Analysis of Sample Correlations for Monte Carlo Rendering

SAMPLING MEASURES & ERROR FORMULATIONS

DISNEY RESEARCH STUDIOS

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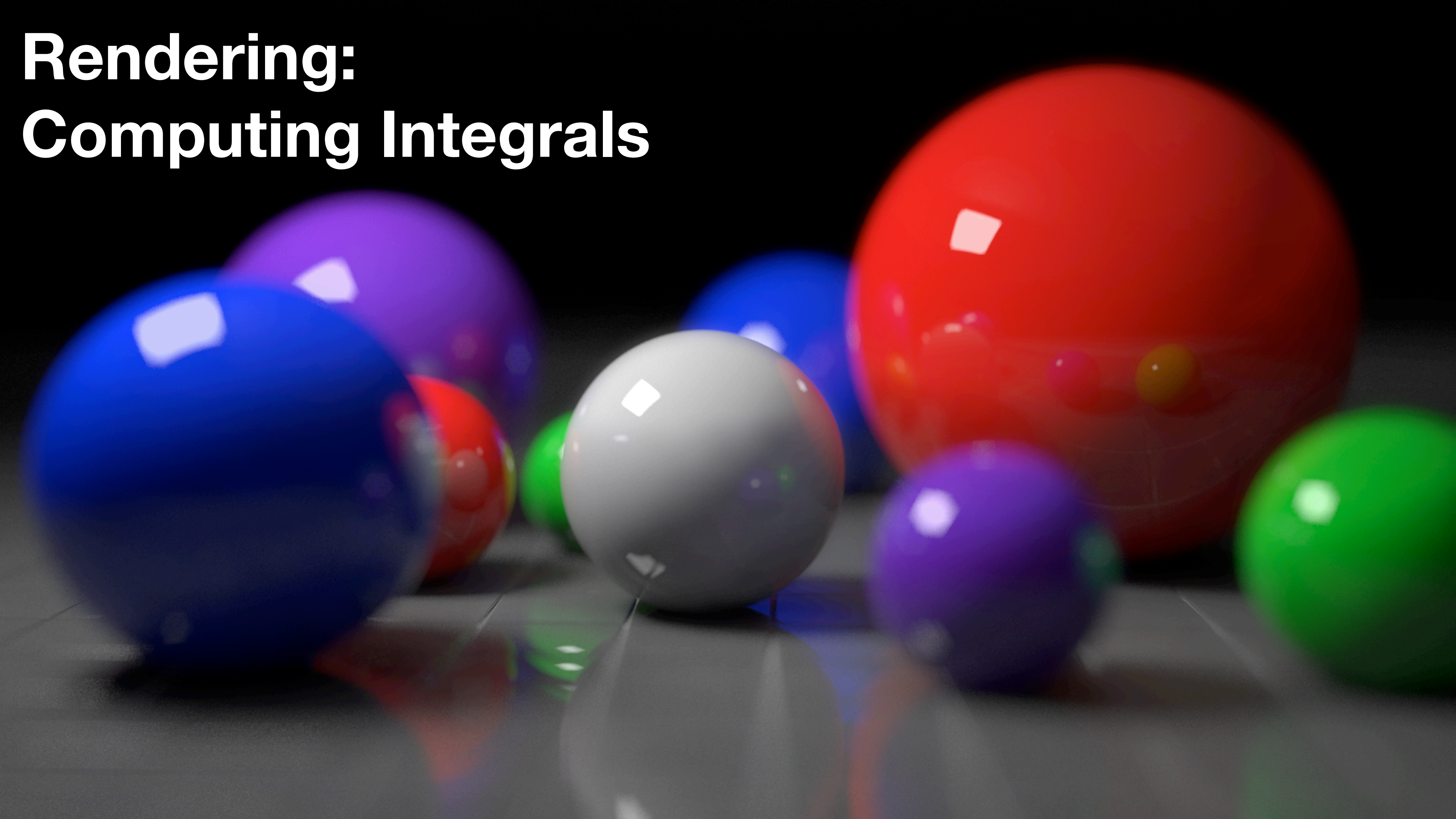
Research Scientist

Rendering: Computing Integrals



Numerical Integration

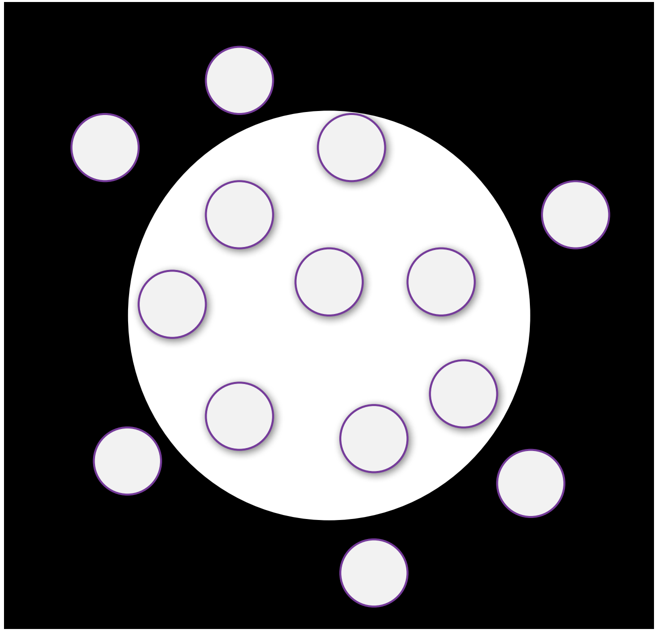
Approximate integrals with weighted sum of samples

$$I := \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} f(\mathbf{x}) d\mathbf{x}$$

$$\hat{I} := \sum_{I=1}^{n} w_i f(\mathbf{x}_i)$$

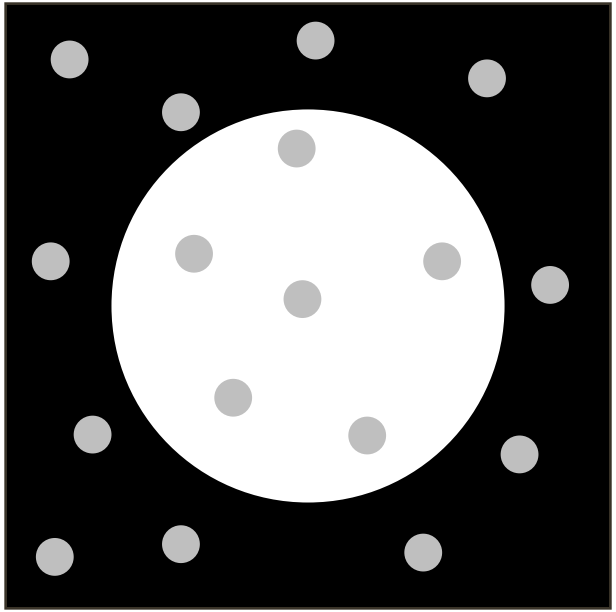
$$bias_{\mathcal{P}}[\hat{I}] = I - \mathbb{E}_{\mathcal{P}}[\hat{I}]$$

$$var_{\mathcal{P}}[\hat{I}] = \mathbb{E}_{\mathcal{P}}[\hat{I}^2] - (\mathbb{E}_{\mathcal{P}}[\hat{I}])^2$$

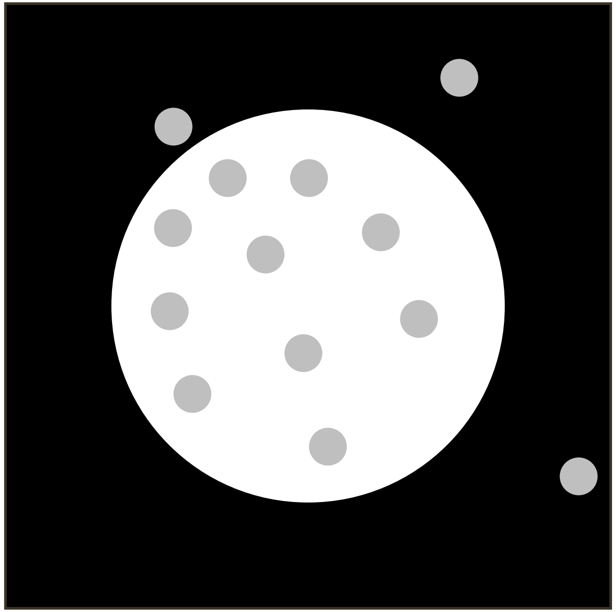


Numerical Integration

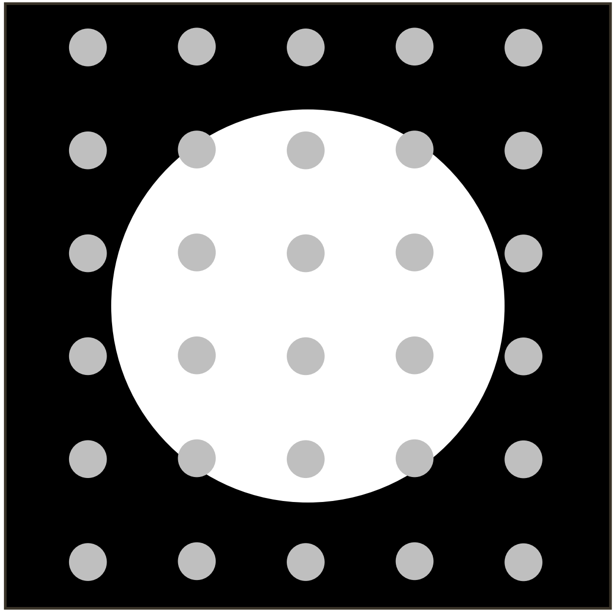
Approximate integrals with weighted sum of samples



Random



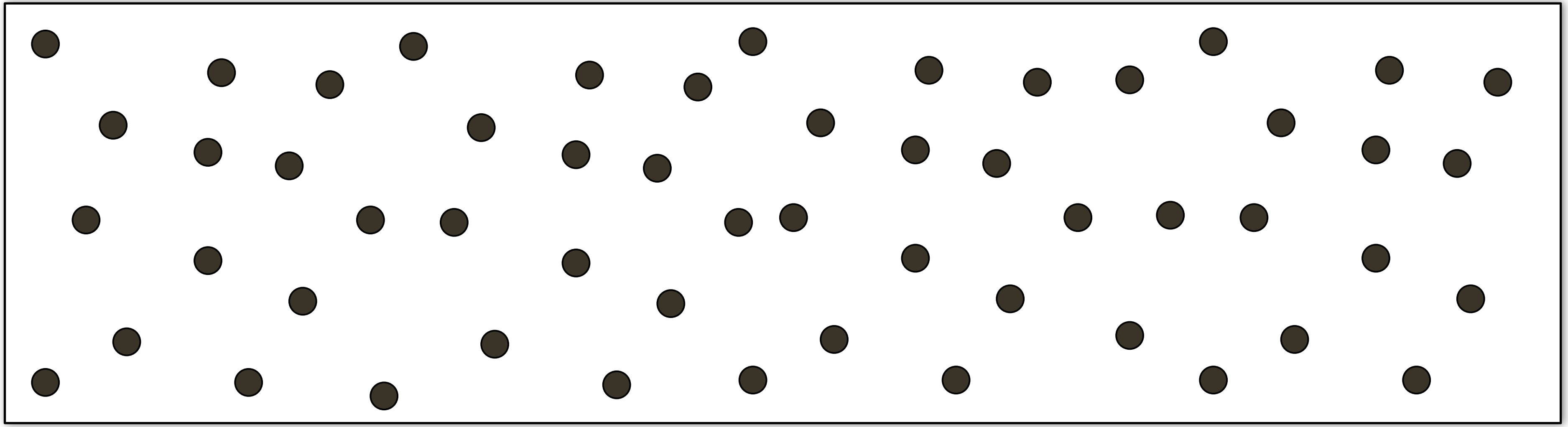
Density



Arrangement

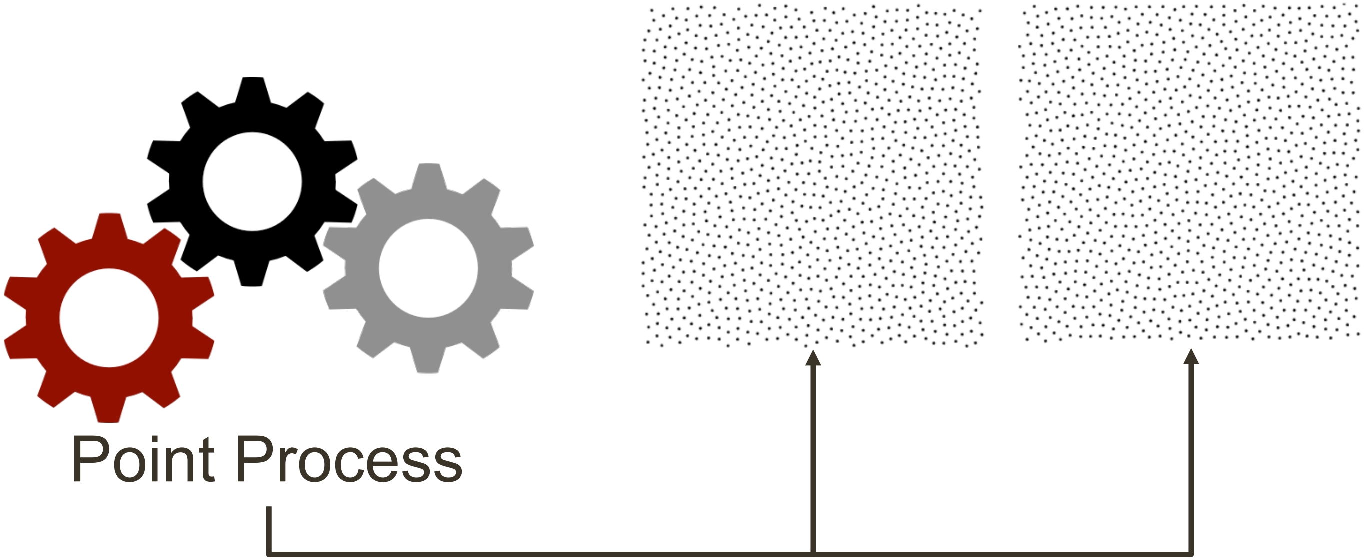
Stochastic Point Processes

Formal characterization of point patterns



Stochastic Point Processes

Formal characterization of point patterns



Stochastic Point Processes

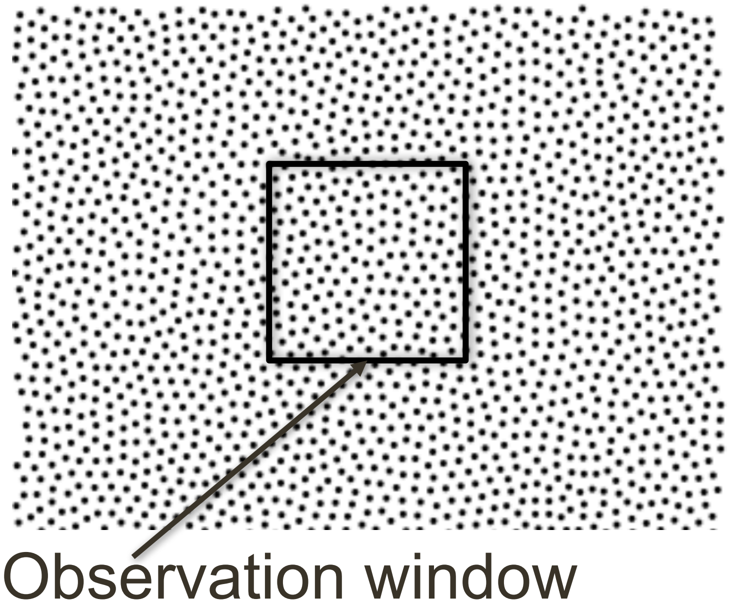
Examples of point processes

Natural Process

Manuel Process

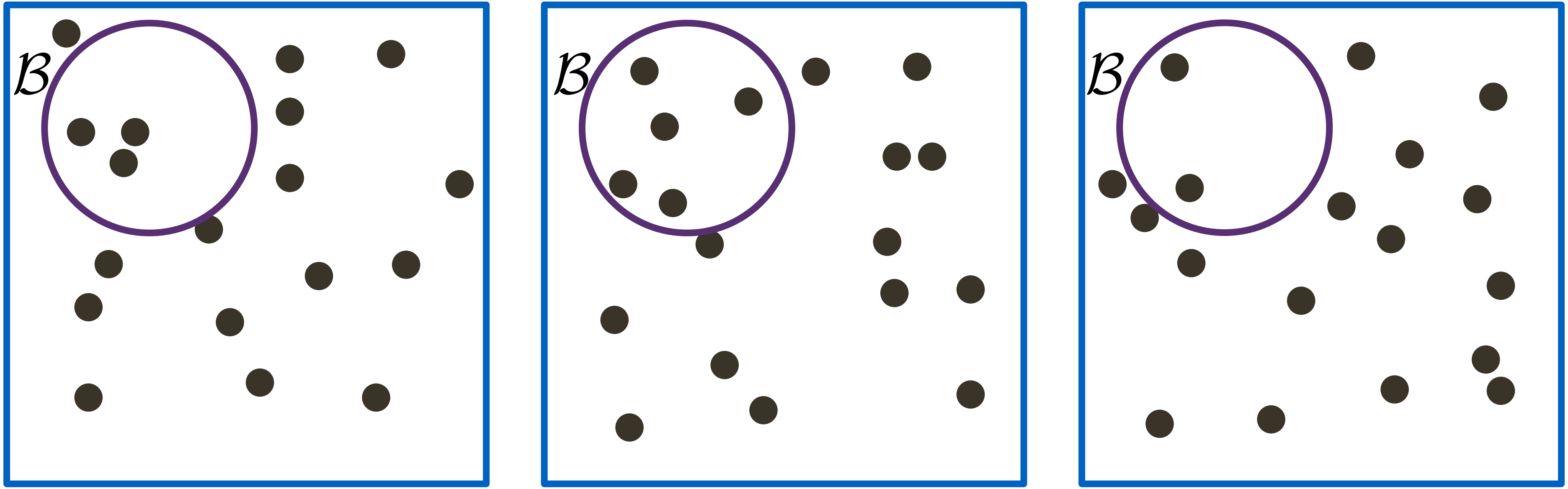
General Point Processes

Infinite point processes



General Point Processes

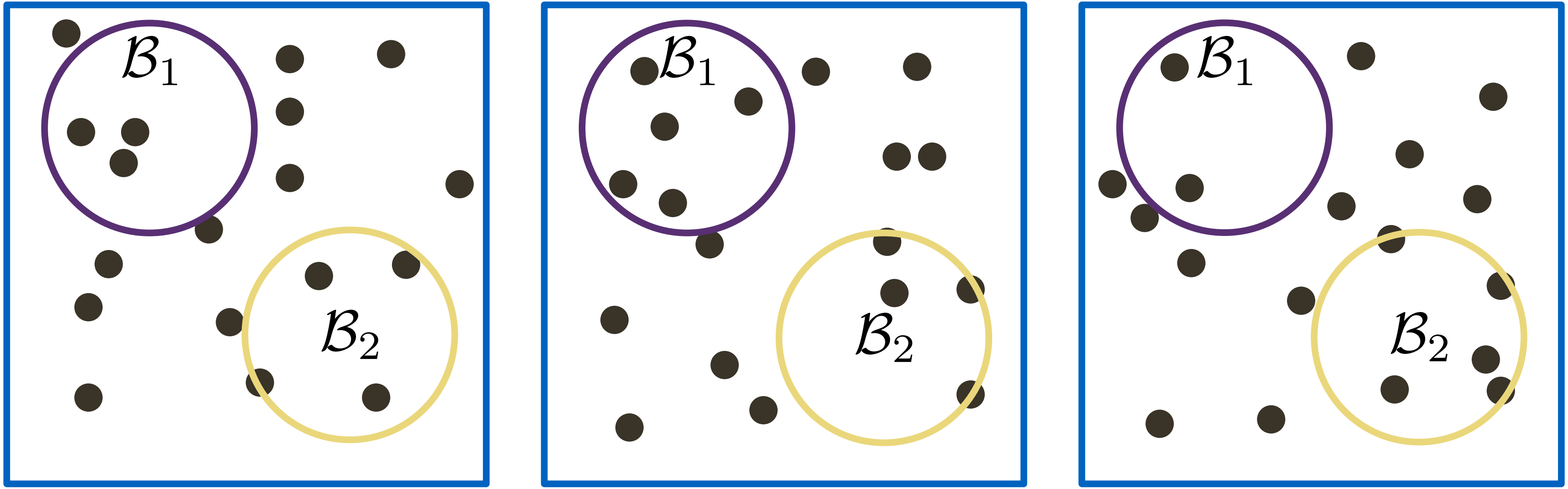
Assign a random variable to each set



$N(\mathcal{B}) = 3$ $\quad$ $N(\mathcal{B}) = 5$ $\quad$ $N(\mathcal{B}) = 2$

General Point Processes

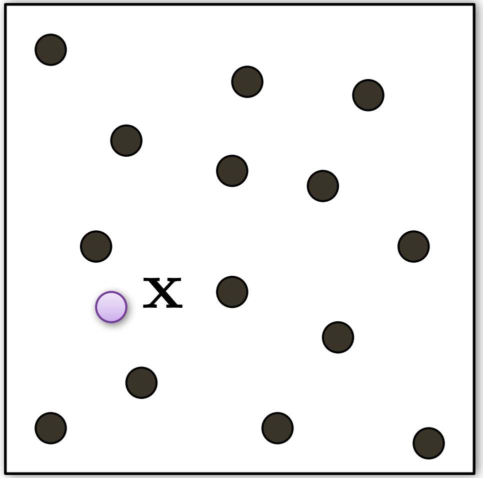
Joint probabilities define the point process



$$p_{N(\mathcal{B}_1),N(\mathcal{B}_2)}$$

Point Process Statistics

First order product density



$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

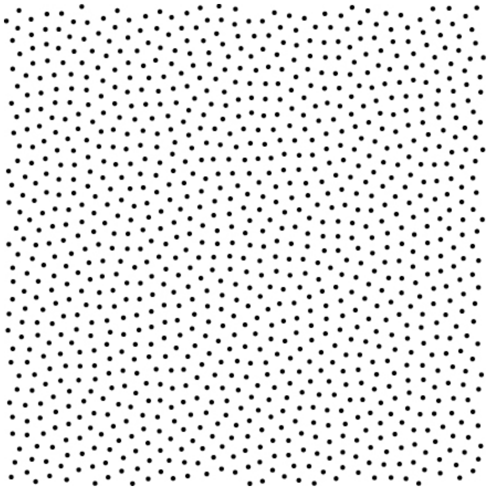
Expected number of points around **x**

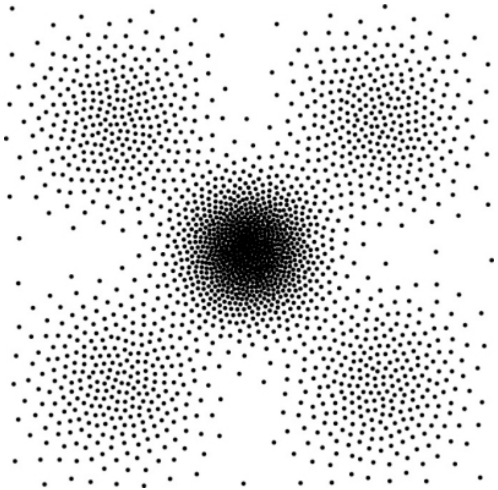
Measures local density

Point Process Statistics

First order product density

$\lambda(\mathbf{x})$



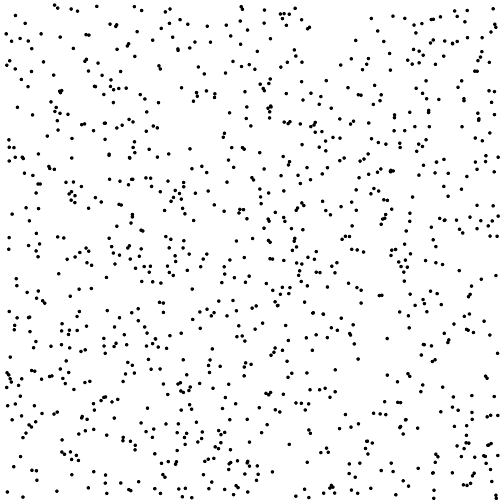


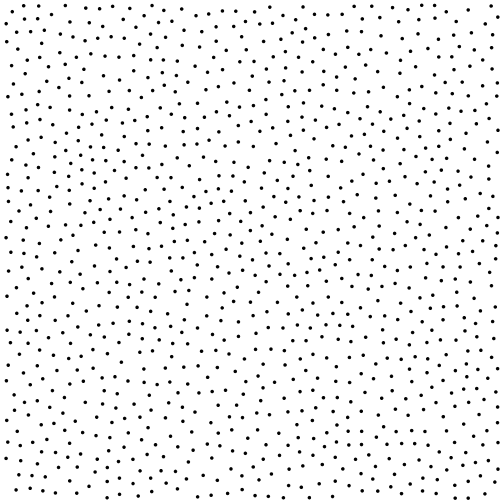
Point Process Statistics

First order product density

$\lambda(\mathbf{x})$

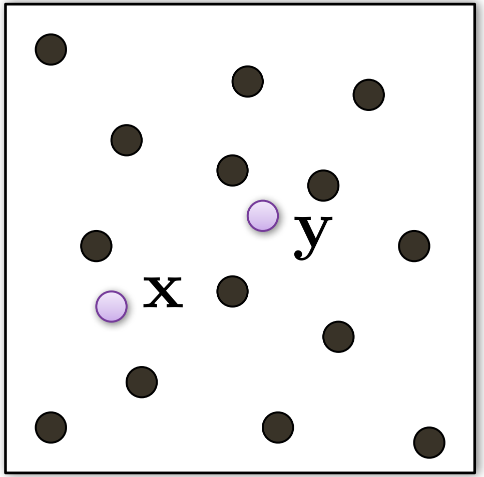
Constant





Point Process Statistics

Second order product density



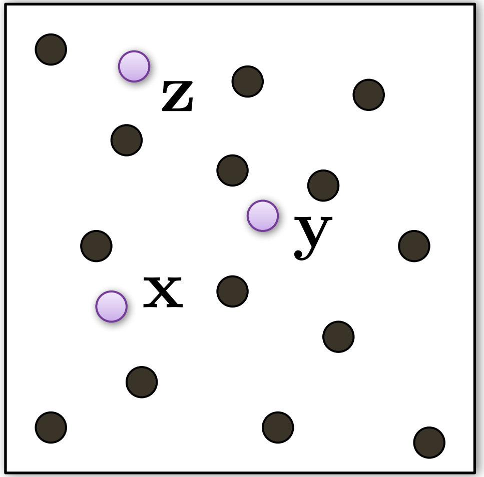
$$\varrho^{(2)}(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x}, \mathbf{y})$$

Expected number of points around **x** & **y**

Measures the joint probability $p(\mathbf{x}, \mathbf{y})$

Point Process Statistics

Higher order product density?



Expected number of points around **x**, **y**, **z**

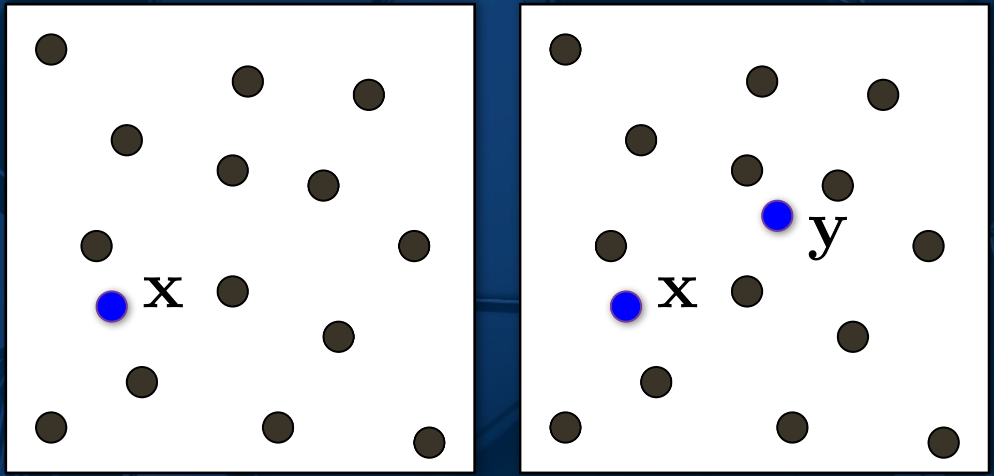
Not necessary: second order dogma

Point Process Statistics

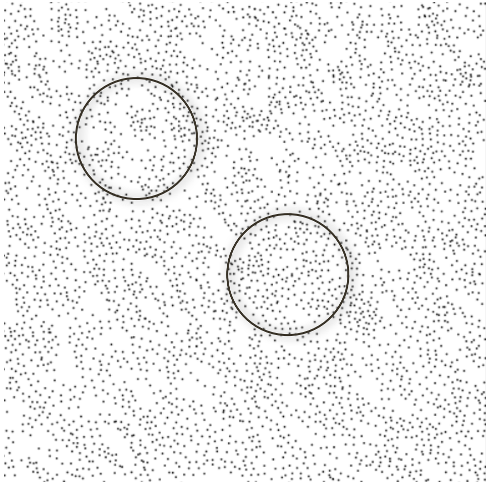
Summary: 1st & 2nd order correlations sufficient

$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

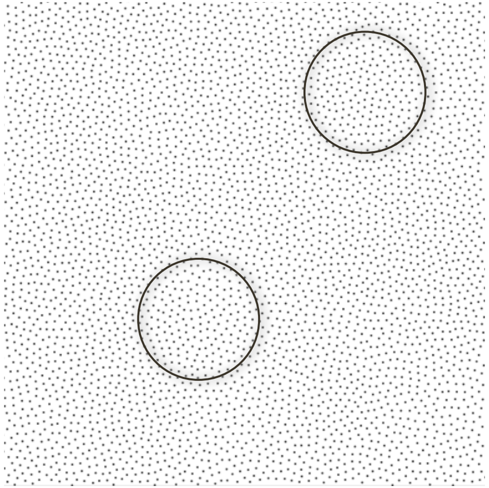
$$\varrho^{(2)}(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x}, \mathbf{y})$$



Stationary Point Processes



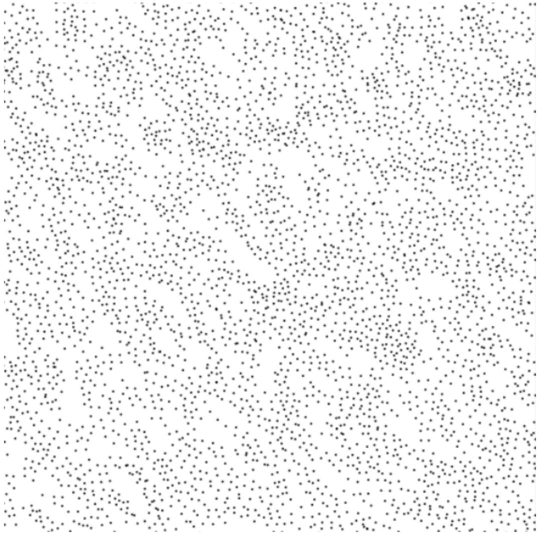
Stationary
(translation invariant)



Isotropic
(translation & rotation invariant)

Stationary Point Processes

Stationary (translation invariant)



$$\lambda(\mathbf{x}) = \lambda$$

$$\varrho(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x} - \mathbf{y})$$

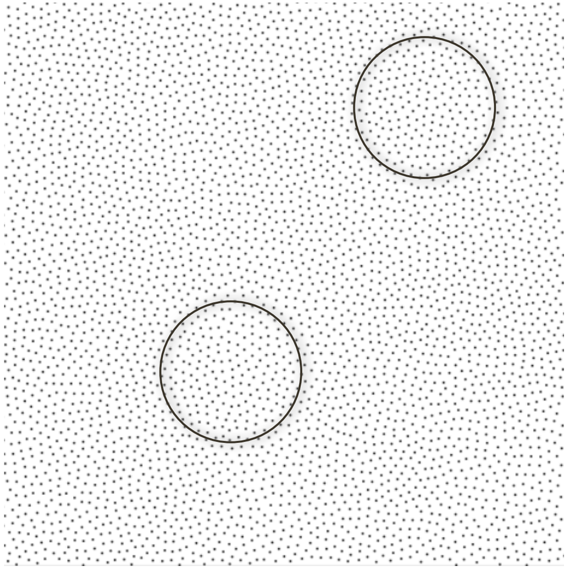
$$= \lambda^2 g(\mathbf{x} - \mathbf{y})$$

Pair Correlation Function (PCF)

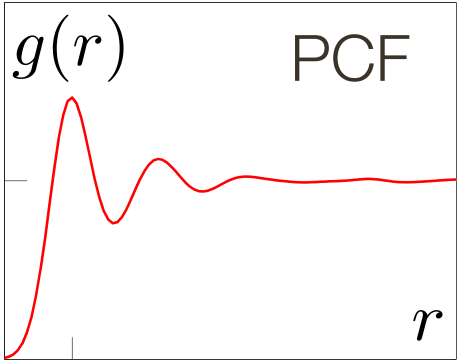
DoF reduced from 2d to d

Stationary Point Processes

Isotropic point process (translation & rotation invariant)



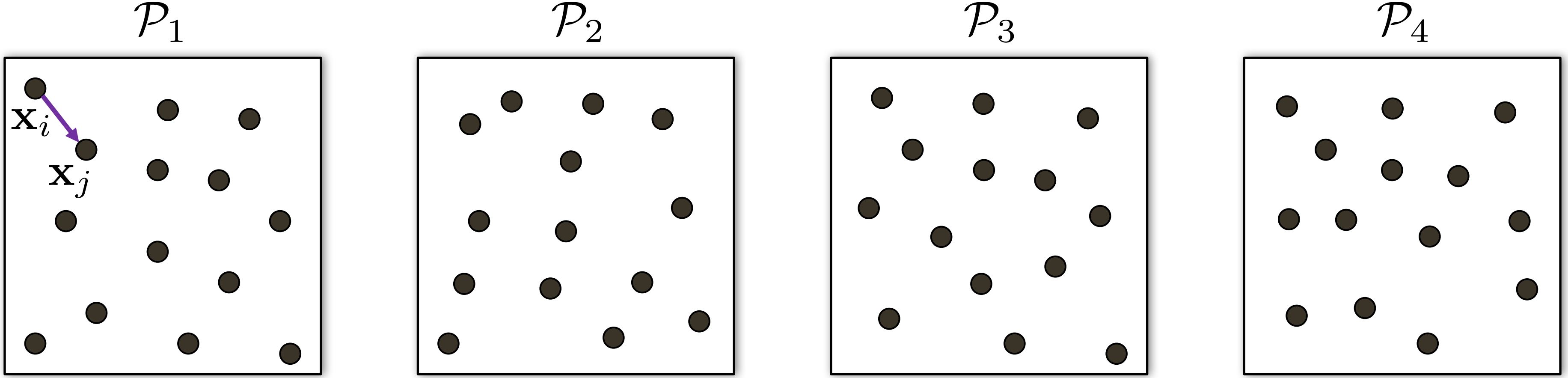
$$\lambda(\mathbf{x}) = \lambda$$
$$g(\mathbf{x} - \mathbf{y}) = g(||\mathbf{x} - \mathbf{y}||)$$



Estimating Correlations

Second order stationary - pair correlation function (PCF)

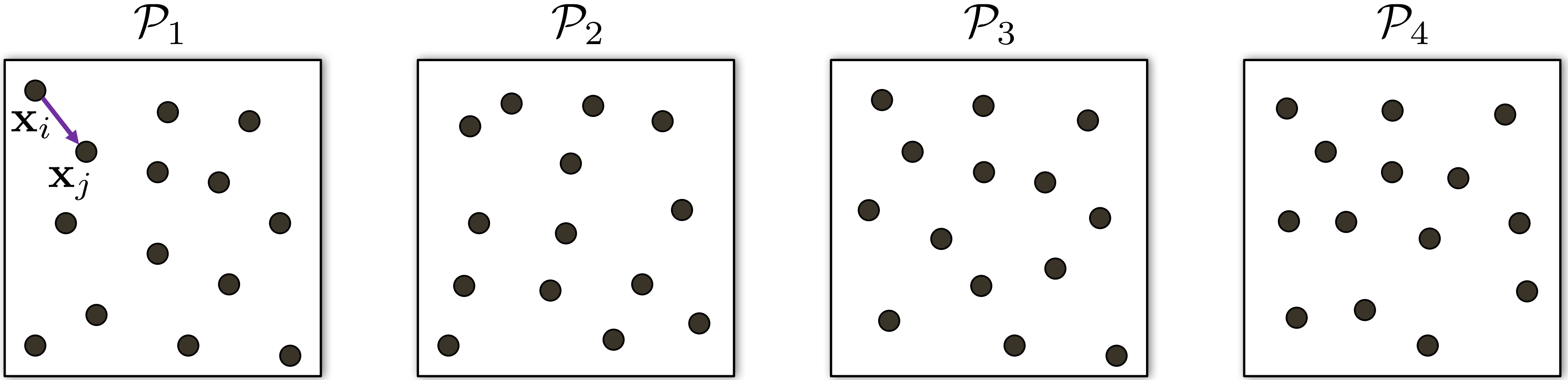
$$\hat{g}(\mathbf{r}) = \frac{1}{K\lambda^2} \sum_{\mathcal{P}_k} \sum_{\mathbf{x}_i, \mathbf{x}_j \in \mathcal{P}_k, i \neq j} \delta(\mathbf{r} - (\mathbf{x}_i - \mathbf{x}_j))$$



Estimating Correlations

Second order stationary - pair correlation function (PCF)

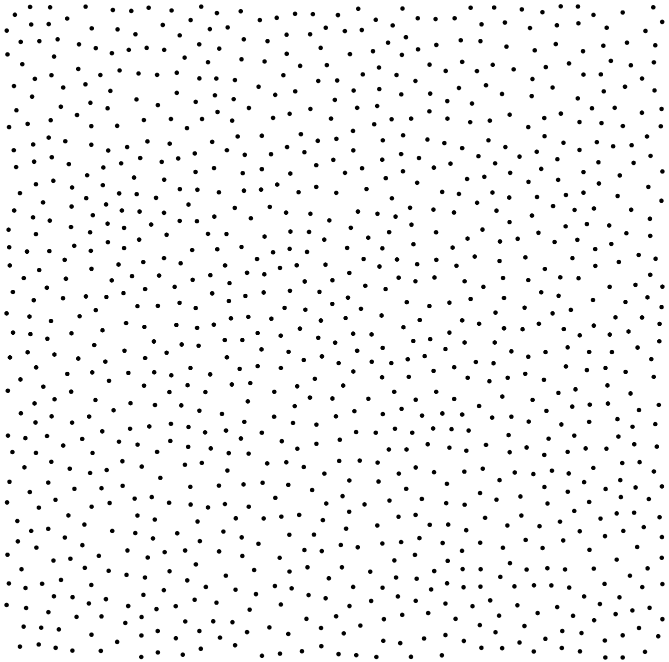
$$\hat{g}(\mathbf{r}) = \frac{1}{K\lambda^2 a_{\mathbb{I}_\mathcal{D}}(\mathbf{r})} \sum_{\mathcal{P}_k} \sum_{\mathbf{x}_i, \mathbf{x}_j \in \mathcal{P}_k, i \neq j} \delta(\mathbf{r} - (\mathbf{x}_i - \mathbf{x}_j))$$



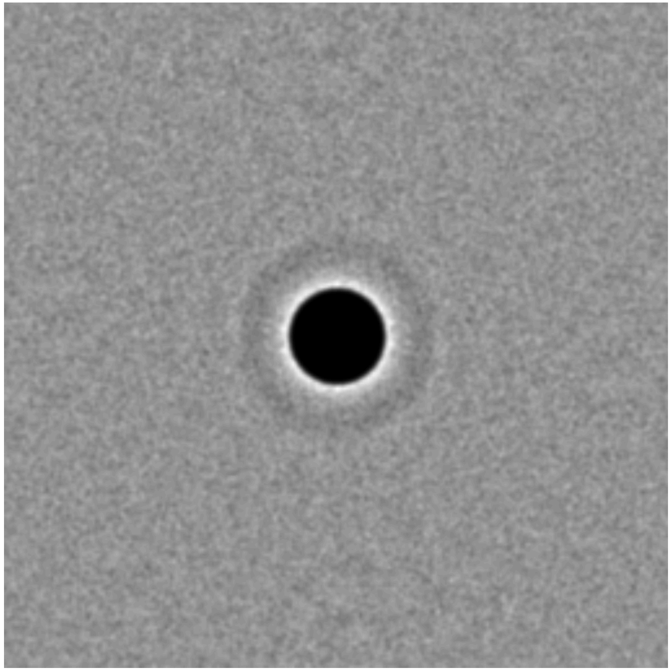
Estimating Correlations

Second order stationary - pair correlation function (PCF)

Point Distribution



Pair Correlation Function



Estimating Correlations

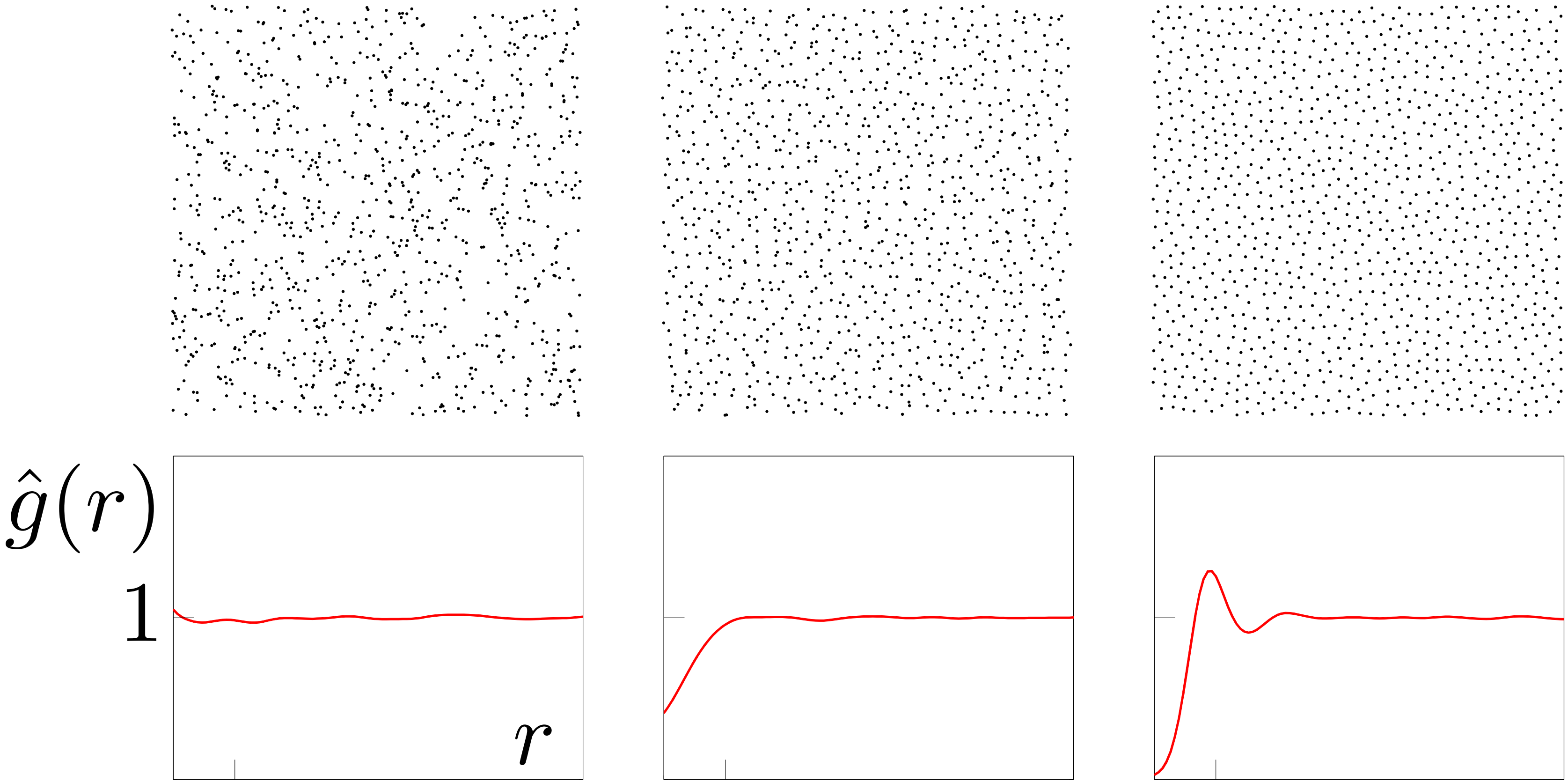
Second order isotropic - pair correlation function (PCF)

$$\hat{g}(r) = \frac{1}{\lambda^2 r^{d-1} |\mathcal{S}_d|} \sum_{i \neq j} k(r - \|\mathbf{x}_i - \mathbf{x}_j\|)$$

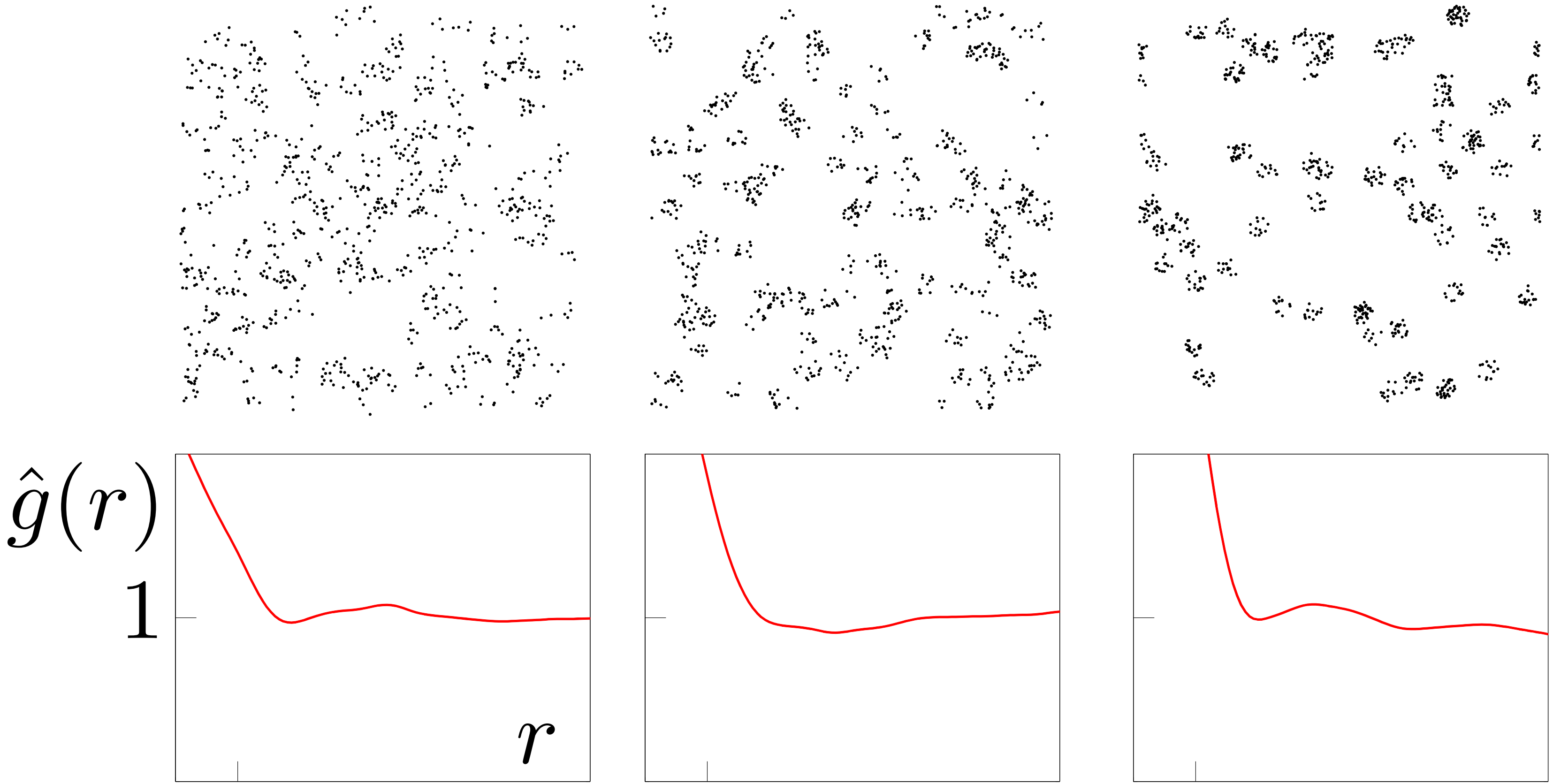
Volume of the unit hypercube in d dimensions

Kernel
e.g. Gaussian

Pair Correlation Function



Pair Correlation Function



Spectral Statistics

$$P(\boldsymbol{\nu}) = \lambda G(\boldsymbol{\nu}) + 1$$

Power spectrum

Fourier transform
of PCF

Spectral Statistics

$$P(\boldsymbol{\nu}) = \lambda G(\boldsymbol{\nu}) + 1$$

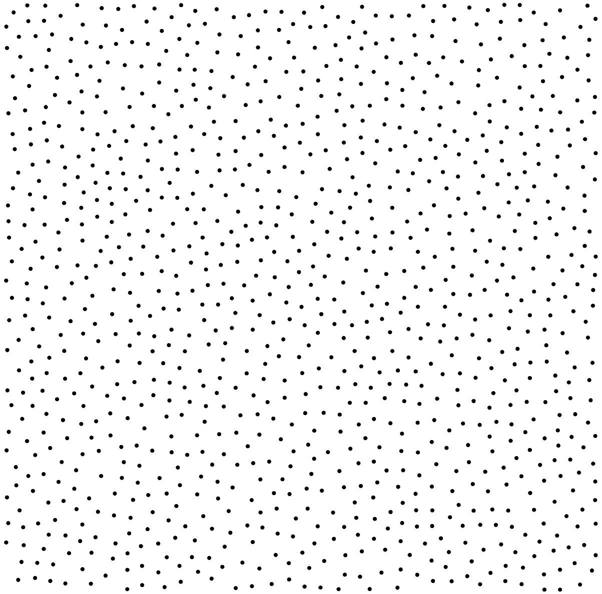
$$s(\mathbf{x}) = \sum \delta(\mathbf{x} - \mathbf{x}_j) \qquad \mathbf{s}_m = \sum e^{-i2\pi \mathbf{m}^T \mathbf{x}_j}$$

$$\mathbf{p}_m = \mathbb{E}_{\mathcal{P}}[\mathbf{s}_m^* \mathbf{s}_m] = \lambda \mathbf{g}_m + 1$$

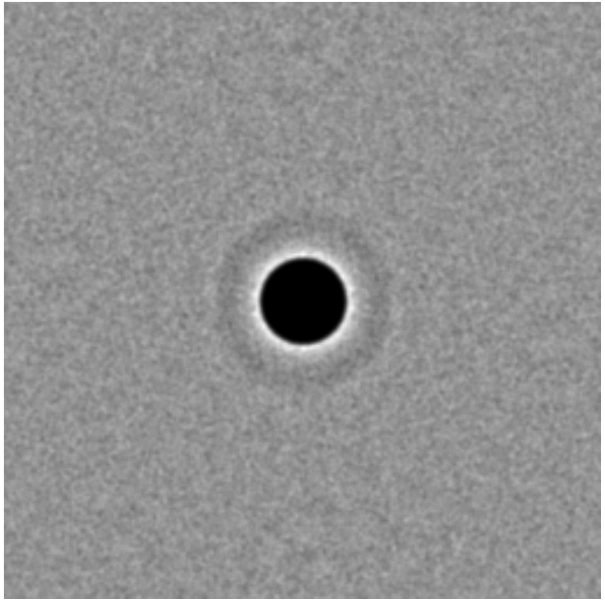
Spectral Statistics

$$\mathbf{p}_m = \mathbb{E}_{\mathcal{P}}[\mathbf{s}_m^* \mathbf{s}_m] = \lambda \mathbf{g}_m + 1$$

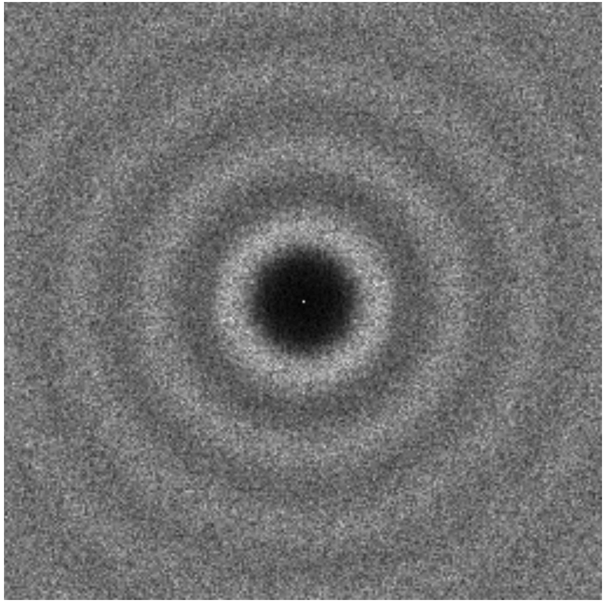
Points



PCF



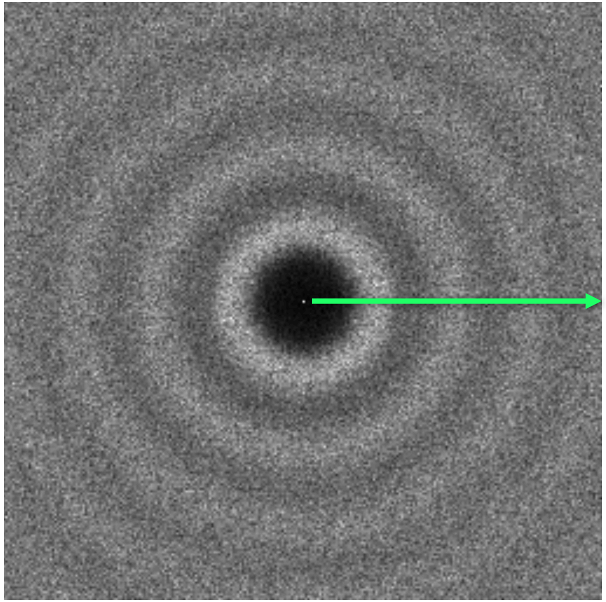
Power spectrum



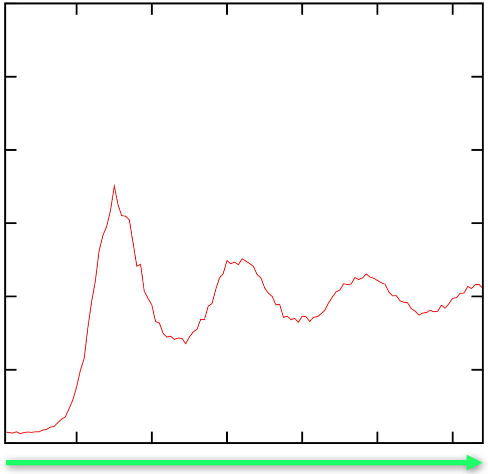
Spectral Statistics

$$\mathbf{p}_m = \mathbb{E}_{\mathcal{P}}[\mathbf{s}_m^* \mathbf{s}_m] = \lambda \mathbf{g}_m + 1$$

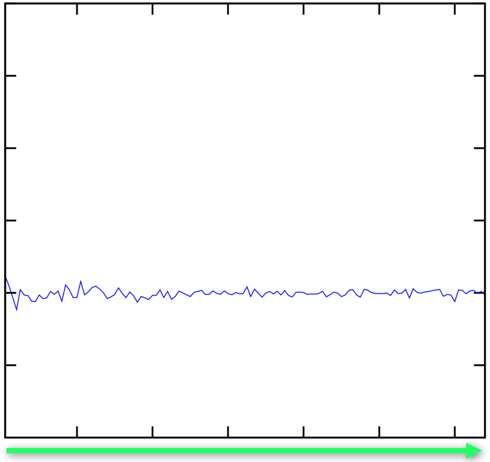
Power spectrum



Radial average



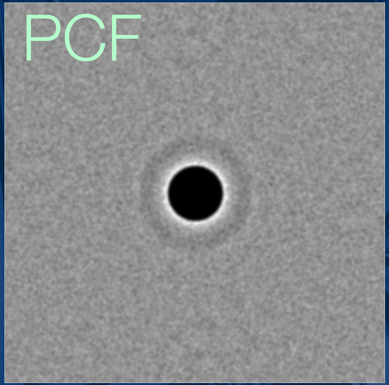
Radial anisotropy

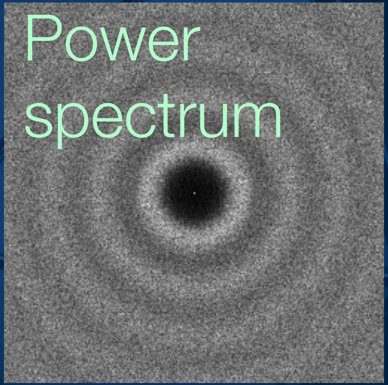


Statistics for Stationary Processes

Summary

Stationary: Spatial (PCF) & spectral (power spectrum)





Isotropic: radial averages

Error in Numerical Integration

Campbell's Theorem

$$\mathbb{E}_{\mathcal{P}}\left[\sum f(\mathbf{x}_i)\right] = \int_{\mathbb{R}^d} f(\mathbf{x})\lambda(\mathbf{x})d\mathbf{x}$$

$$\mathbb{E}_{\mathcal{P}}\left[\sum_{i\neq j} f(\mathbf{x}_i, \mathbf{x}_j)\right] = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(\mathbf{x},\mathbf{y})\varrho(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}$$

Error in Numerical Integration

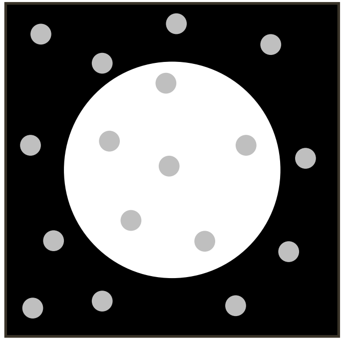
Campbell's theorem for the error of the integral estimator

$$\hat{I} := \sum w_i f(\mathbf{x}_i) \qquad bias(\hat{I}) = I - \mathbb{E}\hat{I} \qquad var(\hat{I}) = \mathbb{E}\hat{I}^2 - (\mathbb{E}\hat{I})^2$$

$$\mathbb{E}\hat{I} = \mathbb{E}\sum w(\mathbf{x}_i) f(\mathbf{x}_i) = \int_V w(\mathbf{x}) f(\mathbf{x}) \lambda(\mathbf{x}) d\mathbf{x} \qquad w_i = w(\mathbf{x}_i)$$

↑ Campbell's theorem

$$w(\mathbf{x}) = 1/\lambda(\mathbf{x}) \rightarrow bias(\hat{I}) = 0$$



Error in Numerical Integration

Campbell's theorem for the error of the integral estimator

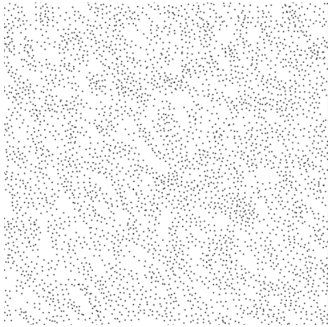
$$\hat{I} := \sum w_i f(\mathbf{x}_i) \qquad bias(\hat{I}) = I - \mathbb{E}\hat{I} \qquad var(\hat{I}) = \mathbb{E}\hat{I}^2 - (\mathbb{E}\hat{I})^2$$

$$\mathbb{E}\hat{I}^2 = \mathbb{E}\sum_{i \neq j} w_i f_i w_j f_j + \mathbb{E}\sum (w_i f_i)^2$$

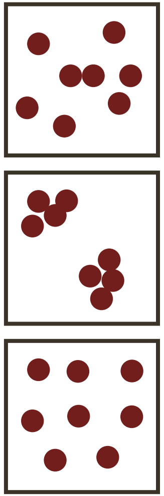
$$= \int_{V \times V} w(\mathbf{x}) f(\mathbf{x}) w(\mathbf{y}) f(\mathbf{y}) \boxed{\varrho(\mathbf{x}, \mathbf{y})} d\mathbf{x} d\mathbf{y} + \int_V w^2(\mathbf{x}) f^2(\mathbf{x}) \boxed{\lambda(\mathbf{x})} d\mathbf{x}$$

Error in Numerical Integration

Stationary point processes



$$\lambda(\mathbf{x}) = \lambda$$
$$\varrho(\mathbf{x},\mathbf{y}) = \lambda^2 g(\mathbf{x}-\mathbf{y})$$



$g = 1$

$g > 1$

$g < 1$

$$bias(\hat{I}) = 0 \quad (w = 1/\lambda)$$

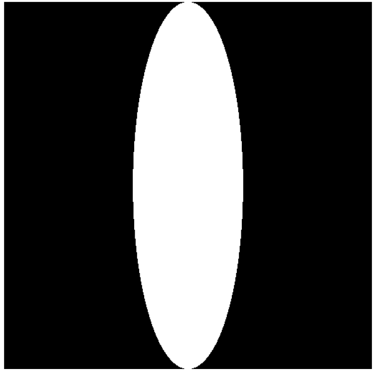
$$var(\hat{I}) = \boxed{\frac{1}{\lambda}\int f^2(\mathbf{x})d\mathbf{x}} + \boxed{\int a_f(\mathbf{r})g(\mathbf{r})d\mathbf{h}} - \left(\int f(\mathbf{x})d\mathbf{x}\right)^2$$

Density   Arrangement

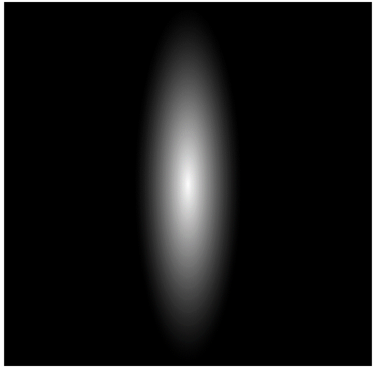
Error in Numerical Integration

Stationary point processes

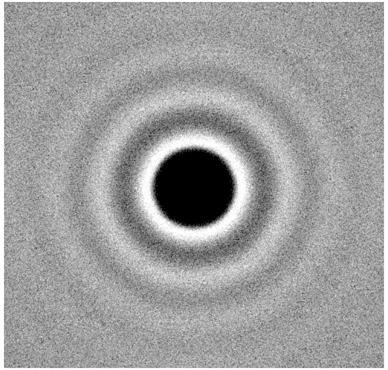
$$var(\hat{I}) = \frac{1}{\lambda}\int f^2(\mathbf{x})d\mathbf{x} + \boxed{\int a_f(\mathbf{r})g(\mathbf{r})d\mathbf{h}} - \left(\int f(\mathbf{x})d\mathbf{x}\right)^2$$



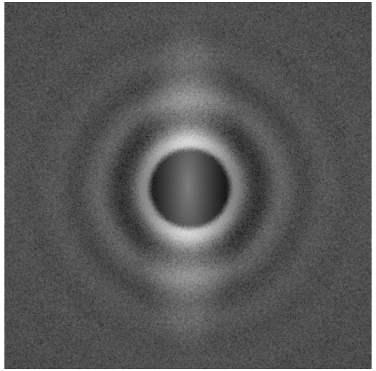
$f(\mathbf{x})$



$a_f(\mathbf{r})$



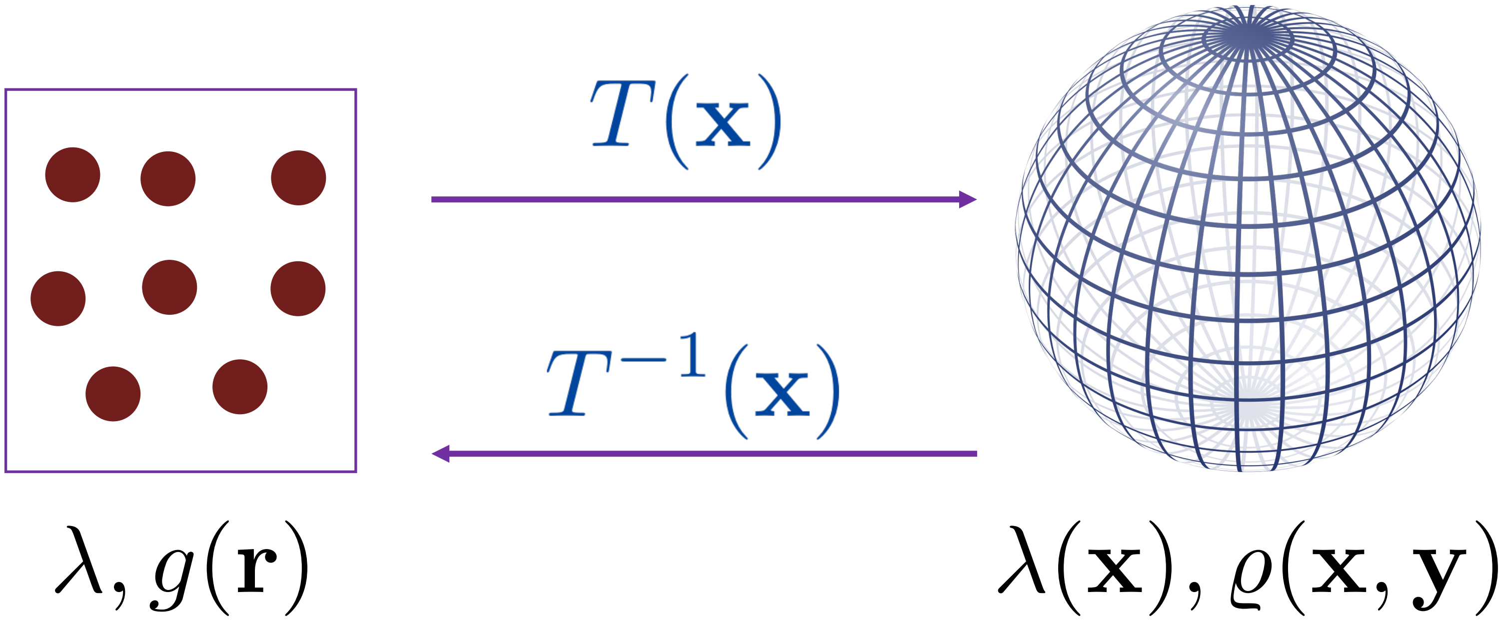
$g(\mathbf{r})$



$a_f(\mathbf{r})g(\mathbf{r})$

Error in Numerical Integration

Importance Sampling – invertible warp



Error in Numerical Integration

Importance Sampling – general unbiased

$$var(\hat{I}) = \int \frac{f^2(\mathbf{x})}{\lambda(\mathbf{x})} d\mathbf{x} + \int f(\mathbf{x}) f(\mathbf{y}) \frac{\varrho(\mathbf{x}, \mathbf{y})}{\lambda(\mathbf{x})\lambda(\mathbf{y})} d\mathbf{x} d\mathbf{y} - \left(\int f(\mathbf{x}) d\mathbf{x} \right)^2$$

Importance Sampling – random add/remove for intensity

$$var(\hat{I}) = \int \frac{f^2(\mathbf{x})}{\lambda(\mathbf{x})} d\mathbf{x} + \int f(\mathbf{x}) f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) d\mathbf{x} d\mathbf{y} - \left(\int f(\mathbf{x}) d\mathbf{x} \right)^2$$

Error in Numerical Integration

Spectral counterparts

$$s(\mathbf{x}) = \sum w(\mathbf{x}_i)\delta(\mathbf{x} - \mathbf{x}_i) \qquad \mathbf{s}_m = \sum w(\mathbf{x}_i)e^{-2\pi \mathbf{m}^T \mathbf{x}_i}$$

$$var(\hat{I}) = I^2 var(\mathbf{s}_0) + \boxed{\sum_{m \neq 0} \mathbf{f}_m^* \mathbf{f}_m \mathbb{E}[\mathbf{s}_m^* \mathbf{s}_m]} + \boxed{\sum_{l \neq m} \mathbf{f}_m^* \mathbf{f}_l \mathbb{E}[\mathbf{s}_m \mathbf{s}_l^*]}$$

Power spectra
Stationary

Phase
Non-stationary

Error in Numerical Integration

Spectral counterparts – stationary point processes

$$s(\mathbf{x}) = \sum w(\mathbf{x}_i)\delta(\mathbf{x} - \mathbf{x}_i) \qquad \mathbf{s}_m = \sum w(\mathbf{x}_i)e^{-2\pi \mathbf{m}^T \mathbf{x}_i}$$

$$var(\hat{I}) = \sum_{m \neq 0} \mathbf{f}_m^* \mathbf{f}_m \mathbb{E}[\mathbf{s}_m^* \mathbf{s}_m]$$

Power spectra

Stationary

Error in Numerical Integration

Spectral counterparts – stationary point processes

$$s(\mathbf{x}) = \sum w(\mathbf{x}_i)\delta(\mathbf{x} - \mathbf{x}_i)$$

$$\mathbf{s}_m = \sum w(\mathbf{x}_i)e^{-2\pi\mathbf{m}^T\mathbf{x}_i}$$

$$var(\hat{I}) = \sum_{m\neq 0} (\mathbf{f}_m^*\mathbf{f}_m)\mathbf{p}_m$$

$$\mathbf{p}_m = \mathbb{E}[\mathbf{s}_m^*\mathbf{s}_m] = \lambda\mathbf{g}_m + 1$$

Error in Numerical Integration

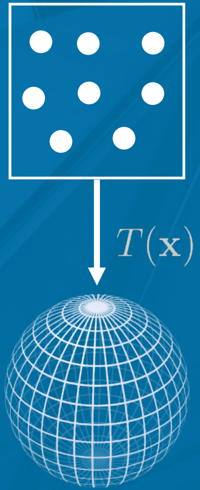
Stationary point processes – spatial vs. spectral

$$var(\hat{I})$$

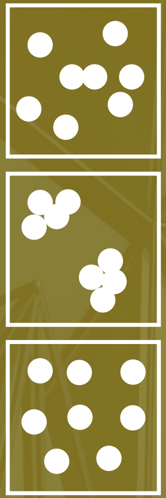
$$\frac{1}{\lambda}\int f^2(\mathbf{x})d\mathbf{x} + \int a_f(\mathbf{r})g(\mathbf{r})d\mathbf{h} - \left(\int f(\mathbf{x})d\mathbf{x}\right)^2$$

$$\sum_{m\neq 0} (\mathbf{f}_m^* \mathbf{f}_m)\mathbf{p}_m$$

1st order **correlations** by **warp/ algorithm**



2nd order **pair-wise correlations**



$g = 1$

$g > 1$

$g < 1$

Study in **spatial or spectral**

$$\int a_f(\mathbf{r}) g(\mathbf{r}) dh$$

$$\sum_{m \neq 0} (\mathbf{f}_m^* \mathbf{f}_m) \mathbf{p}_m$$

Analysis of Sample Correlations for Monte Carlo Rendering

SAMPLING MEASURES & ERROR FORMULATIONS

Disney RESEARCH STUDIOS

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