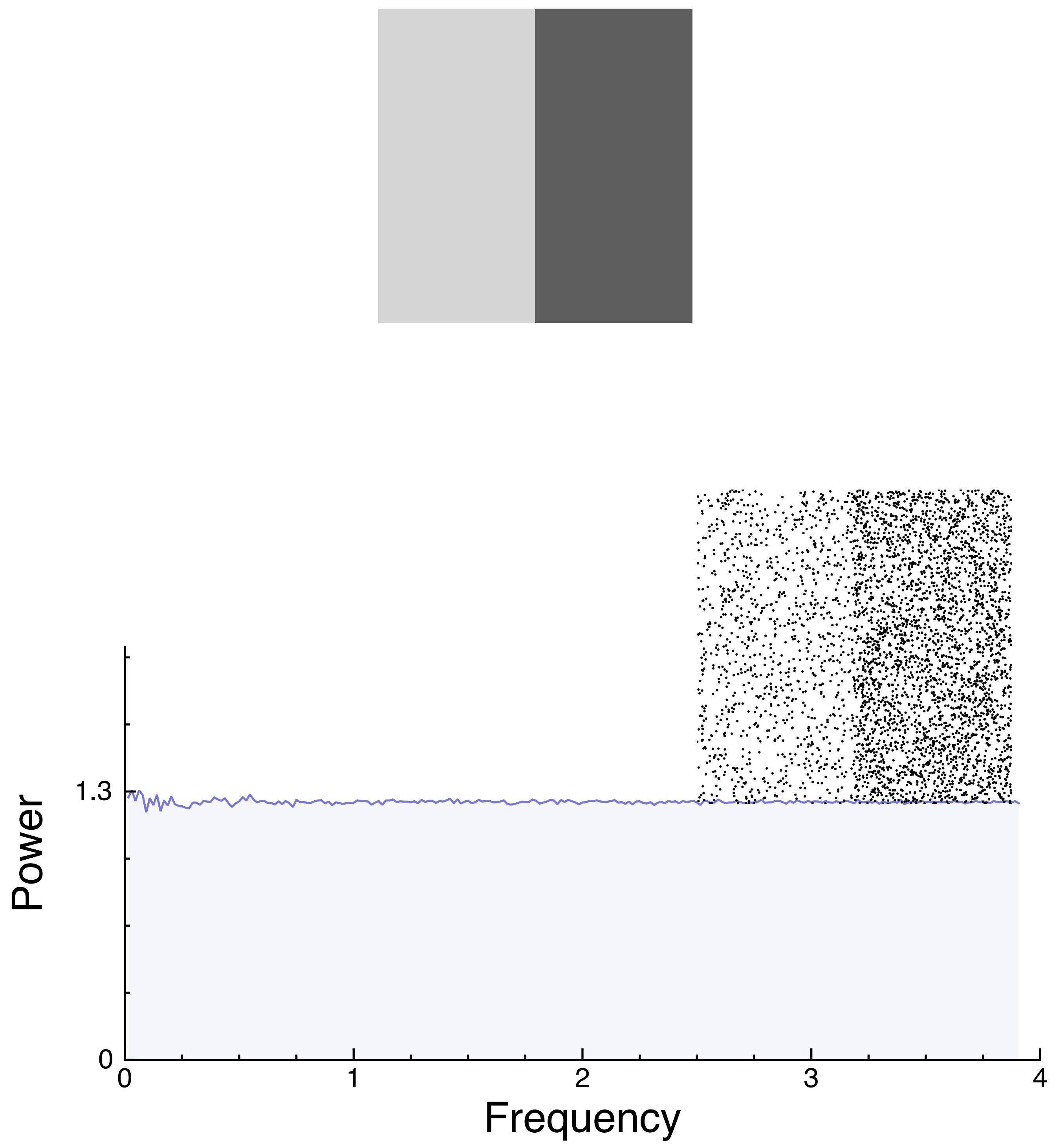


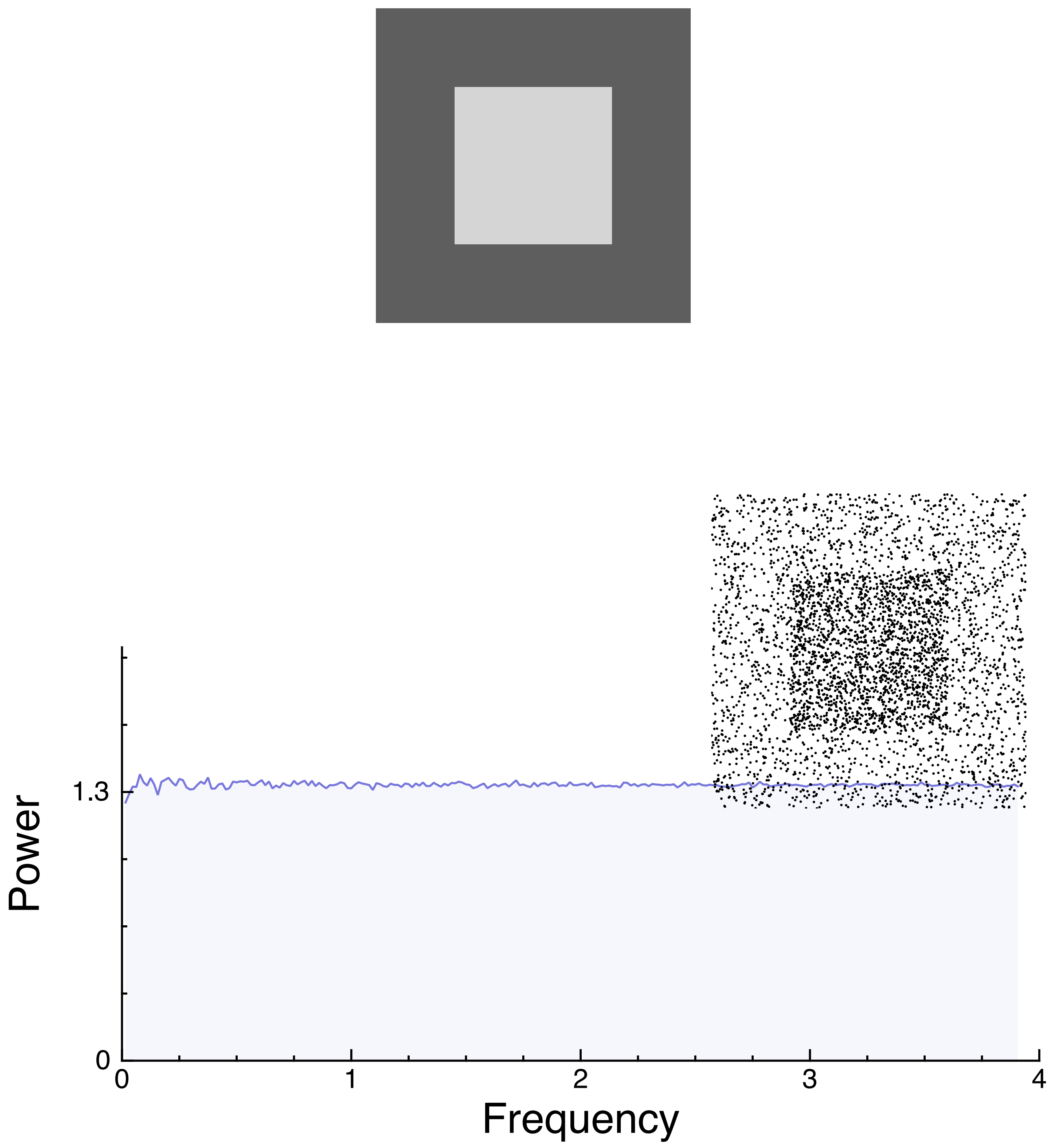


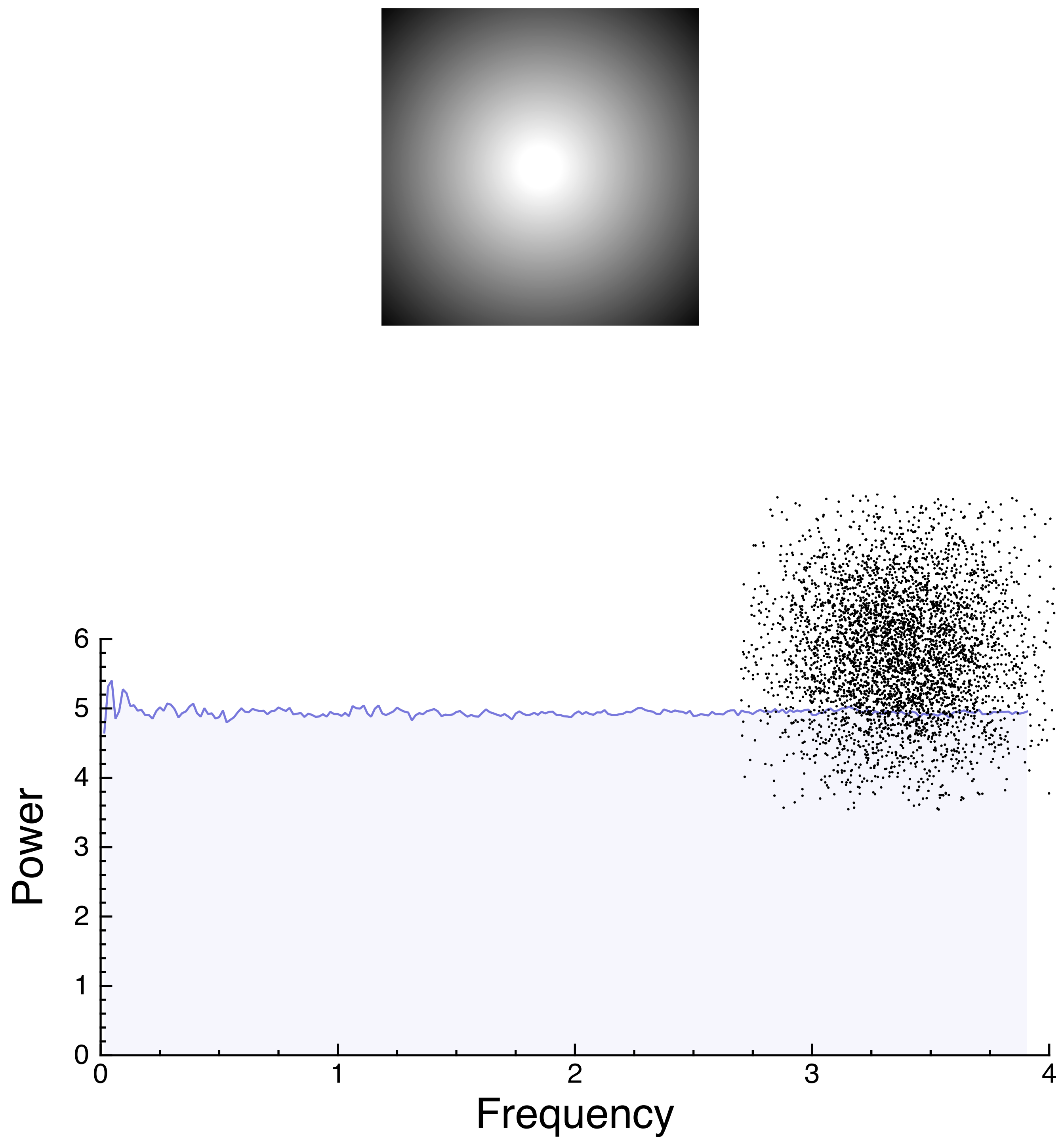
Thank you for your attention!

Questions ?

Power Spectrum of Importance Samples







Power Spectrum of Importance Samples