Error Analysis of Common Sampling Strategies

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Random

![Random Distribution Diagram]

![Random Grid Pattern]
Random vs Jitter

- Random distribution
- Jitter distribution

Images below illustrate the difference in distribution patterns.
Random

Jitter
Random

Jitter

Poisson Disk
Variance Convergence Rate of Samplers

\[ O(N^{-1}) \]

Number of Samples

Variance
Variance Convergence Rate of Samplers

\[ O(N^{-1}) \]

\[ O(N^{-1.25}) \]
Variance Convergence Rate of Samplers

Fredo Durand [2011]
Subr & Kautz [2013]
Pilleboue et al. [2015]
Blue noise sampling and beyond

Importance sampling with correlated samples

Stratification strategies
Regular grid samples

Pauly et al. [2000]

Regular grid
Uniformly jittered regular grid

Pauly et al. [2000]
Randomly jittered samples

Random jitter

Pauly et al. [2000]

Regular
Uniform jitter
Randomly jittered samples

Regular
Uniform jitter
Random jitter

Pauly et al. [2000]
Randomly jittered samples

Pauly et al. [2000]
Randomly jittered samples

Uniform jitter (RMS 13.4%)

Random jitter (RMS 10.4%)

Uniform jitter

Random jitter

Ramamoorthi et al. [2012]
Square area light source

Canonnical square domain

Uniform jitter

Random jitter

Occluded
Visible
Square area light source

Canonical square domain

Uniform jitter

Random jitter

Occluded
Visible
Randomly jittered samples

Uniform jitter (RMS 6.59%)
Random jitter (RMS 8.32%)
Uniform jitter (RMS 13.4%)
Random jitter (RMS 10.4%)

Uniform jitter
Random jitter

Ramamoorthi et al. [2012]
Polar mapping performs better for some samplers compared to concentric mapping observed by Andrew Kensler [2013]
Square area light source

Canonical square domain

Occluded  Visible

Per Christensen [2018]
Polar mapping

Canoncial square domain

Disk area light source

Square area light source

Occluded

Visible

Per Christensen [2018]
Concentric mapping
Canoncial square domain
Occluded  Visible
Square area light source
Disk area light source
Concentric mapping
Shirley and Chiu [1997]
Polar mapping
Per Christensen [2018]
Disk area
light source

Reference

Cengiz Oztireli [2016]
Disk area light source

Reference

Random jitter

Uniform jitter

RMS 11.21%

RMS 10.79%

RMS 10.92%

RMS 11.77%

Cengiz Oztireli [2016]
Disk area
light source

Reference  Random jitter  Uniform jitter  Isotropic jitter

RMS 11.21%  RMS 10.79%  RMS 8.00%

RMS 10.92%  RMS 11.77%  RMS 8.77%

Cengiz Oztireli [2016]
Isotropic jitter = uniform jitter + random rotation
Rotated uniform jitter better for not too complex shadows
Blue noise sampling
and beyond
Fourier analysis of sample correlations
Any sampling pattern with Blue noise characteristics is supposed to be well distributed within the spatial domain without containing any regular structures. The term Blue noise was coined by Ulichney [64 points in 2D. The corresponding sampling power spectra for Halton and Hammersley samples are summarised in Figures 5.8 for arbitrary dimensions, but due to the first dimension being a regular sampling, knowledge of the sequence is called the Hammersley sequence, which can create an even lower discrepancy point set.

Figure 5.6: Illustration of random and some stochastic grid-based sampling patterns with the corresponding Fourier expected power spectra and the corresponding radial mean of their expected power spectra.
Figure 5.6: Illustration of random and some stochastic grid-based sampling patterns with the corresponding Fourier expected power spectra and the corresponding radial mean of their expected power spectra.

5.3 Blue noise

Any sampling pattern with Blue noise characteristics is supposed to be well distributed within the spatial domain without containing any regular structures. The term Blue noise was coined by Ulichney [47], who investigated a radially averaged power spectrum of various sampling patterns. He advocated three important features for an ideal radial power spectrum: First, its peak should be at...
Expected power spectrum for blue noise samples

5.3.3 Tiling-based methods

There are some tile-based approaches that can be used to generate blue noise samples. Tile-based methods overcome the computational complexity of dart-throwing and/or relaxation-based approaches in generating blue noise sampling patterns. In the computer graphics community, two tile-based approaches are well known: First approach uses a set of precomputed tiles, with each tile composed of multiple samples, and later use these tiles, in a sophisticated way, to pave the sampling domain. Second approach employed tiles with one sample per tile and uses some relaxation-based schemes, with look-up tables, to improve the overall quality of samples.

Although many blue noise sample generation algorithms exist, none of them are easily extendable to higher dimensions ($> 3$).

5.4 Interpreting and exploiting knowledge of the sampling spectra

Recently, it has been shown that the low frequency region of the radial power spectrum (of a given sampling pattern) plays a crucial role in deciding the overall variance convergence rates of sampling patterns used for Monte Carlo integration. Since blue noise sampling patterns contain almost no radial energy in the low frequency region, they are of great interest for future research to obtain fast results in rendering problems. Surprisingly, Poisson Disk samples have shown the convergence rate of $O(N^{-1})$ which is the same as given by purely random samples. This can be explained by looking at the low frequency region in the radial power spectrum of Poisson Disk samples (Fig. 5.9) which is not zero. The importance of the shape of the radial mean power spectrum in the low frequency region demands methods and algorithms that could eventually allow sample generation directly from a target Fourier spectrum.

5.4.1 Radially-averaged periodograms

Figures 5.6, 5.8 and 5.9 depict radially averaged periodograms of the various sampling strategies described in this chapter. These spectra reveal two important characteristics of estimators built using the corresponding sampling strategies.
Variance in terms of power spectra

$f(\vec{x})$

Fredo Durand [2011]
Pillebuoe et al. [2015]
Variance in terms of power spectra

\[ \text{Var}(I_N) \propto f(\bar{x}) \]

Samples' expected power spectrum

Integrand power spectrum

Fredo Durand [2011]
Pillebuoe et al. [2015]
Variance in terms of power spectra

\[ \text{Var}(I_N) \propto \]

Samples' expected power spectrum \times Integrand power spectrum

Fredo Durand [2011]
Pillebuoe et al. [2015]
Variance in terms of power spectra

\[ \text{Var}(I_N) \propto \] 

Fredo Durand [2011]  
Pillebuoe et al. [2015]
Variance in terms of power spectra

\[ \text{Var}(I_N) \propto \text{Samples' expected power spectrum} \times \text{Integrand power spectrum} \]

Pillebuoe et al. [2015]
Variance in terms of power spectra

$$\text{Var}(I_N) \propto$$

Samples' expected power spectrum

Integrand power spectrum

Pilleboue et al. [2015]
Convergence rate depends on the low frequency region

\[ \text{Var}(I_N) \propto \frac{\text{Samples' expected power spectrum}}{\text{Integrands power spectrum}} \]

<table>
<thead>
<tr>
<th>Samplers</th>
<th>Worst Case</th>
<th>Best Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random</td>
<td>( O(N^{-1}) )</td>
<td>( O(N^{-1}) )</td>
</tr>
<tr>
<td>Poisson Disk</td>
<td>( O(N^{-1}) )</td>
<td>( O(N^{-1}) )</td>
</tr>
<tr>
<td>Jitter</td>
<td>( O(N^{-1.5}) )</td>
<td>( O(N^{-2}) )</td>
</tr>
<tr>
<td>CCVT</td>
<td>( O(N^{-1.5}) )</td>
<td>( O(N^{-3}) )</td>
</tr>
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</table>
Jittered samples converges faster than Poisson Disk

\[ O(N^{-1.5}) \]

\[ O(N^{-1}) \]
Convergence rate depends on the low frequency region

\[ \text{Var}(I_N) \propto \frac{\text{Samples' expected power spectrum}}{\text{Integrand power spectrum}} \]

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Pilleboue et al. [2015]

Isotropic Spectrum
Poisson Disk
Latin Hypercube Sampler (N-rooks)
Latin Hypercube Sampler (N-rooks)

Initialize
Latin Hypercube Sampler (N-rooks)

Shuffle rows
Latin Hypercube Sampler (N-rooks)
Latin Hypercube Sampler (N-rooks)

Shuffle columns

Slide after Wojciech Jarosz
Latin Hypercube Sampler (N-rooks)
Latin Hypercube Sampler (N-rooks)
Anisotropic Sampling Power Spectra

N-rooks / Latin Hypercube

N-rooks Spectrum
Anisotropic Sampling Power Spectra

N-rooks / Latin Hypercube

N-rooks Spectrum
Anisotropic Sampling Power Spectra

N-rooks / Latin Hypercube  N-rooks Spectrum  Jitter  Jitter Spectrum
Anisotropic Sampling Power Spectra

N-rooks / Latin Hypercube

N-rooks Spectrum

Multi-Jitter

Multi-Jitter Spectrum

Chiu et al. [1993]
Anisotropic Sampling Power Spectra

N-rooks / Latin Hypercube

N-rooks Spectrum

Multi-jitter

Multi-Jitter Spectrum

Chiu et al. [1993]
Anisotropic Sampling Power Spectra

N-rooks / Latin Hypercube  N-rooks Spectrum  Multi-jitter  Multi-Jitter Spectrum

Chiu et al. [1993]
Sampling in Higher Dimensions
4D Sampling

Rob Cook [1986]

2D

\[
\begin{align*}
(x_1, y_1) & \rightarrow (u_1, v_1) \\
(x_2, y_2) & \rightarrow (u_2, v_2) \\
(x_3, y_3) & \rightarrow (u_3, v_3) \\
(x_4, y_4) & \rightarrow (u_4, v_4) \\
& \vdots \\
(\ldots) \\
& \vdots \\
(\ldots) \\
& \vdots \\
& \vdots
\end{align*}
\]

4D

\[
\begin{align*}
(x_1, y_1, u_1, v_1) \\
(x_2, y_2, u_1, v_1) \\
(x_3, y_3, u_4, v_4) \\
(x_4, y_4, u_2, v_2) \\
& \vdots
\end{align*}
\]
Rob Cook [1986]

Uncorrelated Jitter

2D Sampling

\[
\begin{align*}
(x_1, y_1) &\rightarrow (u_1, v_1) \\
(x_2, y_2) &\rightarrow (u_2, v_2) \\
(x_3, y_3) &\rightarrow (u_3, v_3) \\
(x_4, y_4) &\rightarrow (u_4, v_4) \\
&\vdots
\end{align*}
\]

4D Sampling

\[
\begin{align*}
(x_1, y_1, u_3, v_3) \\
(x_2, y_2, u_1, v_1) \\
(x_3, y_3, u_4, v_4) \\
(x_4, y_4, u_2, v_2) \\
&\vdots
\end{align*}
\]
Uncorrelated Poisson Disk

Rob Cook [1986]

4D Sampling

2D

\((x_1, y_1)\) \rightarrow \((u_1, v_1)\)

\((x_2, y_2)\) \rightarrow \((u_2, v_2)\)

\((x_3, y_3)\) \rightarrow \((u_3, v_3)\)

\((x_4, y_4)\) \rightarrow \((u_4, v_4)\)

\vdots

4D

\((x_1, y_1, u_3, v_3)\)

\((x_2, y_2, u_1, v_1)\)

\((x_3, y_3, u_4, v_4)\)

\((x_4, y_4, u_2, v_2)\)

\vdots
4D Sampling Spectra along Projections

Poisson Disk Spectra

UV

XY

Poisson Disk Samples
4D Sampling Spectra along Projections

Poisson Disk Spectra

Poisson Disk Samples

UV

XY

XU
4D Sampling Spectra along Projections

Poisson Disk Spectra

UV

XY

XU

YV

Poisson Disk Samples
Convergence Analysis for Anisotropic Sampling Spectra

Power Spectrum

Radial Power Spectrum

Along canonical axes

Jittered Spectrum Profile

Other directions

Random Spectrum Profile
Convergence Analysis for Anisotropic Sampling Spectra

Power Spectrum

Radial Power Spectrum

Along canonical axes

Jittered Spectrum Profile

Other directions

Random Spectrum Profile
\[ \text{Var}(I_N) = \sum_{\Omega} \left( \langle \mathcal{P}_{SN}(\nu) \rangle \times \mathcal{P}_f(\nu) \right) = \sum_{\Omega} \]  

Variance due to N-rooks Sampler

\[ f(\bar{x}) \]

\[ \mathcal{P}_{SN}(\nu) \]

\[ \mathcal{P}_f(\nu) \]

N-rooks spectrum  

Integrand spectrum
Variance due to N-rooks Sampler

\[ \text{Var}(I_N) = \sum_{\Omega} \langle \mathcal{P}_{S_N}(\nu) \rangle \times \mathcal{P}_f(\nu) \]

\[ = \sum_{\Omega} \sum_{\Omega} \]

\[ \text{Var}(I_N) = \sum_{\Omega} \langle \mathcal{P}_{S_N}(\nu) \rangle \times \mathcal{P}_f(\nu) \]

\[ = \sum_{\Omega} \sum_{\Omega} \]

\[ f(\vec{x}) \]

\[ f(\vec{x}) \]

\[ N \text{-rooks spectrum} \]

\[ N \text{-rooks spectrum} \]

\[ \text{Integrand spectrum} \]

\[ \text{Integrand spectrum} \]
Variance Convergence of Latin Hypercube (N-rooks)

\[ \langle P_{SN}(\nu) \rangle \]

\[ P_f(\nu) \]

Pixel A

Pixel B

\[ O(N^{-1}) \]

\[ O(N^{-2}) \]

Number of Samples

Variance

10^{-4}

10^{-6}

10^{-8}

10^{-10}

10^{-12}

10^1

10^2

10^3

10^4

10^5

10^6
Non-Axis Aligned Integrand Spectra

\[ P_f(\nu) \]

Integrand Spectrum
Non-Axis Aligned Integrand Spectra

Multi-jittered Samples

\[ \mathcal{P}_{SN}(\nu) \]

Sampling Spectrum

\[ \mathcal{P}_f(\nu) \]

Integrand Spectrum
Shearing Multi-Jittered Samples

Singh and Jarosz [SIGGRAPH 2017]
Variance Heatmap

With Original Samples

Multiple images

Uncorrelated Multi-jittered
Variance Heatmap

With Original Samples

With Sheared Samples

Uncorrelated Multi-jittered
So far...

Blue noise samplers can have better convergence compared to stratified samples.

Denser stratification can lead to anisotropic spectra which improves convergence.
What properties we desire in a sampler?

Progressivity ✔️ (Ahmed et al. [2017], Christensen et al. [2018])

High speed (millions of samples per second) ✔️

Extension to dimensions beyond 2D ✗ (Spoke dart throwing, Mitchell [2018])
Low-Discrepancy Sampling

**Deterministic** sets of points specially crafted to be evenly distributed (have low discrepancy).

Entire field of study called Quasi-Monte Carlo (QMC)
# The Van der Corput Sequence

## Radical Inverse $\Phi_b$ in base 2

Subsequent points “fall into biggest holes”

<table>
<thead>
<tr>
<th>$k$</th>
<th>Base 2</th>
<th>$\Phi_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$0.1 = 1/2$</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>$0.01 = 1/4$</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>$0.11 = 3/4$</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>$0.001 = 1/8$</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>$0.101 = 5/8$</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>$0.011 = 3/8$</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>$0.111 = 7/8$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Halton and Hammersley Points

**Halton:** Radical inverse with different base for each dimension:

\[ \tilde{x}_k = (\Phi_2(k), \Phi_3(k), \Phi_5(k), \ldots, \Phi_{p_n}(k)) \]

- The bases should all be relatively prime.
- Incremental/progressive generation of samples

**Hammersley:** Same as Halton, but first dimension is \( k/N \):

\[ \tilde{x}_k = (k/N, \Phi_2(k), \Phi_3(k), \Phi_5(k), \ldots, \Phi_{p_n}(k)) \]

- Not incremental, need to know sample count, \( N \), in advance
The Hammersley Sequence

1 sample in each “elementary interval”
The Hammersley Sequence

1 sample in each “elementary interval”
The Hammersley Sequence

1 sample in each “elementary interval”
The Hammersley Sequence

1 sample in each “elementary interval”
The Hammersley Sequence

1 sample in each “elementary interval”
The Hammersley Sequence

1 sample in each “elementary interval”
Why do we need to scramble?

Halton Projection (29, 31)

Scrambled Halton Projection (29, 31)
Scrambled Low-Discrepancy Sampling
Monte Carlo (16 jittered samples)
Can we combine blue noise properties with low discrepancy?
Low-Discrepancy Blue Noise

Step spectrum

BNOT spectrum
Low-Discrepancy Blue Noise

Step spectrum

BNOT spectrum

LDBN Step

LDBN BNOT

Ahmed et al. [2016]
Low-Discrepancy Blue Noise

Ahmed et al. [2016]
Low-Discrepancy Blue Noise

Ahmed et al. [2016]
Low-Discrepancy Blue Noise 2D-Projections

Sobol

Perrier et al. [2018]
Low-Discrepancy Blue Noise 2D-Projections

Sobol

Special scrambling

Perrier et al. [2018]
Low-Discrepancy Blue Noise 2D-Projections

Sobol → Special scrambling → Blue noise characteristics

Perrier et al. [2018]
Low-Discrepancy Blue Noise 2D Sobol Projections

Perrier et al. [2018]
Blue noise sampling and beyond

Importance sampling with correlated samples
Light IS vs BSDF IS

Light Importance Sampling

BSDF Importance Sampling

Singh et al. [2019]
Scene illuminated by area direct lighting

Reference

Underlying pixel functions

Singh et al. [2019]
Unoccluded pixels' convergence benefit from Light IS

Singh et al. [2019]
Occluded pixels (no improvement in convergence)

Reference

Pixel Q

Singh et al. [2019]
Futuristic sampling target spectrum

Multi-jittered

Future design

Singh and Jarosz [2017]
Future research directions

Direct link between spatial and Fourier statistics needs further investigation

Progressive samplers in higher dimensions

Adapting sample correlations w.r.t. the underlying integrand in high dimensions
Acknowledgments

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